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Local heights of subvarieties over non-archimedean fields

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Introduction

In diophantine geometry, heights of points are an important tool to handle finiteness questions. If X is a projective variety over a number field, then the height of a point $P \in X$ with respect to an embedding of X into $\mathbb{P}^n$ measures the arithmetic complexity of its coordinates. Nesterenko and Philippon generalized this notion to subvarieties Y of X. They defined the height of Y as the height of the Chow form associated to Y as a subvariety of $\mathbb{P}^n$ (cf. [Ph]). The Chow form is a multi-homogeneous polynomial and its height is the height of the vector of coefficients viewed as a point in a suitable projective space. Equivalently, Faltings [Fa] defined the height of Y as an arithmetic intersection number. This definition points out that the height of Y is the arithmetic analogue of the degree of Y in algebraic geometry. For the comparison to the Nestrenko-Philippon height, the reader is referred to [BoGS]. In this paper, Bost-Gillet-Soulé prove a lot of important properties of these heights, e.g. an arithmetic Bézout theorem. If the embedding is changed, then the difference of heights is bounded by a multiple of the degree. This generalization of Weil's theorem is proved in [Gu1].

In his studies of heights of rational points, Weil pointed out that most of the properties are coming from local heights (cf. [We]). For any field K with a fixed absolute value $|\ |_v$, there is a local height of a $\bar{K}$-rational point P with respect to the metrized line bundle $(L, \|\ \|_v)$. It depends on the choice of an invertible meromorphic section s of L and is given by $-\log\|s(P)\|_v$. In the number field case, any global height is the sum of the corresponding local heights with respect to all places of the field. For example, Weil's decomposition theorem is the local version of Weil's theorem mentioned above.

Using Falting's definition, it is quite clear how to define local heights of subvarieties in the archimedean and in the discrete case. If the valuation is archimedean, then it is given by a $*$-product of Green currents of the form $[\log\|s\|^{-2}]$ on the subvariety. For a discrete valuation, the local height of a t-dimensional subvariety Y of X is an intersection number on a model $\mathfrak{X}$ of X over the discrete valuation ring. More precisely, let $D_0, \ldots, D_t$ be Cartier divisors on $\mathfrak{X}$, then the local height of Y is the intersection number $D_0 \ldots D_t.\bar{Y}$

where $\bar{Y}$ is the closure of Y in $\mathfrak{X}$. Note that D_j induces an invertible meromorphic section s_j and a metric on the line bundle $L_j = O(D_j|_X)$ of X. It was noted by Zhang [Zh] that the dependence of the local height of Y on the model is measured by the metrics.

If the non-archimedean valuation v on K is not necessarily discrete, then intersection numbers on a model are not defined since the valuation ring K° has not to be noetherian. In [Gu 2], local heights of t-dimensional subvarieties are characterized by five properties. They depend on $(\hat{L}_j, s_j)_{j=0,\ldots,t}$ where s_j is an invertible meromorphic section of the metrized line bundle $\hat{L}_j$ on X. The first property is multi-linearity and symmetry in (L_j, s_j). Then it is assumed that local heights are functorial. Furthermore, the behaviour of the local heights under change of metrics or sections is described. Finally, the local height of subvarieties on a multi-projective space is given in terms of the Chow forms. Clearly, in the discrete and in the archimedean situation, the local heights described above satisfy these five properties ([Gu 2], Theorem 1.10, Theorem 1.14).

If the ground field K is an M-field, then the existence of local heights of subvarieties leads to a theory of global heights of subvarieties generalizing Weil's results for points. (An M-field is a field with a family of absolute values as number fields, function fields or the field of meromorphic functions on the unit disc playing a fundamental role in Nevanlinna theory.) This gives a unified theory of global heights. For example, Weil's theorem leads to a generalization of the first main theorem in Nevalinna theory. For details we refer to [Gu 2].

The goal of this article is to show the existence of local heights in the non-archimedean situation. The idea is to replace the algebraic K°-models of X by formal K°-models of the rigid analytic variety associated to X. First we define an intersection product of a Cartier divisor with a cycle on a rigid analytic variety and then we extend it to admissible formal K°-models using the theory of Bosch-Lütkebohmert ([BL 3], [BL 4]) initiated by Raynaud. It is shown that the dependence of this intersection product on the model is measured by the metric. Similarly as in non-archimedean Arakelov Theory [BlGS], we consider cycles as projective limits over the family of all models. For a quasi-compact and quasi-separated rigid analytic variety, we get a model-free description of the intersection product.

The paper is organized as follows. In section 1, we give the necessary definitions and results from rigid and formal geometry. The basic reference for rigid geometry is [BGR]. Let K be a field with a complete non-archimedean absolute value and corresponding valuation ring K°. For an admissible formal scheme $\mathfrak{X}$ over K°, there is a canonical rigid analytic variety X associated to $\mathfrak{X}$ called the generic fibre of X. We call $\mathfrak{X}$ a formal K°-model of X. Instead of formal models of X, we can consider formal analytic structures on X. The latter are given by topologies on X allowing reductions of X over the residue field. They are coarser than the Grothendieck topology of X. Very important for the following sections is the interplay between formal analytic structures and formal models described in 1.10 and Proposition 1.11. Finally, we state Raynaud's description of rigid geometry in terms of formal K°-models.

In section 2, we define the proper intersection product of a Cartier divisor and a cycle on X. Locally, X is given by a K-affinoid algebra $\mathscr{A}$. On the noetherian scheme Spec $\mathscr{A}$, we can use [EGA IV] or [Fu] to define this intersection product. Locally, this induces a cycle on X and by a gluing process, we get our intersection product. It is bilinear and satisfies projection formula. The action of Cartier divisors is commutative. By the above localization process, these properties follow immediately from the corresponding properties on noetherian schemes. Finally, we consider the effect of base change.

