Moving Lemma for K_1-Chains

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1 Introduction

1.1 Arithmetic intersection theory plays the same important role in arithmetic geometry as the intersection theory in algebraic geometry. It was introduced in higher dimensions by Gillet-Soulé [GS]. This work is still the main reference for the basic theorems and proofs. The goal of the present article is to fill up a gap in the proof of compatibility of arithmetic intersection product and arithmetic rational equivalence. In the introduction, we explain first the construction of Gillet-Soulé and the gap, then we outline the article.

1.2 Let X be an irreducible smooth quasiprojective variety over a number field K and let $\mathscr{X}$ be a regular model of X of finite type, separated and flat over the ring of integers $\mathscr{O}_K$. More generally, we could work over an arithmetic ring as in [GS]. An arithmetic cycle on $\mathscr{X}$ is a pair $(\mathscr{Z}, g_Z)$ where $\mathscr{Z}$ is a cycle on $\mathscr{X}$ with generic fibre Z and g_Z is a Green's current for Z (cf. 4.2 for details). The Green's currents are only considered up to $\mathrm{im}(d) + \mathrm{im}(d^c)$. For a closed irreducible subvariety $\mathscr{W}$ of $\mathscr{X}$ and a non-zero rational function $f_{\mathscr{W}}$ on $\mathscr{W}$, we have an arithmetic cycle

$$\widehat{\mathrm{div}}(f_{\mathscr{W}}) = (\mathrm{div}(f_{\mathscr{W}}), \log |f_{\mathscr{W}}|^{-2})$$

on $\mathscr{X}$. The arithmetic Chow group $\widehat{CH}(\mathscr{X})$ is the quotient of the group of arithmetic cycles with respect to the subgroup generated by $\widehat{\mathrm{div}}(f_{\mathscr{W}})$ where $f_{\mathscr{W}}$ and $\mathscr{W}$ are ranging over all possible choices. A K_1-chain on $\mathscr{X}$ is a finite formal sum $\mathbf{f} = \sum_{\mathscr{W}} f_{\mathscr{W}}$ and hence the subgroup above is the set of all $\widehat{\mathrm{div}}(\mathbf{f})$.

1.3 If $(\mathscr{Y}, g_Y)$ and $(\mathscr{Z}, g_Z)$ are arithmetic cycles on $\mathscr{X}$ such that the generic fibres Y and Z intersect properly on X, then a proper intersection product

$$(\mathscr{Y}, g_Y).(\mathscr{Z}, g_Z) = (\mathscr{Y}.\mathscr{Z}, g_Y * g_Z)$$

is explained in [GS]. The product of cycles $\mathscr{Y}.\mathscr{Z}$ is well-defined as a cycle up to vertical rational equivalence and by introducing $\mathbb{Q}$-coefficients (cf. 4.9). The $*$-product $g_Y * g_Z$ is a Green's current for $Y.Z$ (cf. 4.2). To get a ring structure on $\widehat{CH}(\mathscr{X}) \otimes \mathbb{Q}$, one has to solve the following problem (cf. [GS], proof of theorem 4.2.3 on p. 144):

1.4 Let $\mathbf{f}$ be a K_1-chain on $\mathscr{X}$ such that the generic fibre of $\mathrm{div}(\mathbf{f})$ and Z intersect properly on X. Then one has to construct a K_1-chain $\mathbf{f}.Z$ (simply notation) on $\mathscr{X}$ such that

$$\widehat{\mathrm{div}}(\mathbf{f}.Z) = \widehat{\mathrm{div}}(\mathbf{f}).(\mathscr{Z}, g_Z)$$

up to vertical rational equivalence and up to $\mathrm{im}(d) + \mathrm{im}(d^c)$. The gap occurs in the construction of $\mathbf{f}.Z$ which we explain next.

1.5 One can show that it is enough to construct $\mathbf{f}.Z$ on the generic fibre X (cf. 4.9)), so we may ignore the model $\mathscr{X}$ over $\mathscr{O}_K$. In [GS] on p. 140, a K_1-chain $\mathbf{f} = \sum_W f_W$ is said to intersect the cycle Z properly on X if for all W with $f_W \neq 1$, we have

(a) W intersects Z properly on X;

(b) $\mathrm{div}(f_W)$ intersects Z properly on X.

For the definition of a K_1-chain, we refer to definition 2.2. If we require only (b), then we say that $\mathbf{f}$ intersects Z almost properly. To reduce the construction of $\mathbf{f}.Z$ to the case of a single f_W, one uses the following result.

1.6 Moving Lemma for K_1-chains ([GS], p. 142). Let $\mathbf{f}$ be a K_1-chain on X such that $\mathrm{div}(\mathbf{f})$ intersects Z properly on X. Then there is a K_1-chain $\mathbf{g}$ on X such that $\mathbf{g}$ is equal to $\mathbf{f}$ up to boundaries of K_2-chains (cf. 4.4) and such that $\mathbf{g}$ intersects Z almost properly.

1.7 Since the boundaries do not change the problem 1.4 in an essential way, the remark in 1.5 and the moving lemma for K_1-chains lead to the following reduction of 1.4: Let f_W be a non-zero rational function on the irreducible closed subvariety W of X and let Z be a cycle on X such that $\mathrm{div}(f_W)$ and Z intersect properly. Then one has to construct a K_1-chain $f_W.Z$ on X such that

$$\mathrm{div}(f_W.Z) = \mathrm{div}(f_W).Z$$

on X and such that

$$\log|f_W.Z| = \log|f_W| \wedge \delta_Z$$

up to $\mathrm{im}(d) + \mathrm{im}(d^c)$. In 1.8, the case of proper intersection is handled under an additional assumption. The general case of proper intersection is explained in 1.9 and example 1.10 shows where the gap occurs.

1.8 Suppose that f_W intersects Z properly on X. Let $\widetilde{f_W}$ be an extension of f_W to a rational function on X. Additionally, we assume that $\mathrm{div}(\widetilde{f_W})$ intersects $W.Z$ properly. Then we have

$$\mathrm{div}(f_W).Z = (\mathrm{div}(\widetilde{f_W}).W).Z = \mathrm{div}(\widetilde{f_W}).(W.Z)$$

and

$$\log|f_W|^{-2} * g_Z = (\log|\widetilde{f_W}|^{-2} * g_W) * g_Z = \log|\widetilde{f_W}|^{-2} \wedge \delta_{W.Z}$$

as an identity of Green's currents holding up to $\mathrm{im}(d) + \mathrm{im}(d^c)$. In this case, it is clear that $f_W.Z$ is the restriction of f_W to $W.Z$.

1.9 If f_W intersects Z properly, then $f_W.Z$ is defined as the restriction of f_W to $W.Z$ ([GS], p. 140). But without the additional hypothesis of 1.8, f_W is not necessarily a unit in the generic points of the components of $W.Z$. This will be shown in the following example. Hence the restriction doesn't make sense.