In section 3, we define the Weil divisor $\mathrm{cyc}(D)$ associated to a Cartier divisor D on the formal K°-model $\mathfrak{X}$ of X. A cycle on $\mathfrak{X}$ is a formal sum of a cycle on the generic fibre of X called the horizontal component and of a cycle on the special fibre $\tilde{\mathfrak{X}}$ of $\mathfrak{X}$ called the vertical component. The horizontal part of $\mathrm{cyc}(D)$ is given by the considerations of section 2. To define the multiplicity of $\mathrm{cyc}(D)$ in an irreducible component W of $\tilde{\mathfrak{X}}$, we assume first that K is algebraically closed and that $\mathfrak{X}$ is formal affine with reduced special fibre $\tilde{\mathfrak{X}}$. By considerations of Berkovich [Be1], there is a distinguished multiplicative semi-norm on the algebra $\mathscr{A}$ of X associated to W. For $a \in \mathscr{A}$, it is given by $\sup|a(x)|$ where x is ranging over all points of X with reduction not contained in any other component than W. Applying $-\log$ to this semi-norm, we get the multiplicity of a in W. Using this construction locally, we obtain Weil divisors associated to Cartier divisors for K algebraically closed and for formal K°-models $\mathfrak{X}$ with reduced special fibre. By base change, projection formula and linearity, we can always reduce to this situation. Let us give an explanation for this construction. The semi-norms of Berkovich occur in the frame work of formal analytic structures on X. If X is reduced and K is algebraically closed, then there is a one-to-one correspondence between formal-analytic structures on X and formal K°-models of X with reduced special fibre. If K is not algebraically closed, then reductions of formal analytic structures behave not well with respect to base change and so we can not use them to define the multiplicities. However, if the reduction of $\mathfrak{X}$ is geometrically reduced, then it is shown in Lemma 3.21 that we may compute the multiplicites in the special fibre directly on $\mathfrak{X}$. If the base field is stable, then we deduce in Lemma 5.7 a formula for these multiplicities (without any assumptions on the reduction) by using ramification indices and residue degrees.

In section 4, we define the intersection product of a Cartier divisor D and a cycle $\mathscr{Z}$ on $\mathfrak{X}$ intersecting properly in the generic fibre X. This is now immediate using sections 2 and 3. Then we prove projection formula. The idea is to reduce to the case of a finite field extension. Then projection formula is just the well-known formula relating the degree of the extension to ramification indices and residue degrees.

Commutativity of the intersection product is proved in section 5. Let D, D' be Cartier divisors on $\mathfrak{X}$ intersecting properly in the generic fibre. Then we have to prove $D.\mathrm{cyc}(D') = D'.\mathrm{cyc}(D)$. If the intersection is not proper on $\mathfrak{X}$, the vertical parts seem to be only equivalence classes with respect to rational functions on the special fibre. Since we deal with divisors, we can give a refined interpretation of $D.\mathrm{cyc}(D')$ as a cycle. The idea is to normalize the equation of D with respect to the semi-norm corresponding to a vertical prime component of $\mathrm{cyc}(D')$. So commutativity is an identity of cycles. Let us sketch the proof of commutativity. By base change, we may assume that X is a curve over an algebraically closed field. Then we show that X may be replaced by an admissible open subset of a non-singular projective curve. The semi-stable reduction theorem gives us a formel K°-model with semi-stable reduction. By projection formula, it is allowed to change models and so we use the semi-stable model to compute explicitly the multiplicities of both sides in an irreducible component of the special fibre.

Now let Y be an algebraic variety over K. It induces a rigid analytic variety Y^{an} over K and any cycle on Y gives rise to a cycle on Y^{an}. In section 6, we show that the proper intersection product of Cartier divisors and cycles on Y is compatible with the corresponding intersection product on Y^{an} considered in section 2. Moreover, if the valuation on K is discrete and $\mathscr{Y}$ is an algebraic flat K°-model of Y, then we have an intersection product on $\mathscr{Y}$. It is shown that under formal completion it is compatible with the intersection product of section 4.

Back to the general case, we study in section 7 metrized line bundles on the quasi-compact and quasi-separated rigid analytic variety X. We introduce formal metrics and we show that they are induced by formal K°-models and that any line bundle has a formal metric. A metric is called approximable if it is the uniform limit of roots of formal metrics. By the Stone-Weierstrass theorem, we prove that a metric is approximable if and only if it has a continuous extension to the Berkovich compactification of X.

In section 8, we consider the directed family of all formal K°-models of the quasi-compact and quasi-separated rigid analytic variety X. The projective limit of the group of vertical cycles gives us a definition of vertical cycles on X not depending on a particular choice of a formal K°-model. Let D be a Cartier divisor on a formal K°-model $\mathfrak{X}$. It induces a line bundle L on X with an invertible meromorphic section s and a formal metric $\| \ \|$. The vertical part of the intersection product of D with a horizontal cycle $\mathscr{Z}$ on $\mathfrak{X}$ gives rise to a vertical cycle on X depending only on the generic fibre of Z and $(L, \| \ \|, s)$. So we get a model-free description of intersection product.

In section 9, we define local heights of cycles on a proper algebraic variety Y over K. They depend on line bundles $L_0, \ldots, L_t$ of Y, on invertible meromorphic sections s_j of L_j and on formal metrics $\| \ \|$ of L_j. Such a local height is defined for all t-dimensional cycles Z on Y such that the intersection of Z, $\operatorname{div}(s_0), \ldots, \operatorname{div}(s_t)$ is empty. Using the theory of section 8, it is given by an intersection number. We show that these local heights have the five characteristic properties described above.

In the appendix, we deduce some basic properties of coherent $\mathscr{O}_{\mathfrak{X}}$-modules on admissible formal affine schemes over K°. They are used in section 3 to study the closure of a subvariety of X in a formal K°-model $\mathfrak{X}$.

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1. Basic facts from rigid and formal geometry

Let K be a field with a non-trivial non-archimedean complete absolute value $|\ |$.

1.1. On the polynomial ring $K[x_1, \ldots, x_n]$, we have the Gauss norm

$$\left| \sum_{\nu_1, \ldots, \nu_n \geqq 0} a_{\nu_1 \cdots \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n} \right| = \max_{\nu_1, \ldots, \nu_n \geqq 0} |a_{\nu_1 \cdots \nu_n}| \, .$$

The completion of $K[x_1, \ldots, x_n]$ with respect to the Gauss norm is called the Tate algebra and is denoted by $K\langle x_1, \ldots, x_n \rangle$. A K-affinoid algebra $\mathscr{A}$ is a quotient of a Tate algebra. Using the quotient norm, $\mathscr{A}$ is a Banach algebra. Banach's open mapping theorem implies that any algebra homomorphism between K-affinoid algebras is continuous and bounded ([BGR], 6.1.3). So the Banach algebra structure of $\mathscr{A}$ is uniquely determined by the algebraic structure. A homomorphism of K-affinoid varieties is simply an algebra homomorphism. Any K-affinoid algebra is noetherian ([BGR], Proposition 6.1.1/3).

1.2. A K-affinoid variety is a pair $\mathrm{Sp}\,\mathscr{A} := (\mathrm{Max}\,\mathscr{A}, \mathscr{A})$ where $\mathscr{A}$ is a K-affinoid algebra and $\mathrm{Max}\,\mathscr{A}$ is its spectrum of maximal ideals. The category of K-affinoid varieties is the dual category of K-affinoid algebras, i.e. a morphism $\mathrm{Sp}\,\mathscr{A} \to \mathrm{Sp}\,\mathscr{B}$ is an algebra homomorphism $\mathscr{B} \to \mathscr{A}$ together with the induced morphism $\mathrm{Max}\,\mathscr{A} \to \mathrm{Max}\,\mathscr{B}$. The K-affinoid variety induced by the Tate-algebra $K\langle x_1, \ldots, x_n \rangle$ is called the unit ball and is denoted by $\mathbb{B}^n$. Any K-affinoid algebra may be viewed as a subspace of a unit ball defined by finitely many power series.

1.3. Let $X = \mathrm{Sp}\,\mathscr{A}$ be a K-affinoid variety and let $x \in X$, i.e. x is a maximal ideal m_x of $\mathscr{A}$. The residue field $\mathscr{A}/m_x$ is a finite extension of K ([BGR], Corollary 6.1.2/3). Since K is complete, there is a unique extension of $|\ |$ to an absolute value on $\mathscr{A}/m_x$. For $a \in \mathscr{A}$, we define $|a(x)|$ as the absolute value of the class of a in $\mathscr{A}/m_x$. The supremum semi-norm ([BGR], 6.2.1) on $\mathscr{A}$ is defined by

$$|a|_{\sup} := \sup_{x \in \mathrm{Sp}\,\mathscr{A}} |a(x)| \, .$$

It is a norm if and only if $\mathscr{A}$ is reduced ([BGR], Proposition 6.2.1/4). Consider

$$\mathscr{A}^\circ := \{a \in \mathscr{A}; |a|_{\sup} \leqq 1\}$$

and

$$\mathscr{A}^{\circ\circ} := \{a \in \mathscr{A}; |a|_{\sup} < 1\} \, .$$

The reduction $\tilde{\mathscr{A}}$ of $\mathscr{A}$ is defined by $\tilde{\mathscr{A}} := \mathscr{A}^\circ / \mathscr{A}^{\circ\circ}$. The reduction $\tilde{X}$ of X is the affine scheme $\tilde{X} := \mathrm{Spec}\,\tilde{\mathscr{A}}$ of finite type over $\tilde{K}$ ([BGR], Corollary 6.3.4/3). The algebra $\tilde{\mathscr{A}}$ is reduced since the supremum semi-norm is power multiplicative.

If there is an epimorphism $K\langle x_1, \ldots, x_n \rangle \to \mathscr{A}$ such that the residue norm coincides with the supremum semi-norm on $\mathscr{A}$, then $\mathscr{A}$ is called distinguished ([BGR], Definition 6.4.3/2). If K is algebraically closed, then $\mathscr{A}$ is distinguished if and only if $\mathscr{A}$ is reduced ([BGR], Theorem 6.4.3/1).

1.4. A rational domain in $X = \mathrm{Sp}\,\mathscr{A}$ is a set

$$X\left(\frac{\mathbf{f}}{g}\right) := \{x \in X;\, |f_j(x)| \leqq |g(x)|, j = 1, \ldots, n\}$$

where $g, f_1, \ldots, f_n$ are elements of $\mathscr{A}$ without common zeros. There is a canonical K-affinoid algebra $\mathscr{A}\left\langle\frac{\mathbf{f}}{g}\right\rangle$ satisfying a certain universal property ([BGR], 7.2.2, 7.2.3) such that $X\left(\frac{\mathbf{f}}{g}\right)$ is the spectrum of maximal ideals in $\mathscr{A}\left\langle\frac{\mathbf{f}}{g}\right\rangle$. As a special case ([BGR], Corollary 7.2.3/8) of rational domains, we have the Laurent domains

$$X(\mathbf{f}, \mathbf{g}^{-1}) := \{x \in X;\, |f_i(x)| \leqq 1, |g_j(x)| \geqq 1, i = 1, \ldots, m, j = 1, \ldots, n\}$$

in X where $f_1, \ldots, f_m, g_1, \ldots, g_n \in \mathscr{A}$. A strict Laurent domain is a Laurent domain as above with $f_1 = \cdots = f_m = 1$ and $g_1, \ldots, g_n \in \mathscr{A}^{\circ}$. A strict Laurent domain is denoted by $X(\mathbf{g}^{-1})$.