1.10 Example. Let $X = \mathbb{P}^2_K$ with affine coordinates x, y and let Z be the hyperplane $x = 0$. On the singular cubic W given by the affine equation $y^2 = x^2(x+1)$, we consider the restriction f_W of $\widetilde{f_W}(x,y) = \frac{y-x}{y+x}$. Note that $W \cap Z$ is the node $P_0 = (0,0)$ of W and the point at infinity of W. Moreover, $\mathrm{div}(f_W)$ is the zero-divisor and hence intersects Z properly. But f_W is not a unit at P_0.

1.11 In the case of almost proper intersection, the construction of $f_W.Z$ ([GS], p. 141) uses 1.9 in the proper components of intersection of W by Z. In the union T of non proper components of intersection, one restricts f_W to a representative of the intersection product $W.Z$. It is used there that $f_W|_T$ is a unit which is also unclear.

1.12 Pointing out the gap from 1.9 to Gillet-Soulé, they proposed in a letter to the author to replace the Weil divisor $\mathrm{div}(f_W)$ in the definition of (almost) proper intersection by a suitable larger set $D(f_W)$. The idea is to consider f_W as a function on the normalization W' of W, taking there the support of the divisor and defining its image in W as $D(f_W)$. In example 1.10,

$D(f_W)$ contains also P_0. If f_W intersects Z properly, then one can define $f_W.Z$ by working on W' where restriction is well-defined and then using projection formula. Hence we need a stronger moving lemma for K_1-chains for the new definition of almost proper intersection and we have to handle the case of almost proper intersection. The goal of this article is to supply the necessary details.

1.13 Here is a summary of the present article. The definition of a K_1-chain and its basic properties are given in the second section. The new moving lemma for K_1-chains is proved in the third section. The proof still follows the lines of Roberts' proof of Chow's moving lemma (cf. [Ro]). The idea is to write the K_1-chain $\mathbf{f}$ as a sum of cones over K_1-chains in projective space intersected with X and with a remaining summand which intersects Z properly (cf. (8) in the proof of lemma 3.3). This construction is based on the excess-lemma 3.2. Now we perform a $\mathbb{P}^1$-deformation of the cones in projective space to get a K_1-chain on X intersecting Z properly (cf. (9) in the proof of lemma 3.3). In the fourth section, we first summarize basic facts from the $*$-product of Green's currents. In the case of proper intersection, lemma 4.3 (resp. lemma 4.10) shows that $f_W.Z$ is what we want in the analytic (resp. geometric) setting. To omit the problem mentioned in 1.11, the new moving lemma for K_1-chains does not use the notion of almost proper intersection but is directly stated on $X \times \mathbb{P}^1$. To get the version 1.6 with the new definition of almost proper intersection, one can just use the K_1-chain $\mathbf{g}$ from 4.6. Applying the moving lemma for K_1-chains, we give the solution of problem 1.4 first in the analytic setting (proposition 4.7) and then refined intersection theory gives also the geometric side (4.9).
As the result of this paper is basic for arithmetic intersection theory, the proofs are kept elementary and are quite detailed. The use of higher algebraic K-theory is kept to a minimum, but a good understanding of intersection theory on the level of the first half of [Fu] is assumed.

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2 K_1-chains

2.1 Let X be a variety over a field K, i.e. a reduced scheme of finite type over K. For an irreducible closed subvariety W, we denote by $K(W)$ the field of rational functions on W.

2.2 Definition. A K_1-chain on X is an element

$$\mathbf{f} = \sum_W f_W \in \bigoplus_W K(W)^*$$

where W ranges over all irreducible closed subvarieties of X.

2.3 Clearly, the K_1-chains form a group. For an irreducible closed subvariety W of X, let $\pi_W : W' \to W$ be the normalization of W. Then W' is an irreducible normal variety over K and π_W is a finite surjective morphism which is birational. For $f_W \in K(W)^*$, let $f'_W := f_W \circ \pi_W$.

2.4 Definition. If f is a non-zero rational function on an irreducible normal variety, then $D(f)$ denotes the support of the Weil divisor of f. For any K_1-chain $\mathbf{f} = \sum f_W$, let

$$D(\mathbf{f}) := \bigcup_W \pi_W(D(f'_W)).$$

2.5 Definition. Let Z be a cycle on X and let $\mathbf{f} = \sum f_W$ be a K_1-chain on X. Then we say that $\mathbf{f}$ intersects Z properly if for any irreducible closed subvariety W with $f_W \neq 1$, W and $D(f_W)$ both intersect Z properly.

2.6 Let $\varphi : X \to Y$ be a proper morphism of varieties over K and let $\mathbf{f} = \sum f_V$ be a K_1-chain on X. We are going to define the push-forward of $\mathbf{f}$ as a K_1-chain of Y. For any irreducible closed subvariety V of X with $W := \varphi(V)$, there is a rational function $\varphi_*(f_V)$ on W given as follows. If the field extension $K(V)/K(W)$ is finite, then $\varphi_*(f_V)$ is the norm $N_{K(V)/K(W)}(f_V)$, otherwise $\varphi_*(f_V) := 1$. Finally, the push-forward of $\mathbf{f}$ is given by

$$\varphi_*(\mathbf{f}) := \sum_V \varphi_*(f_V).$$

2.7 Lemma. *Let $\varphi : X \to Y$ be a proper morphism of varieties over K and let $\mathbf{f} = \sum f_V$ be a K_1-chain on X. Then we have*

$$D(\varphi_*(\mathbf{f})) \subset \varphi(D(\mathbf{f})).$$

Proof: First, we may reduce to a single f_V. Then we may assume that V maps finitely onto W. By the universal property of normalizations, there is a unique morphism $\varphi' : V' \to W'$ over φ. Since φ' is proper, we have

$$\varphi'_*(\operatorname{div}(f'_V)) = \operatorname{div}(\varphi'_*(f'_V))$$

proving easily the claim. □

2.8 Now let X be a closed subvariety of $\mathbb{P}^n_K$ and let L be a linear subspace of $\mathbb{P}^n_K$ disjoint from X. For any irreducible closed subvariety W of X, we consider the projecting cone $C_L(W)$ over W with vertex L. Geometrically, it is the union of lines joining L and W. Another way to define it, is to use the linear projection $p_L : \mathbb{P}^n_K - L \to L'$ where L' is any linear subspace of $\mathbb{P}^n_K$ disjoint from L with $\dim(L) + \dim(L') = n - 1$. Then $C_L(W)$ is the closure of $p_L^{-1}(p_L(W))$ in $\mathbb{P}^n_K$. The projecting cone $C_L(W)$ is an irreducible closed subvariety of dimension equal to $\dim(W) + \dim(L) + 1$.

2.9 Under the same hypothesis as in 2.8, let $\mathbf{f} = \sum f_W$ be a K_1-chain on X. Then we define a rational function on $C_L(W)$ by

$$C_L(f_W) := ((p_L)_*(f_W)) \circ p_L.$$

By additivity, we extend this to define the projecting cone $C_L(\mathbf{f})$ of $\mathbf{f}$ with vertex L as a K_1-chain on $\mathbb{P}^n_K$. Clearly, this construction does not depend on the choice of L'.