More generally, one defines affinoid subdomains of X ([BGR], 7.2.2). They are characterized by the universal property mentioned above. By a theorem of Gerritzen and Grauert, they are finite unions of rational domains ([BGR], Corollary 7.3.5/3). So it is enough for our purposes to know rational domains.

1.5. On the K-affinoid variety $X = \mathrm{Sp}\,\mathscr{A}$, there is the (strong) Grothendieck topology ([BGR], 9.1.4). The rational domains in X form a basis of this Grothendieck topology and any admissible open covering of a rational domain has a refinement by finitely many rational domains. There is a canonical sheaf of rings $\mathscr{O}_X$ on the Grothendieck topology of X determined by

$$\mathscr{O}_X\left(X\left(\frac{\mathbf{f}}{g}\right)\right) = \mathscr{A}\left\langle\frac{\mathbf{f}}{g}\right\rangle$$

for any rational domain $X\left(\frac{\mathbf{f}}{g}\right)$ in X.

1.6. We have a canonical map $\pi : X = \mathrm{Sp}\,\mathscr{A} \to \tilde{X}$ of sets. A subset of X is called formal open if it is the inverse image of a Zariski open subset of $\tilde{X}$. This gives a quasi-compact topology on X ([Bo 2], §1). Obviously, the strict Laurent domains form a basis of this so-called formal topology. The Grothendieck topology is finer than the formal topology and so we can restrict $\mathscr{O}_X$ to get a ringed space $\mathrm{Spf}\,\mathscr{A}$ on the formal topology of X. Such spaces are called formal K-affinoid varieties. Any homomorphism $\mathscr{A} \to \mathscr{B}$ of K-affinoid varieties induces a morphism $\mathrm{Spf}\,\mathscr{B} \to \mathrm{Spf}\,\mathscr{A}$ of ringed spaces. Only such morphisms are said to be morphisms of formal K-affinoid varieties.

1.7. A locally G-ringed space over K is a pair $(X, \mathscr{O}_X)$ where X is a set with a Grothendieck topology and $\mathscr{O}_X$ is a sheaf of K-algebras on the Grothendieck topology such that the stalks are local rings. The morphisms of locally G-ringed spaces over K are similarly defined as morphisms of locally ringed spaces ([BGR], 9.3.1). A K-affinoid variety is a locally G-ringed space over K and morphisms of K-affinoid varieties are the same as morphisms of locally G-ringed spaces over K ([BGR], Proposition 9.3.1/1).

A rigid analytic variety over K is a locally G-ringed space $(X, \mathcal{O}_X)$ over K which has an admissible open covering by K-affinoid varieties and which satisfies (G_0), (G_1) and (G_2) of [BGR], 9.1.2. The last three conditions are of minor importance here and if they are not satisfied, we may construct a slightly finer Grothendieck topology with the same basis such that (G_0), (G_1), (G_2) are fulfilled. The category of rigid analytic varieties over K is viewed as a full subcategory of the locally G-ringed spaces over K, i.e. they have the same morphisms.

1.8. A formal analytic variety over K is a K-ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ (i.e. $\mathfrak{X}$ is a topological space and $\mathcal{O}_{\mathfrak{X}}$ is a sheaf of K-algebras on $\mathfrak{X}$) such that for any $x \in \mathfrak{X}$, there is an open neighbourhood U such that $(U, \mathcal{O}_{\mathfrak{X}}|_U)$ is a formal K-affinoid variety. Such an U is called a formal open affinoid subspace of $\mathfrak{X}$.

A morphism of formal analytic varieties over K is a morphism of K-ringed spaces which induces locally a morphism of formal K-affinoid varieties. Since the category of K-affinoid varieties is equivalent to the category of formal K-affinoid varieties, we can paste the formal affinoid open subspaces together as rigid analytic varieties over K ([BGR], Proposition 9.3.2/1) to get a rigid analytic variety $\mathfrak{X}^{\mathrm{an}}$ over K. A morphism $\varphi : \mathfrak{X} \to \mathcal{Y}$ of formal analytic varieties over K induces a morphism $\varphi^{\mathrm{an}} : \mathfrak{X}^{\mathrm{an}} \to \mathcal{Y}^{\mathrm{an}}$ of rigid analytic varieties over K since it is determined by local formal K-affinoid data.

An important property of a formal analytic variety $\mathfrak{X}$ over K is that it has a canonical reduction $\tilde{\mathfrak{X}}$: For any formal open affinoid subspace U, we have the reduction $\tilde{U}$. The schemes $\tilde{U}$ paste together to give a reduced scheme $\tilde{\mathfrak{X}}$ locally of finite type over $\tilde{K}$ [Bo2]. The following statement is proved in [Bo2], § 3.

Theorem 1.9 (Bosch). (i) *A formal analytic variety over K is formal K-affinoid if and only if its reduction is affine.*

(ii) *A morphism of formal analytic varieties over K is finite if and only if its reduction is a finite morphism of schemes.*

(iii) *A formal analytic variety over K is separated if and only if its reduction is a separated scheme.*

1.10. A K°-algebra A is called admissible if it is isomorphic to the quotient of $K^\circ\langle x_1, \ldots, x_n\rangle$ by an ideal I and if A has no K°-torsion. The latter is equivalent to flatness. If A is admissible, then I is finitely generated ([BL3], Proposition 1.1).

A formal scheme $\mathfrak{X}$ over K° is said to be admissible if it is locally isomorphic to a formal affine scheme $\operatorname{Spf} A$ where A is an admissible K°-algebra. Note that $\mathcal{A} = A \otimes_{K^\circ} K$ is a K-affinoid algebra. Moreover, we can glue the formal K-affinoid varieties $\operatorname{Spf} \mathcal{A}$ together and we obtain a formal analytic variety $\mathfrak{X}^{f-\mathrm{an}}$ over K. We denote the rigid analytic variety corresponding to $\mathfrak{X}^{f-\mathrm{an}}$ by $\mathfrak{X}^{\mathrm{an}}$.