2.10 Lemma. *Under the hypothesis as in 2.8, we have $D(C_L(\mathbf{f})) \subset C_L(D(\mathbf{f}))$.*

Proof: We reduce to the case of a single f_W. By lemma 2.7, we have

$$D((p_L)_*(f_W)) \subset p_L(D(f_W)).$$

On the other hand, the definition of $C_L(f_W)$ and the universal property of normalizations lead to

$$D(C_L(f_W)) - L \subset (p_L)^{-1}(D((p_L)_*(f_W))).$$

This proves $D(C_L(f_W)) - L \subset C_L(D(f_W))$ and by dimensionality reasons, we get the claim. □

2.11 Let X be a smooth variety over K. We consider a K_1-chain $\mathbf{f} = \sum f_W$ and a cycle Z on X intersecting properly. The goal is to define a product $\mathbf{f}.Z$ as a K_1-chain on X. By bilinearity, we may assume that $\mathbf{f} = f_W$ for a single irreducible closed subvariety W of X and that Z is prime. Since the normalization $\pi_W : W' \to X$ is a finite morphism into a smooth variety and since Z intersects W properly, the pull back $(\pi_W)^*(Z)$ is well-defined as a cycle on the normalization W' (cf. [Fu], chapter 8). Now the assumption that $D(f_W)$ and Z intersect properly implies that $\mathrm{div}(f'_W)$ intersects $(\pi_W)^*(Z)$ properly. Using normality, we see that the restriction of f'_W to every component V of $(\pi_W)^*(Z)$ is a well-defined rational function on V. Using linearity in the components of $(\pi_W)^*(Z)$, we define $f'_W.(\pi_W)^*(Z)$ as the restriction of f'_W to $(\pi_W)^*(Z)$. This leads to the K_1-chain

$$f_W.Z := (\pi_W)_*(f'_W.\pi_W^*(Z))$$

on X.

2.12 Let $\mathbf{f} = \sum f_W$ be a K_1-chain on X. Then $\mathrm{div}(\mathbf{f})$ is the cycle on X given by the sum of the Weil divisors of all the f_W's, viewed as cycles on X. If X is smooth and Z is a cycle on X intersecting $\mathbf{f}$ properly, then we can easily prove $\mathrm{div}(\mathbf{f}).Z = \mathrm{div}(\mathbf{f}.Z)$.

2.13 Lemma. *Let $\mathbf{f} = \sum f_W$ be a K_1-chain intersecting the cycle Z properly on the smooth variety X over K. Then we have $D(\mathbf{f}.Z) \subset D(\mathbf{f}) \cap \mathrm{supp}(Z)$.*

Proof: We may reduce to the case of a single f_W and a prime cycle Z. It is easy to see that

$$D(f'_W.\pi_W^*(Z)) \subset D(f'_W) \cap \pi_W^{-1}(Z)$$

and applying π_W, we get the claim. □

2.14 Lemma. *Let W and Z be irreducible closed subvarieties of the smooth variety X over K. Let f_W be a non-zero rational function on W intersecting Z properly. If V is an irreducible component of $W \cap Z$ and if f_W restricts to a well-defined rational function on V, then for the V-component of $f_W.Z$ and for the multiplicity m_V of $W.Z$ in V, we have*

$$(f_W.Z)_V = (f_W|_V)^{m_V}.$$

Proof: Let V' be a component of $\pi_W^*(Z)$ with multiplicity $m_{V'}$ lying over V. Note that

$$(\pi_W)_*(f'_W|_{V'}) = (f_W|_V)^{[K(V'):K(V)]}$$

and hence

$$(f_W.Z)_V = (f_W|_V)^{\sum_{V'} m_{V'}[K(V'):K(V)]}.$$

By projection formula, we have

$$W.Z = (\pi_W)_*(\pi_W^*(Z))$$

proving

$$m_V = \sum_{V'} m_{V'}[K(V') : K(V)].$$

□

2.15 Lemma. *Let Y and X be smooth varieties over K and let $i : Y \to X$ be a closed embedding. Suppose that Z is a prime cycle on X intersecting Y properly on X. Let $\mathbf{f}$ be a K_1-chain on Y intersecting $Y \cap \mathrm{supp}(Z)$ properly on Y. Then $i_*(\mathbf{f})$ intersects Z properly on X and we have the projection formula*

$$i_*(\mathbf{f}).Z = i_*(\mathbf{f}.i^*(Z)).$$

The proof is trivial since we use on both sides the same normalizations.

2.16 Lemma. *Let $\varphi : X \to Y$ be a morphism of irreducible normal varieties over K and let Z be a prime cycle on X. Let f be a non-zero rational function on Y such that $\varphi(Z)$ is* *not contained in the support of* $\mathrm{div}(f)$. *We assume that the restriction of* φ *to* Z *is a proper morphism* $Z \to Y$. *Then we have*

$$\varphi_*((f \circ \varphi).Z) = f.\varphi_*(Z).$$

Here, the products are defined by restriction. The claim even holds for arbitrary varieties as long as the restrictions are well-defined.

Proof: In fact, the claim is a trivial consequence of

$$N_{K(Z)/K(\varphi(Z))}(f \circ \varphi) = f^{[K(Z):K(\varphi(Z))]}.$$

$\square$

Let $\varphi : X \to Y$ be a flat morphism of smooth varieties. For a non-zero rational function f_W on an irreducible closed subvariety W of Y, we define $\varphi^*(f_W)$ to be the K_1-chain on X given as the restriction of $f_W \circ \varphi$ to the cycle $\varphi^*(W)$.

2.17 Lemma. *Suppose that* Z *is a cycle on* X *intersecting both* $\varphi^{-1}(W)$ *and* $\varphi^{-1}(D(f_W))$ *properly. We assume that the restriction of* φ *to the support of* Z *is proper. Then* f_W *intersects* $\varphi_*(Z)$ *properly and we have the projection formula*

$$\varphi_*(\varphi^*(f_W).Z) = f_W.\varphi_*(Z).$$

Proof: Clearly, $\varphi^*(f_W)$ intersects Z properly. To see that f_W intersects $\varphi_*(Z)$ properly, we may assume that Z is prime. Note that

$$\begin{aligned}
\mathrm{codim}(W, Y) &\geq \mathrm{codim}(\varphi(Z) \cap W, \varphi(Z)) \\
&\geq \mathrm{codim}(Z \cap \varphi^{-1}(W), Z) \\
&= \mathrm{codim}(\varphi^{-1}(W), X) \\
&= \mathrm{codim}(W, Y)
\end{aligned}$$

where we have used flatness in the last step. Hence we have equality everywhere. Similarly, we argue for $D(f_W)$ instead of W proving that f_W intersects $\varphi_*(Z)$ properly. Let $V := \varphi^{-1}(W)$. By the universal property of normalization, we have a morphism $\varphi' : V' \to W'$ such that the diagram

$$\begin{tikzcd}
V' \arrow[r, "\varphi'"] \arrow[d, "\pi_V"] & W' \arrow[d, "\pi_W"] \\
X \arrow[r, "\varphi"] & Y
\end{tikzcd}$$

is commutative. From lemma 2.16, we get

$$\varphi_*(\varphi^*(f_W).Z) = (\pi_W)_*(f'_W|_{\varphi'_*(\pi_V^*(Z))}).$$

Note that there is a birational morphism $V' \to X \times_Y W'$. Using projection formula and the fibre square rule, we get

$$\varphi'_*(\pi_V^*(Z)) = \pi_W^*(\varphi_*(Z))$$

proving our projection formula. $\square$

2.18 Lemma. *Let* $\varphi : X \to Y$ *be a finite surjective morphism of irreducible normal varieties over* K *and let* D *be an effective Cartier divisor of* Y. *Let* f *be a non-zero rational function on* X *whose divisor does not contain any component of* $\varphi^*(D)$. *Then* $\mathrm{div}(\varphi_*(f))$ *does not contain any component of* D *and we have*

$$\varphi_*(f.\varphi^*(D)) = \varphi_*(f).D.$$

Note that in the case of Cartier divisors, the product with a K_1-chain is well-defined even if the variety is not smooth(using [Fu], chapter 2).