Conversely, let $\mathfrak{X}$ be a formal analytic variety over K. Locally, $\mathfrak{X}$ looks like $\mathrm{Spf}\,\mathscr{A}$ for a K-affinoid algebra $\mathscr{A}$. Then $\mathscr{A}^\circ$ has no K°-torsion and we can glue the formal affine varieties $\mathrm{Spf}\,\mathscr{A}^\circ$ together to obtain a formal scheme $\mathfrak{X}^{f-\mathrm{sch}}$ over K°.

For a formal scheme $\mathfrak{X}$ over K°, we denote the special fibre by $\tilde{\mathfrak{X}}$. It is the $\tilde{K}$-scheme whose underlying topological space is the same as the one of $\mathfrak{X}$ and whose sheaf of regular functions is given by $\mathcal{O}_{\tilde{\mathfrak{X}}} := \mathcal{O}_{\mathfrak{X}} \otimes_{K^\circ} \tilde{K}$.

We have a natural finite morphism $(\mathfrak{X}^{f-\mathrm{an}})^\sim\ 1\ \ \tilde{\mathfrak{X}}$ of K-schemes, locally induced by $A \otimes_{K^\circ} \tilde{K} \to (A \otimes_{K^\circ} K)^\sim$ ([BL2], §1). Clearly, the above maps $\mathfrak{X} \to \mathfrak{X}^{f-\mathrm{an}}$, $\mathfrak{X} \to \mathfrak{X}^{f-\mathrm{sch}}$, $\mathfrak{X} \to \tilde{\mathfrak{X}}$ are all functorial. The morphism $(\mathfrak{X}^{f-\mathrm{an}})^\sim \to \tilde{\mathfrak{X}}$ is surjective if the formal scheme $\mathfrak{X}$ over K° is admissible ([BL2], §1).

The next result of Bosch, Lütkebohmert ([BL2], Lemma 1.1) was stated in the case of a discrete complete valuation, but the proof holds in general. A formal analytic variety is called distinguished if it is locally isomorphic to $\mathrm{Spf}\,\mathscr{A}$ for a distinguished K-affinoid algebra $\mathscr{A}$.

Proposition 1.11. *The functors $\mathfrak{X} \to \mathfrak{X}^{f-\mathrm{sch}}$ and $\mathfrak{X} \to \mathfrak{X}^{f-\mathrm{an}}$ give an equivalence between the category of distinguished formal analytic varieties over K and the category of admissible formal schemes over K° with reduced special fibre. For any distinguished formal analytic variety $\mathfrak{X}$ over K, the canonical morphism $\tilde{\mathfrak{X}} \to (\mathfrak{X}^{f-\mathrm{sch}})^\sim$ is an isomorphism.*

1.12. We denote by I a principal ideal of K°, $I \neq K^\circ$. Let $\mathfrak{X}$ be an admissible formal scheme over K° and let $\mathscr{J} := I\mathcal{O}_X$ be an ideal of definition. The formal scheme $\mathfrak{X}$ may be viewed as a direct limit of schemes $\mathfrak{X}_\lambda := \mathfrak{X} \otimes_{K^\circ} (K^\circ / I^{\lambda+1})$ $(\lambda \in \mathbb{N})$. An admissible formal blowing up is a morphism $p: \mathfrak{X}' \to \mathfrak{X}$ obtained by the following construction: Let $\mathscr{K}$ be a coherent ideal of $\mathcal{O}_{\mathfrak{X}}$ which is locally on a formal affine neighbourhood $\mathrm{Spf}\,A$ induced by a finitely generated ideal of A containing a power of I. Then

$$\mathfrak{X}' := \varinjlim_{\lambda \in \mathbb{N}} \mathrm{Proj} \bigoplus_{n=0}^{\infty} (\mathscr{K}^n \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathscr{J}^{\lambda+1}))$$

and p is the natural projection. Note that $\mathfrak{X}'$ is an admissible formal scheme over K° ([BL3], Proposition 2.1). A morphism $\mathfrak{X} \to \mathfrak{Y}$ in the category of admissible formal schemes over K°, localized by admissible formal blowing ups, is a diagram

$$\begin{array}{ccc} & \mathfrak{X}' & \\ {}^{p}\swarrow & & \searrow^{\varphi} \\ \mathfrak{X} & & \mathfrak{Y} \end{array}$$

where φ is a morphism of admissible formal schemes and p is an admissible formal blowing up.

Composition of two morphisms is given by

$$\left(\begin{array}{ccc} & \mathfrak{X}' & \\ {}^{p}\swarrow & & \searrow^{\varphi} \\ \mathfrak{X} & & \mathscr{Y}\end{array}\right) \circ \left(\begin{array}{ccc} & \mathscr{Y}' & \\ {}^{p''}\swarrow & & \searrow^{\varphi'} \\ \mathscr{Y} & & \mathfrak{Z}\end{array}\right)$$

$$= \left(\begin{array}{ccc} & \mathfrak{X}' \times_{\mathscr{Y}} \mathscr{Y}' & \\ \swarrow & & \searrow \\ \mathfrak{X} & & \mathfrak{Z}\end{array}\right).$$

This is well defined since admissible formal blowing ups are stable under composition and base change ([BL 3], Remark 2.4).

For a detailed proof of the following result of Raynaud [R], the reader may consult [BL 3], § 4.

Theorem 1.13. *There is an equivalence between the category of quasi-compact admissible formal schemes over K°, localized by admissible formal blowing ups, and the category of rigid analytic varieties which are quasi-compact and quasi-separated. It is induced by the functor $\mathfrak{X} \to \mathfrak{X}^{\mathrm{an}}$.*

Proposition 1.14 (Kiehl). *Let $\varphi : X \to Y$ be a proper morphism of rigid analytic varieties over K. For $r \in \mathbb{N}$, the set $\{y \in Y \mid \dim f^{-1}(y) \geqq r\}$ is an analytic subset of Y.*

Proof. This follows from [Ki 2] and the proper mapping theorem [Ki 1]. □

2. Divisors on rigid analytic varieties

Let K be a field with a non-trivial non-archimedean complete absolute value $|\ |$ and let X be a rigid analytic variety over K.