Proof: Obviously, $\varphi(\mathrm{div}(f))$ does not contain any component of D and hence both sides of the projection formula are well-defined. Note that the claim is local in D, so we may assume that the support Z of D is irreducible. Let ζ be the generic point of Z. Then we have to show that both sides of the projection formula have the same Z-component. Since $\mathscr{O}_{Y,\zeta}$ is a discrete valuation ring, we may assume that $D = \mathrm{div}(t)$ for a local parameter t in ζ. We may assume that $Y = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(A)$ are affine varieties and that $f \in A$. Note that $A \otimes_B \mathscr{O}_{Y,\zeta}$ is a free $\mathscr{O}_{Y,\zeta}$-module of rank $\deg(\varphi)$. The Z-component of $\varphi_*(f).D$ equals

$$N_{A\otimes_B \mathscr{O}_{Y,\zeta}/\mathscr{O}_{Y,\zeta}}(f)|_Z = N_{A\otimes_B K(Z)/K(Z)}(\bar{f}) \tag{1}$$

where $\bar{f}$ denotes the canonical element in $A \otimes_B K(Z)/K(Z)$ induced by f. By the theory of artinian rings, we have

$$A \otimes_B K(Z) \cong \prod_{\xi\in\varphi^{-1}(\zeta)} \mathscr{O}_{\varphi^*(D),\xi}$$

and so we deduce

$$N_{A\otimes_B K(Z)/K(Z)}(\bar{f}) = \prod_{\xi\in\varphi^{-1}(\zeta)} N_{\mathscr{O}_{\varphi^*(D),\xi}/K(Z)}(\bar{f}). \tag{2}$$

Let t_ξ be a local parameter in the discrete valuation ring $\mathscr{O}_{X,\xi}$. Then

$$\mathscr{O}_{\varphi^*(D),\xi} \cong \mathscr{O}_{X,\xi}/(\mathscr{O}_{X,\xi} t_\xi^{m_\xi})$$

where m_ξ is the multiplicity of $\varphi^*(D)$ in ξ. This proves easily

$$N_{\mathscr{O}_{\varphi^*(D),\xi}/K(Z)}(\bar{f}) = N_{K(\xi)/K(Z)}(\bar{f})^{m_\xi}. \tag{3}$$

On the other hand, the Z-component of $\varphi_*(f.\varphi^*(D))$ equals

$$\prod_{\xi\in\varphi^{-1}(\zeta)} N_{K(\xi)/K(Z)}(\bar{f})^{m_\xi}.$$

Using (1)-(3), we get the claim. □

3 Moving Lemma

3.1 We assume that X is a smooth irreducible variety over a field K and that Z is an irreducible closed subvariety of X. Recall that the excess of an irreducible closed subset Y of X with respect to Z is

$$e_Z(Y) := \mathrm{codim}(Y,X) + \mathrm{codim}(Z,X) - \mathrm{codim}(Y \cap Z, X).$$

If Y is any closed subset of X, then the excess of Y with respect to Z is the maximum of the excesses of the irreducible components of Y with respect to Z. The excess is always non-negative and it is zero if and only if Z and Y intersect properly.
Now let $\mathbf{f} = \sum f_W$ be a K_1-chain on X. The excess of f_W with respect to Z is defined by

$$e_Z(f_W) := e_Z(W) + e_Z(D(f_W))$$

and the excess $e_Z(\mathbf{f})$ is the maximum of all $e_Z(f_W)$ with W ranging over all irreducible closed subvarieties of X with $f_W \neq 1$. Note that $e_Z(\mathbf{f}) = 0$ if and only if $\mathbf{f}$ intersects Z properly. We can also extend the definition to cycles Z by using again the maximum over all components.

The next lemma was used by J. Roberts [Ro] for his proof of Chow's moving lemma. Gillet and Soulé adapted it to the the case of K_1-chains. We denote by $G(n,r)$ the space of projective linear subspaces of $\mathbb{P}^n_K$ of dimension r.

3.2 Lemma. *Let X be an irreducible smooth subvariety of $\mathbb{P}^n_K$ and let Z be a cycle on X. For a K_1-chain $\mathbf{f} = \sum f_W$ with $f_X = 1$, there is an open dense subset of $G(n, n-1-\dim(X))$ whose $\overline{K}$-rational points L satisfy*

(a) $L \cap X = \emptyset$;

(b) $e_Z(C_L(\mathbf{f}).X - \mathbf{f}) \leq \max\{0, e_Z(\mathbf{f}) - 1\}$.

Proof: Clearly, we may assume K algebraically closed and that $\mathbf{f} = f_W$ for a single W. By the main lemma of [Ro] on p. 93, there is an open dense subset U of $G(n, n-1-\dim(X))$ with

$$L \cap X = \emptyset \tag{4}$$

$$e_Z(C_L(W).X - W) \leq \max\{0, e_Z(W) - 1\} \tag{5}$$

and

$$e_Z(C_L(D(f_W)).X - D(f_W)) \leq \max\{0, e_Z(D(f_W)) - 1\} \tag{6}$$

for all $L \in U$. Note that $C_L(f_W)$ intersects X properly. Shrinking U a little bit, we may assume that the linear projection p_L from 2.8 maps W birationally onto $p_L(W)$. Here, we have to assume $W \neq X$. There is an open dense subset W_0 of W mapping isomorphically onto $p_L(W_0)$ such that f_W is a regular function on W_0 without zeros. By construction, $C_L(f_W)$ is a regular function without zeros on the open dense subset $p_L^{-1}(p_L(W_0))$ of $C_L(W)$. If $P \in p_L^{-1}(p_L(W_0))$, then there is a unique $Q \in W_0$ with $p_L(P) = p_L(Q)$ and we have $C_L(f_W)(P) = f_W(Q)$. In particular, the restriction of $C_L(f_W)$ to W is equal to f_W. If $e_Z(W) \geq 1$, then it follows from (5) that W has multiplicity one in $C_L(W).X$. In fact, this was one of the main steps in [Ro] in the proof of lemma 6. If $e_Z(W) = 0$, then one can not directly conclude from (5) that W has multiplicity 1. But the arguments of Roberts still apply for generic $L \in U$. By lemma 2.14, we conclude that the W-component of the K_1-chain $C_L(f_W).X$ equals f_W. Using (6) and lemma 2.10, this leads to