2.1. An analytic subset of X is a subset Y of X which is locally on a K-affinoid neighbourhood given as the zero set of finitely many regular functions. A cycle on X is a locally finite formal sum

$$\sum n_Y Y$$

where $n_Y \in \mathbb{Z}$ and Y ranges over all irreducible analytic subsets of X. Locally finite means that it exists a covering $\{U_k\}$ of X, admissible with respect to the Grothendieck topology, such that any element U_k of the covering intersects only finitely many Y with $n_Y \neq 0$. A Weil divisor is a cycle on X such that all Y with $n_Y \neq 0$ have codimension 1 in X.

Definition 2.2. Extending [EGA IV], § 21 to G-ringed spaces, one has a theory of Cartier divisors. More precisely, let $\mathscr{S}$ be the subsheaf of $\mathscr{O}_X$ consisting of the elements which are not zero divisors. The sheaf of meromorphic functions is given by $\mathscr{M}_X := \mathscr{O}_X(\mathscr{S}^{-1})$. It is a sheaf of K-algebras. Let $\mathscr{M}_X^*$ (resp. $\mathscr{O}_X^*$) be the sheaf of invertible elements in $\mathscr{M}_X$

(resp. $\mathcal{O}_X$). A Cartier divisor on X is a global section of $\mathcal{M}_X^*/\mathcal{O}_X^*$. Global sections of $\mathcal{M}_X^*$ are called invertible meromorphic functions.

Remark 2.3. Let $\mathcal{L}$ be an invertible sheaf on X and let s be an invertible meromorphic section of $\mathcal{L}$, i.e. locally, under a trivialization, s corresponds to a section of $\mathcal{M}_X^*$. This local section is independent of the trivialization up to $\mathcal{O}_X^*$ and so we get a well-defined Cartier divisor $\operatorname{div}(s)$. The results and proofs of [EGA IV], 21.1–4 remain true if we replace ringed spaces by G-ringed spaces.

Remark 2.4. Consider a K-affinoid variety $\operatorname{Sp}\mathcal{A}$. Then there is a one-to-one correspondence between analytic subsets of $\operatorname{Sp}\mathcal{A}$ and closed subsets of $\operatorname{Spec}\mathcal{A}$ using the same ideal of vanishing. More generally, there is a one-to-one correspondence between closed subspaces of $\operatorname{Sp}\mathcal{A}$ and closed subschemes of $\operatorname{Spec}\mathcal{A}$. This will be used to reduce the study of cycles on rigid analytic varieties to problems of cycles on affine noetherian schemes.

2.5. Given a Cartier divisor D on the rigid analytic variety X over K, we associate to D a canonical Weil divisor $\operatorname{cyc}(D)$ by the following construction: Locally, X is isomorphic to a K-affinoid variety $\operatorname{Sp}\mathcal{A}$. We may assume that the restriction of D to $\operatorname{Sp}\mathcal{A}$ is given by a single equation γ. We may view γ as a rational function on the noetherian scheme $\operatorname{Spec}\mathcal{A}$. We consider its associated Weil divisor on $\operatorname{Spec}\mathcal{A}$. Using Remark 4, it induces a Weil divisor on the K-affinoid variety $\operatorname{Sp}\mathcal{A}$. To define a Weil divisor $\operatorname{cyc}(D)$ on X, we have to check that these locally defined Weil divisors agree on overlapping charts. It is enough to prove $\operatorname{div}(\gamma)\cap\operatorname{Sp}\mathcal{A}'=\operatorname{div}(\gamma|_{\operatorname{Sp}\mathcal{A}'})$ for a K-affinoid subdomain $\operatorname{Sp}\mathcal{A}'$ of $\operatorname{Sp}\mathcal{A}$. Note that $\mathcal{A}'$ is a flat $\mathcal{A}$-algebra ([BGR], Corollary 7.3.2/6). Then the claim follows from the fact that, for a flat morphism of noetherian schemes, the formation of Weil divisor is compatible with pull-back ([EGA IV], Proposition 21.10.6).

2.6. Next we define proper push-forward of cycles. Let $\varphi: X \to X'$ be a proper morphism of rigid analytic varieties over K ([BGR], 9.6.2).

First we define the push-forward of an irreducible analytic subset Y of X. By the proper mapping theorem ([Ki1], Satz 4.1), $Y' := \varphi(Y)$ is an irreducible analytic subset of X'. If the dimension of Y' is smaller than the dimension of Y, then let $\varphi_*(Y) := 0$. Now we assume that both dimensions are equal. Let $\psi: Y \to Y'$ be the restriction of φ. Outside a lower dimensional analytic subset W of Y', ψ is a finite morphism (by using Proposition 1.14 and [BGR], Lemma 9.6.3/4). Let $\operatorname{Sp}\mathcal{A}'$ be an admissible open K-affinoid subset of $Y'\setminus W$ and let $\operatorname{Sp}\mathcal{A} := \psi^{-1}(\operatorname{Sp}\mathcal{A}')$. Then the corresponding morphism $\operatorname{Spec}\mathcal{A} \to \operatorname{Spec}\mathcal{A}'$ is also finite. Using Remark 4 and the definition of finite push-forward for noetherian schemes ([EGA IV], 21.10.14), we get a push-forward for $\operatorname{Sp}\mathcal{A}$. We claim that this locally defined push-forwards can be glued to get $n(Y'\setminus W)$ for some $n \in \mathbb{N}$. Then we define $\varphi_*(Y) := nY'$. Let $\operatorname{Sp}\mathcal{B}'$ be an affinoid subdomain of $\operatorname{Sp}\mathcal{A}'$ and let $\operatorname{Sp}\mathcal{B} := \psi^{-1}(\operatorname{Sp}\mathcal{B}')$. It is enough to show that the push-forward of $\operatorname{Sp}\mathcal{A}$ coincides on $\operatorname{Sp}\mathcal{B}'$ with the push-forward of $\operatorname{Sp}\mathcal{B}$. Since $\mathcal{A}$ is a finite $\mathcal{A}'$-algebra, we have $\mathcal{B} = \mathcal{B}' \otimes_{\mathcal{A}'} \mathcal{A}$ ([BGR], Proposition 3.7.3/6). Now the claim follows from the fact that, for noetherian schemes, finite push-forward commutes with flat pull-back in a cartesian diagram ([Fu], Proposition 1.7, § 20).