$$e_Z(D(C_L(f_W).X - f_W)) \leq \max\{0, e_Z(D(f_W)) - 1\}. \tag{7}$$

The claim is now a consequence of (4),(5) and (7). □

The next statement is the moving lemma for K_1-chains. It is done in [GS], however with a weaker notion of proper intersection. As mentioned in 1.13, it is better to state the moving lemma on $X \times \mathbb{P}^1_K$ rather than on X. Given a K_1-chain $\mathbf{F}$ on $X \times \mathbb{P}^1_K$ intersecting the fibre over $t \in \mathbb{P}^1_K$ properly, we denote by $\mathbf{F}_t := \mathbf{F}.(X \times t)$ the fibre over t. We call $\mathbf{F}$ horizontal if $F_W \neq 1$ only for irreducible closed subvarieties W of $X \times \mathbb{P}^1_K$ mapping onto $\mathbb{P}^1_K$ by the second projection.

3.3 Lemma. *Let X be a smooth quasiprojective variety over an infinite field K. Let Z be a cycle on X and let $\mathbf{f} = \sum f_W$ be a K_1-chain on X such that $\mathrm{div}(\mathbf{f})$ intersects Z properly on X. Then there is a horizontal K_1-chain $\mathbf{F}$ on $X \times \mathbb{P}^1_K$ such that*

(a) $\mathbf{F}$ *intersects $X \times 0$ and $X \times \infty$ properly on $X \times \mathbb{P}^1_K$;*

(b) $\mathbf{F}_0 = \mathbf{f}$;

(c) $\mathbf{F}_\infty$ *intersects Z properly on X;*

(d) The non-proper components of intersection of $\mathrm{div}(\mathbf{F})$ by $\mathrm{supp}(Z) \times \mathbb{P}^1_K$ are contained in finitely many fibres $p_2^{-1}(t_1), \ldots, p_2^{-1}(t_r)$ with $t_i \in \mathbb{P}^1_K - \{0, 1\}$.

Proof: Clearly, we may assume that X is an irreducible subvariety of $\mathbb{P}^n_K$ and that all W with $f_W \neq 1$ have the same dimension d. If $d = \dim(X)$, then $\mathbf{f}$ is a single rational function on X and its pull back to $X \times \mathbb{P}^1_K$ fullfills the claim. So we may assume $d < \dim(X)$. We choose a generic subspace $L \subset \mathbb{P}^n_K$ of dimension $n-1-\dim(X)$. For any $j \in \mathbb{N}$, we define a K_1-chain $\mathbf{f}_j$ on X by starting with

$$\mathbf{f}_0 := \mathbf{f}$$

and then going on recursively with

$$\mathbf{f}_j := C_L(\mathbf{f}_{j-1}).X - \mathbf{f}_{j-1}.$$

Note that $C_L(\mathbf{f}_{j-1})$ intersects X always properly. By lemma 3.2, we have

$$e_Z(\mathbf{f}_0) > e_Z(\mathbf{f}_1) > \cdots > e_Z(\mathbf{f}_N) = 0$$

for some $N \in \mathbb{N}$. A trivial algebraic calculation shows

$$\mathbf{f} = \sum_{j=0}^{N-1} (-1)^j C_L(\mathbf{f}_j).X + (-1)^N \mathbf{f}_N. \tag{8}$$

The group $\mathrm{PGL}(n+1,K)$ operates on K_1-chains of $\mathbb{P}^n_K$ by $\rho(\mathbf{g}) := (\rho^{-1})^*(\mathbf{g}) = \rho_*(\mathbf{g})$. We choose a generic $\rho \in \mathrm{PGL}(n+1,K)$ such that $\rho(C_L(\mathbf{f}_j))$ intersects Z and also X properly on $\mathbb{P}^n_K$. There is a rational map γ from $\mathbb{P}^1_K$ to $\mathrm{PGL}(n+1,K)$ with γ_0 equal to the identity and $\gamma_\infty = \rho$. Let U be a sufficiently small open subset of the domain of γ containing 0 and ∞. We consider the morphism

$$\varphi : \mathbb{P}^n_K \times U \to \mathbb{P}^n_K, (P,t) \mapsto \gamma_t^{-1}(P).$$

For a K_1-chain $\mathbf{h}$ on $\mathbb{P}^n_K$, we get a K_1-chain $\varphi^*(\mathbf{h})$ on $\mathbb{P}^n_K \times U$ with the fibre property

$$\varphi^*(\mathbf{h})_t = \gamma_t(\mathbf{h})$$

for all $t \in U$. This follows from the projection formula 2.17.
We may assume that $\varphi^*(C_L(\mathbf{f}_j))$ intersects $X \times U$ properly on $\mathbb{P}^n_K \times U$. To see it, just check the fibres over 0 and ∞ where it holds trivially by the fibre property above and then we may pass to a smaller open subset U. We may also assume that $\varphi^*(C_L(\mathbf{f}_j)).(X \times U)$ is a horizontal K_1-chain. Let $\mathbf{F}_j$ be the canonical extension of $\varphi^*(C_L(\mathbf{f}_j)).(X \times U)$ to a horizontal K_1-chain of $X \times \mathbb{P}^1_K$. We define a K_1-chain

$$\mathbf{F} := \sum_{j=0}^{N-1} (-1)^j \mathbf{F}_j + (-1)^N p_1^*(\mathbf{f}_N) \tag{9}$$

on $X \times \mathbb{P}^1_K$. Here p_1 is the first projection.
Using lemma 2.13 and lemma 2.10, we easily get

$$D(\mathbf{F}_j) \cap (X \times U) \subset \varphi^{-1}(C_L(D(\mathbf{f}_j))) \cap (X \times U).$$

This leads to (a). It is also obvious that

$$\mathrm{div}(\mathbf{F}_j).(X \times 0) = \mathrm{div}(C_L(\mathbf{f}_j))$$

and

$$\mathrm{div}(\mathbf{F}_j).(X \times \infty) = \mathrm{div}(C_L(\rho(\mathbf{f}_j))).X.$$