The above considerations defined the push-forward of an irreducible analytic subset of X as a multiple of the image. By linearity, we extend φ_* to all cycles. This is well-defined since the image of a locally finite family of analytic subsets in X is locally finite in X' (use the definition of properness). If $\varphi': X' \to X''$ is also a proper morphism of rigid analytic varieties over K, then $(\varphi' \circ \varphi)_* = \varphi'_* \circ \varphi_*$.

2.7. We associate to a closed analytic subspace W of X a canonical cycle

$$\mathrm{cyc}(W) = \sum_Y n_Y Y$$

where Y ranges over all irreducible components of W and n_Y is the multiplicity of W in Y. To be more precise, let Y be an irreducible component of W and let $U = \mathrm{Sp}\,\mathscr{A}$ be an admissible open K-affinoid subset of X. Then $W \cap U$ corresponds to a closed subscheme of $\mathrm{Spec}\,\mathscr{A}$ (Remark 4) which has a well defined cycle $\mathrm{cyc}(W \cap U)$. Since this construction is compatible with pull back with respect to a flat morphism of noetherian schemes ([Fu], Lemma 1.7.1, Example 20.1.3), the cycles $\mathrm{cyc}(W \cap U)$ induce a global cycle $\mathrm{cyc}(W)$.

2.8. Let $\varphi : X' \to X$ be a flat morphism of rigid analytic varieties over K. For an irreducible analytic subset Y of X, let $\varphi^*(Y) := \mathrm{cyc}(\varphi^{-1}(Y))$. Here $\varphi^{-1}(Y)$ is the analytic subspace of X given by the image of the ideal of Y. By linearity, we extend the definition to define a homomorphism φ^* mapping cycles on X to cycles on X'. Let $U = \mathrm{Sp}\,\mathscr{A}$ be an admissible open K-affinoid subset of X. Then $Y \cap U$ is given by a K-affinoid algebra $\mathscr{A}_Y$. If $U' = \mathrm{Sp}\,\mathscr{A}'$ is an admissible open K-affinoid subset of X' mapping into U, then $\varphi^{-1}(Y) \cap U'$ is given by the K-affinoid algebra $\mathscr{A}_Y \otimes_{\mathscr{A}} \mathscr{A}'$. Using Remark 4, we see that $\varphi^*(Y) \cap U'$ is the same as the pull-back of the subscheme corresponding to $Y \cap U$. Using this local consideration and [Fu], Lemma 1.7.1, Example 20.1.3, it follows that

$$\varphi^*(\mathrm{cyc}(W)) = \mathrm{cyc}(\varphi^{-1}(W))$$

for any closed analytic subspace W of X.

Definition 2.9. Let D be a Cartier divisor of X and let Y be an irreducible analytic subset of X. As usual, we equip Y with the induced structure. The support of D is the closed analytic subset $|D|$ of X consisting of the points $x \in X$ where a local equation of D is not a unit in $\mathcal{O}_{X,x}$. We say that D intersects Y properly if Y is not contained in a component of $|D|$. For such Y, the restriction of D to Y is a well-defined Cartier divisor with associated Weil divisor $D.Y$. More generally, D intersects a cycle Z properly if D intersects all components of Z properly. For such $Z = \Sigma n_Y Y$, the cycle

$$D.Z := \Sigma n_Y D.Y$$

is called the intersection product of D and Z.

Proposition 2.10. *Let $\varphi : X \to X'$ be a morphism of rigid analytic varieties and let D' be a Cartier divisor on X'.*

(a) *Suppose that φ is proper. If Z is a cycle on X such that $\varphi^*(D')$ intersects Z properly, then*

$$\varphi_*(\varphi^*(D').Z) = D'.\varphi_*(Z) \quad (\textit{Projection formula}).$$

(b) *Assume that φ is flat and that D' intersects a cycle Z' properly in X'. Then*

$$\varphi^*(D'.Z') = \varphi^*(D').\varphi^*(Z').$$

Proof. To prove (a), we may assume that Z is an irreducible analytic subset, $Z = \mathrm{cyc}(X)$ and $\varphi(X) = X'$. Let $[X' : X]$ be the degree of φ, then we have to prove

$$\varphi_*\big(\mathrm{cyc}\big(\varphi^*(D')\big)\big) = [X' : X]\,\mathrm{cyc}(D').$$

Clearly, we may assume that X and X' have the same dimension, otherwise both sides of the identity are zero. Using Stein factorization ([BGR], Proposition 9.6.3/5), it is enough to prove the claim for finite φ and for φ with $\mathcal{O}_{X'} \cong \varphi_* \mathcal{O}_X$.

For finite φ, we have seen in 2.6 that push-forward is locally induced by push-forward of schemes. Therefore the projection formula for finite φ is a consequence of the projection formula for finite morphisms of noetherian schemes.