Using flatness of the projection p_L, we get

$$\mathrm{div}(C_L(\mathbf{f}_j)) = C_L(\mathrm{div}(\mathbf{f}_j)).$$

For a generic choice of L, we prove by induction that $\operatorname{div}(\mathbf{f}_j)$ intersects Z properly on X and that $\operatorname{div}(C_L(\mathbf{f}_j))$ intersects Z properly on $\mathbb{P}^n_K$. Here, we use the excess lemma for cycles ([Ro], main lemma on p. 93). Hence there is an open neighbourhood V of 0 in $\mathbb{P}^1_K$ such that $\operatorname{div}(C_L(\gamma_t(\mathbf{f}_j)))$ intersects Z properly on $\mathbb{P}^n_K$ for $t \in V$. Similarly, we argue over an open neighbourhood of ∞. therefore the non-proper components of intersection of $\operatorname{div}(\mathbf{F}_j)$ by $Z \times \mathbb{P}^1_K$ are lying over finitely many points of $\mathbb{P}^1_K - \{0, \infty\}$. This proves (d).
Next, we handle (b). We prove first

$$(\mathbf{F}_j)_0 = C_L(\mathbf{f}_j).X. \tag{10}$$

We may assume that $\mathbf{f}_j$ is given by a single rational function f_j on an irreducible closed subvariety W_j. Then $g := \varphi^*(C_L(f_j))$ lives on the irreducible closed subvariety $V := \varphi^{-1}(C_L(W_j))$ of $\mathbb{P}^n_K \times U$. Furthermore, we put $Y := X \times U$ and $D := \mathbb{P}^n_K \times 0$. By definition, we have

$$(\mathbf{F}_j)_0 = (p_1)_*((g.Y).(X \times 0))$$

where the first product is computed on $\mathbb{P}^n_K \times U$ and the second product is on $X \times U$. By lemma 2.15, we get

$$(\mathbf{F}_j)_0 = (p_1)_*((g.Y).D)).$$

We prove below that the following associativity holds

$$(g.Y).D = g.(Y.D). \tag{11}$$

We conclude that

$$\begin{aligned} (\mathbf{F}_j)_0 &= (p_1)_*(g.(Y.D)) \\ &= \varphi_*(g.(X \times 0)). \end{aligned}$$

Using the definition of g and projection formula 2.17, we get (10). Moreover, projection formula 2.17 shows

$$p_1^*(\mathbf{f}_N)_0 = (p_1)_*(p_1^*(\mathbf{f}_N).(X \times 0)) = \mathbf{f}_N. \tag{12}$$

Then (b) follows from (9), (10), (12) and (8).
Now we claim that

$$\mathbf{F}_\infty = \sum_{j=0}^{N-1} (-1)^j \rho(C_L(\mathbf{f}_j)).X + (-1)^N \mathbf{f}_N. \tag{13}$$

The proof is similar as for (10). With the same notation and assumption as above but with $D = \mathbb{P}^n_K \times \infty$, we have

$$(\mathbf{F}_j)_\infty = (p_1)_*((g.Y).D)) = (p_1)_*(g.(Y.D))$$

as before. We conclude

$$\begin{aligned} (\mathbf{F}_j)_\infty &= (p_1)_*(g.(Y.D)) \\ &= (p_1)_*(\varphi^* C_L(f_j).(X \times \infty)). \end{aligned}$$

Now using $p_1 = \rho \circ \varphi$ on $X \times \infty$ and two times projection formula 2.17, we get

$$(\mathbf{F}_j)_\infty = \rho_*(C_L(f_j)).X$$

proving (13). If B is an irreducible closed subvariety of X and C is an irreducible closed subvariety of $\mathbb{P}^n_K$ intersecting X properly on $\mathbb{P}^n_K$, then it is easy to see that B and C intersect properly on $\mathbb{P}^n_K$ if and only if B and $C \cap X$ intersect properly on X. This general remark, our assumptions on ρ and lemma 2.13 imply that $(\mathbf{F}_j)_\infty$ intersects Z properly on X. This proves (c).

It remains to prove (11). We have a morphism ψ and a commutative diagram

$$\begin{tikzcd}
\pi_V^{-1}(Y)' \arrow[r, "\pi_2"] \arrow[d, "\psi"] & V' \arrow[d, "\pi_V"] \\
(Y \cap V)' \arrow[r, "\pi_1"] & \mathbb{P}^n_K \times U
\end{tikzcd}$$

of normalizations. Here, π_1 and π_2 are the normalization morphisms for $Y \cap V$ and $\pi_V^{-1}(Y)$, respectively. The existence of ψ follows from the fact that $\pi_V \circ \pi_2$ maps $\pi_V^{-1}(Y)'$ dominantly onto $Y \cap V$ and from the universal property of normalizations. Since g is a rational function on V, both $g.Y$ and $g.(Y.D)$ are computed on V'. By birationality of normalization, there is a rational function h on $\pi_V^{-1}(Y)'$ with $(\pi_2)_*(h) = g'.\pi_V^*(Y)$. By lemma 2.16, we have

$$(\pi_2)_*(h.\pi_2^*\pi_V^*D) = (g'.\pi_V^*Y).\pi_V^*D.$$

On the right hand side, associativity follows immediately from lemma 2.14 since we have simply to restrict functions without passing to the normalizations. We conclude that

$$(\pi_2)_*(h.\pi_2^*\pi_V^*D) = g'.(\pi_V^*(Y.D)). \tag{14}$$

Lemma 2.18 shows that

$$\psi_*(h.\pi_2^*\pi_V^*D) = \psi_*h.\pi_1^*(D) \tag{15}$$

holds. Now the obvious equality

$$(\pi_1)_*\psi_*h = (\pi_V)_*(g'.\pi_V^*Y) = g.Y$$

implies

$$(g.Y)' = \psi_*h. \tag{16}$$

Applying (16), (15) and (14), we get

$$\begin{aligned}
(g.Y).D &= (\pi_1)_*(\psi_*h.\pi_1^*D) \\
&= (\pi_1)_*\psi_*(h.\pi_2^*\pi_V^*D) \\
&= (\pi_V)_*(g'.\pi_V^*(Y.D)).
\end{aligned}$$

By definition, the last term is $g.(Y.D)$ proving (11). □

4 Rational Equivalence in Arithmetic Intersection Theory

4.1 Let X be a smooth complex quasi-projective variety endowed with its structure as a complex manifold. Let f_W be a non-zero rational function on the irreducible closed analytic subvariety W. Then $\log|f_W|$ denotes the current on X which maps the smooth differential form ρ to

$$(\log|f_W|)(\rho) := \int_W \log|f_W| \wedge \rho.$$

By additivity, we define $\log|\mathbf{f}|$ for any K_1-chain $\mathbf{f} = \sum f_W$.

4.2 Now we recall some basic facts from the $*$-product of Gillet and Soulé. For details, the reader is referred to [GS]. For a cycle Z on X, a Green's current g_Z is a current on X such that

$$\omega_Z := dd^c g_Z + \delta_Z$$

is (the current associated to) a smooth differential form on X. Here, δ_Z is the current of integration over the cycle Z. In 4.1, the current $-2\log|\mathbf{f}|$ is a Green's current for $\mathrm{div}(\mathbf{f})$ with $\omega_{\mathrm{div}(\mathbf{f})} = 0$. Every cycle has a Green's current. In [GS], logarithmic Green's forms for Z are also defined. They are smooth differential forms outside the support of Z with logarithmic growth conditions along Z. It is proved that for every Green's current g_Z, there is a logarithmic Green's form η_Z with $\eta_Z \equiv g_Z$, where $\equiv$ means equality up to $\mathrm{im}(d) + \mathrm{im}(d^c)$. If Y is a cycle with no component contained in the support of Z and η_Y is a logarithmic Green's form for Y, then we define a current $\eta_Y \wedge \delta_Z$ on X by mapping the smooth differential form ρ to

$$(\eta_Y \wedge \delta_Z)(\rho) := \int_Z \eta_Y \wedge \rho.$$

This is well-defined because the restriction of η_Y to Z is integrable on Z. Finally, we define $g_Y \wedge \delta_Z := \eta_Y \wedge \delta_Z$. This is only well-defined up to the relation $\equiv$.
If Y and Z intersect properly, then the $*$-product of Green's currents g_Y and g_Z is defined by

$$g_Y * g_Z := g_Y \wedge \delta_Z + \omega_Y \wedge g_Z.$$

This is a Green's current for the proper intersection product $Y.Z$, again only well-defined up to $\equiv$. Now let $\mathbf{f}$ be a K_1-chain on X such that $\mathrm{div}(\mathbf{f})$ and Z intersect properly on X. Then we have

$$\log(|\mathbf{f}|^{-2}) * g_Z \equiv \log(|\mathbf{f}|^{-2}) \wedge \delta_Z.$$

It is a basic problem of arithmetic intersection theory to show that this is equivalent to the Green's current of a K_1-chain (cf. 1.3 and 1.4). This will follow below from our moving lemma for K_1-chains 3.3. First, we handle an easier case.

4.3 Lemma. *If $\mathbf{f}$ intersects Z properly, then we have*

$$\log|\mathbf{f}.Z| \equiv \log|\mathbf{f}| \wedge \delta_Z.$$

Proof: We may assume that $\mathbf{f}$ is given by a single $f_W \in K(W)^*$. On the normalization W', we have

$$\log|f'_W.\pi^*(Z)| \equiv \log|f'_W| \wedge \delta_{\pi^*(Z)}.$$

This is almost by definition. Note that we may work directly on W' using differential forms on singular spaces from Bloom-Herrera [BH]. Then the projection formula will give the claim. For the convenience of the reader, we sketch the argument. For a logarithmic Green's form η_Z, we get

$$\log|f'_W.\pi_W^*(Z)|^{-2} \equiv \pi_W^*(\eta_Z) * \log|f'_W|^{-2} \equiv \pi_W^*(\eta_Z) \wedge \delta_{\mathrm{div}(f'_W)} + \pi_W^*(\omega_Z) \wedge \log|f'_W|^{-2}.$$

Then the projection formula as a consequence of the transformation formula of integrals shows

$$\log|f_W.Z|^{-2} \equiv \eta_Z \wedge \delta_{\mathrm{div}(f_W)} + \omega_Z \wedge \log|f_W|^{-2} \equiv \eta_Z * \log|f_W|^{-2}.$$

Using again commutativity, we get the claim. □

4.4 Let X be a variety over a field K. Then there is a boundary map

$$d : \bigoplus_V K_2(K(V)) \longrightarrow \bigoplus_V K_1(K(V))$$

from the E_1-term of the Quillen spectral sequence where V ranges over all irreducible closed subvarieties of X (cf. [GS], p. 128). Since $K_1(K(V)) = K(V)^*$, the range of the map d is a subgroup of the K_1-chains. The quotient relation with respect to the range of d is denoted by $\equiv$. We have the following commutativity result for our product of K_1-chains.

4.5 Lemma. *Let* $\mathbf{f}$ *be a* K_1*-chain and let* g *be a non-zero rational function on an irreducible normal variety* X *such that* $\mathbf{f}$ *intersects* $\mathrm{div}(g)$ *properly. Then we have*

$$\mathbf{f}.\mathrm{div}(g) \equiv g.\mathrm{div}(\mathbf{f}).$$

Proof: We may assume that $\mathbf{f}$ is given by a single f_W. Using lemma 2.16, it is enough to prove the claim on the normalization W'. So we have reduced the claim to the case where $\mathbf{f} = f$ is also a non-zero rational function on X. Then the claim follows from the fact that $f.\mathrm{div}(g) - g.\mathrm{div}(f)$ is up to a sign equal to the tame-symbol which is a boundary (cf. [GS], p. 129). $\square$

4.6 Let X be a smooth quasi-projective variety over an infinite field K. Let $\mathbf{f}$ be a K_1-chain on X such that $\mathrm{div}(\mathbf{f})$ intersects the cycle Z properly on X. We choose a horizontal K_1-chain $\mathbf{F}$ on $X \times \mathbb{P}^1_K$ satisfying (a)-(d) of Lemma 3.3. Let t be the coordinate on $\mathbb{P}^1_K$ and let p_1 be the first projection of $X \times \mathbb{P}^1_K$. By (d) and refined intersection theory (cf. [Fu], chapter 8), there is a finite subset $S \subset \mathbb{P}^1_K - \{0,1\}$ such that $\mathrm{div}(\mathbf{F}).p_1^*(Z)$ is well-defined as a cycle on $X \times \mathbb{P}^1_K$ up to rational equivalence in the fibres over S. Since t is invertible on the fibres over S, lemma 4.5 shows easily that

$$\mathbf{h} := t.(\mathrm{div}(\mathbf{F}).p_1^*(Z))$$

is well-defined as a K_1-chain up to $\equiv$ by choosing a representative for $\mathrm{div}(\mathbf{F}).p_1^*(Z)$. Then we define our K_1-chain on X by

$$\mathbf{g} := \mathbf{F}_\infty.Z + (p_1)_*(\mathbf{h}).$$

4.7 Proposition. *Under the hypothesis of 4.6, we have*

$$\log|\mathbf{g}| \equiv \log|\mathbf{f}| \wedge \delta_Z.$$

This equality of currents has to be understood on the complex analytic variety associated to X and $\mathbf{f}$, $\mathbf{g}$ and Z have to be replaced by the corresponding base changes.

Proof: If $\mathbf{f}_1$ and $\mathbf{f}_2$ are K_1-chains on X equivalent with respect to the relation $\equiv$ defined in 4.4, then we have $\log|\mathbf{f}_1| \equiv \log|\mathbf{f}_2|$ (cf. p.129 of [GS]). Hence the left hand side of the claim is well-defined. We have

$$\log|\mathbf{g}| = \log|\mathbf{F}_\infty.Z| + (p_1)_*(\log|\mathbf{h}|).$$

Using lemma 4.3, we get

$$\log|\mathbf{F}_\infty.Z| \equiv \log|\mathbf{F}_\infty| \wedge \delta_Z.$$

Almost by definition, we have

$$\log|\mathbf{h}| \equiv \log|t| \wedge \delta_{\mathrm{div}(\mathbf{F}).p_1^*(Z)}.$$

Suppose for the moment that the intersection of $\mathrm{div}(\mathbf{F})$ and $p_1^*(Z)$ is proper. Using the basic rules of the $*$-product and a logarithmic Green's form η_Z for Z, we get

$$\begin{aligned}
\log|\mathbf{h}| &\equiv \log|t| * (\log|\mathbf{F}|^{-2} * p_1^*(\eta_Z)) \\
&\equiv (\log|t| * (\log|\mathbf{F}|^{-2})) * p_1^*(\eta_Z) \\
&\equiv (\log|\mathbf{F}| \wedge \delta_{\mathrm{div}(t)}) * p_1^*(\eta_Z).
\end{aligned}$$

By projection formula and lemma 2.14, we conclude

$$(p_1)_*(\log|\mathbf{h}|) \equiv (\log|\mathbf{F}_0| - \log|\mathbf{F}_\infty|) \wedge \delta_Z.$$

Finally, lemma 3.3(b) proves

$$\log|\mathbf{g}| \equiv \log|\mathbf{f}| \wedge \delta_Z$$

as claimed. In general, there are finitely many non-proper components of the intersection of $\mathrm{div}(\mathbf{F})$ by $p_1^*(Z)$ lying over the finite set $S \subset \mathbb{P}^1_K - \{0,1\}$. Then we use Theorem 2.2.2 on p. 116 of [GS] instead of associativity to justify the above considerations. □

4.8 Remark. Now we apply our result to the arithmetic situation described in 1.4. The goal is to show that arithmetic rational equivalence is compatible with the arithmetic intersection product. Let X be a quasiprojective K-variety with a regular separated and flat $\mathscr{O}_K$-model $\mathscr{X}$ of finite type, where $\mathscr{O}_K$ is the ring of integers of a number field K.
Recall that there is a canonical extension of a cycle Y on X to a horizontal cycle $\overline{Y}$ on $\mathscr{X}$. For a prime cycle, it is just the Zariski closure in $\mathscr{X}$. Clearly, we have a similar extension for K_1-chains on X. For example, if f_W is a rational function on an irreducible closed subset W of X, then $\overline{f_W}$ is the unique extension of f_W to a rational function on $\overline{W}$. By additivity, we extend this notion to all K_1-chains of X.

4.9 Proposition. *Let $\mathbf{f} = \sum f_{\mathscr{W}}$ be a K_1-chain on $\mathscr{X}$ where $\mathscr{W}$ is ranging over all irreducible closed subsets of $\mathscr{X}$. Let $\mathscr{Z}$ be a cycle on $\mathscr{X}$ with generic fibre Z and let g_Z be a Green's current for Z (to be understood after base change to $\mathbb{C}$). We assume that $\mathrm{div}(\mathbf{f}|_X)$ and Z intersect properly on X. On the generic fibre X, we are in the situation of 4.6 where the moving lemma for K_1-chains gives us a K_1-chain $\mathbf{g}$ on X. Then we claim that*

$$\widehat{\mathrm{div}}(\bar{\mathbf{g}}) \equiv \widehat{\mathrm{div}}(\mathbf{f}).(Z, g_Z).$$

Proof: The claim means that

$$\log|\mathbf{g}| \equiv \log|\mathbf{f}| \wedge \delta_Z$$

and

$$\mathrm{div}(\bar{\mathbf{g}}) \equiv \mathrm{div}(\mathbf{f}).\mathscr{Z}.$$

As usual, the first identity has to be understood after base change to $\mathbb{C}$. It follows from proposition 4.7. In the second identity, $\equiv$ means up to vertical rational equivalence, i.e. up to the divisors of rational functions living on closed irreducible subvarieties contained in the finite fibres of $\mathscr{X}$. We denote by $CH_{\mathrm{fin}}(\mathscr{X})$ the group of cycles on $\mathscr{X}$ modulo the equivalence relation $\equiv$. Then the second identity takes place in $CH_{\mathrm{fin}}(\mathscr{X}) \otimes \mathbb{Q}$ because one needs the isomorphism to K-theory to justify the product on the right hand side (cf. [GS], 4.1). The proof of the second identity follows the arguments of 4.7.
It is easy to see that the canonical extensions of equivalent K_1-chains on X have equivalent divisors on $\mathscr{X}$. Hence the left hand side of the claim is well-defined. We have

$$\mathrm{div}(\overline{\mathbf{g}}) \equiv \mathrm{div}(\overline{\mathbf{F}_\infty.Z}) + (p_1)_*(\mathrm{div}(\overline{\mathbf{h}})).$$

By lemma 4.10 below, we obtain

$$\mathrm{div}(\overline{\mathbf{F}_\infty.Z}) \equiv \mathrm{div}(\overline{\mathbf{F}_\infty}).\mathscr{Z}.$$

Clearly, we have

$$\mathrm{div}(\overline{\mathbf{h}}) \equiv \mathrm{div}(t).\overline{\mathrm{div}(\mathbf{F}).p_1^*(Z)}.$$

Using the rules of refined intersection theory, we get

$$\mathrm{div}(\overline{\mathbf{h}}) \equiv \mathrm{div}(t).\mathrm{div}(\overline{\mathbf{F}}).p_1^*(\mathscr{Z}).$$

Using

$$(p_1)_*(\mathrm{div}(\overline{\mathbf{F}}).\mathrm{div}(t)) \equiv \mathrm{div}(\overline{\mathbf{F}_0}) - \mathrm{div}(\overline{\mathbf{F}_\infty}) \equiv \mathrm{div}(\mathbf{f}) - \mathrm{div}(\overline{\mathbf{F}_\infty})$$

and projection formula, we get the claim. $\square$

4.10 Lemma. *Suppose that* $\mathbf{f} = \sum f_W$ *is a* K_1*-chain on* X *and* Z *is a cycle on* X *such that* $\mathbf{f}$ *intersects* Z *properly on* X*. Then*

$$\mathrm{div}(\bar{\mathbf{f}}).\bar{Z} \equiv \mathrm{div}(\overline{\mathbf{f}.Z}) \in CH_{\mathrm{fin}}(\mathscr{X}) \otimes \mathbb{Q}.$$

Proof: It is enough to consider a single rational function f_W and a prime cycle Z. Let $\bar{\pi} : \overline{W}' \to \overline{W}$ be the normalization of $\bar{W}$. Clearly, the generic fibre π is the normalization of W. Let $f_W' := f_W \circ \pi$. By assumption, the intersection of $\mathrm{div}(f_W')$ and $\pi^{-1}(Z)$ is proper on W'. We use first

$$\mathrm{div}(\overline{f_W'}).\bar{\pi}^*(\bar{Z}) \equiv \mathrm{div}(\overline{f_W'}).\overline{\pi^*(Z)} \in CH_{\mathrm{fin}}(\overline{W}') \otimes \mathbb{Q}.$$

Then note that $\mathrm{div}(\overline{f_W'}).\overline{\pi^*(Z)}$ is the divisor of the restriction of $\overline{f_W'}$ to $\overline{\pi^*(Z)}$. The latter is the horizontal extension of the restriction of f_W' to $\pi^*(Z)$. By the very definition, this gives

$$\mathrm{div}(\overline{f_W'}).\overline{\pi^*(Z)} = \mathrm{div}(\overline{f_W'.\pi^*(Z)}).$$

We conclude that

$$\mathrm{div}(\overline{f_W'}).\bar{\pi}^*(\bar{Z}) \equiv \mathrm{div}(\overline{f_W'.\pi^*(Z)}) \in CH_{\mathrm{fin}}(\overline{W}') \otimes \mathbb{Q}.$$

Applying $\bar{\pi}_*$, we get the claim by using the projection formula on the left hand side and the definition of product on the right hand side. $\square$

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