Now assume that φ satisfies $\mathcal{O}_{X'} \cong \varphi_* \mathcal{O}_X$. Then φ is surjective and has connected fibres ([BGR], Lemma 9.6.3/4). There is a lower dimension analytic subspace W' of X' such that φ is an isomorphism outside of W' (Proposition 1.14 and [BGR], Lemma 9.6.3/4). We claim that W' may be chosen of codimension at least 2. If this is true then it is sufficent to check the projection formula outside of W'. Since the restriction of φ to $\varphi^{-1}(X' \backslash W')$ is an isomorphism onto $X' \backslash W'$, the claim follows immediately. It remains to prove that W' may be chosen of codimension at least 2. Assume that W' has a component Y' of codimension 1 in X'. Consider the morphism

$$\psi : Y := \varphi^{-1}(Y') \to Y'$$

induced by φ. The codimension of Y in X is one. Since the fibres of ψ are connected, there is a lower dimensional analytic subset $W_{Y'}$ of Y' such that ψ is bijective outside of $W_{Y'}$. In W', we replace the irreducible components Y' by the sets $W_{Y'}$. This leads to an analytic subset W'' of codimension at least 2 in X. Moreover, φ is bijective outside of W''. We conclude that φ is an isomorphism outside W'' ([BGR], Lemma 9.6.3/4).

To prove (b), note that the statement is local in X' and in X. So we can reduce to the case where X and X' are K-affinoid varieties. But then, as we have seen in 2.8, the flat pull-back is induced by the corresponding flat pull-back of noetherian schemes. Therefore, the claim follows from the corresponding statement for noetherian schemes ([EGA IV], 21.10.6). □

Proposition 2.11. *Let D, D' be Cartier divisors on X with proper intersection.* (*Note that this always means that the supports intersect properly.*) *Then*

$$D.\mathrm{cyc}(D') = D'.\mathrm{cyc}(D).$$

Proof. This is a local statement. So we may assume that X is a K-affinoid variety. Then the claim is a consequence of the corresponding statement for schemes ([Fu], Theorem 2.4 generalizes to this situation). □

Proposition 2.12. *Let*

$$\begin{array}{ccc} X_2' & \xrightarrow{\psi'} & X_2 \\ \big\downarrow{\scriptstyle\varphi'} & & \big\downarrow{\scriptstyle\varphi} \\ X_1' & \xrightarrow{\psi} & X_1 \end{array}$$

be a Cartesian diagram of rigid analytic varieties over K with φ proper and ψ flat. Then φ' is proper, ψ' is flat and $\psi^* \circ \varphi_* = \varphi'_* \circ \psi'^*$.

Proof. As properness is preserved under base change ([BGR], 9.6.2), the morphism φ' is proper. To prove flatness of ψ', we may assume that all varieties are K-affinoid. By [BL4], Theorem 5.2, there is a flat morphism $\mathfrak{X}_1' \to \mathfrak{X}_1$ of admissible formal schemes over K° with generic fibre ψ. Using Theorem 1.13, there is an admissible formal scheme $\mathfrak{X}_2$ over K° and a K°-morphism $\mathfrak{X}_2 \to \mathfrak{X}_1$ with generic fibre φ. Let $\mathfrak{X}_2' := \mathfrak{X}_2 \times_{\mathfrak{X}_1} \mathfrak{X}_1'$, then the first projection extends ψ'. The reduction of $\mathfrak{X}_2' \to \mathfrak{X}_2$ modulo powers of some fixed $\pi \in K^{\circ\circ}$ is flat. We conclude that $\mathfrak{X}_2' \to \mathfrak{X}_2$ is flat ([BL3], Lemma 1.6). This proves flatness of ψ'.

Finally, we have to prove

$$\psi^*\big(\varphi_*(Y)\big) = \varphi'_*\big(\psi'^*(Y)\big)$$

for an irreducible analytic subset Y of X_2. We may assume $Y = \mathrm{cyc}(X_2)$ and $\varphi(X_2) = X_1$. Since a flat morphism is open ([BL4], Corollary 5.11), and since φ is finite outside a lower dimensional closed analytic subset of X_1, we may assume that φ is finite. Then the diagram in Proposition 12 is cartesian in the sense of noetherian schemes. Then the claim follows exactly in the same way as [Fu], Proposition 1.7. □

Let K' be a complete field which is an extension of K such that the absolute values coincide on K. Then we have the base change $X \hat{\otimes}_K K'$ of X to K' ([BGR], 9.3.6). It is a rigid analytic variety over K', given lcally over the admissible open K-affinoid subset $\mathrm{Sp}\,\mathscr{A}$ of X by $\mathrm{Sp}\,\mathscr{A} \hat{\otimes}_K K'$ where $\mathscr{A} \hat{\otimes}_K K'$ is the completion of $\mathscr{A} \otimes_K K'$ with respect to the tensor product semi-norm.

Proposition 2.13. *The following constructions commute with base extension to K':*

(a) *associated Weil-divisor of a Cartier-divisor*;

(b) *proper push-forward*;

(c) *cycle associated to a closed analytic subspace*;

(d) *flat pull-back*;

(e) *proper intersection product of a Cartier divisor with a cycle.*

Proof. As usual, it is enough to prove the claim for K-affinoid varieties. Then $\mathscr{A} \hat{\otimes}_K K'$ is a flat $\mathscr{A}$-algebra. To prove this, let A be an admissible K°-algebra such that $A \otimes_{K^\circ} K \cong \mathscr{A}$. For the $(K')^\circ$-algebra $B := A \hat{\otimes}_{K^\circ} (K')^\circ$, we have $B \otimes_{(K')^\circ} K' \cong \mathscr{A} \hat{\otimes}_K K'$. It is enough to show that B is a flat $\mathscr{A}$-algebra. Since B/IB is a flat A/IA-algebra for any principal ideal $I \subset K^{\circ\circ}$, this follows from [BL3], Lemma 1.6. (There it is assumed that A and B are of topological finite presentation over the same valuation ring, but the proof also applies to our case.)

Now (a) follows from [EGA IV], Proposition 21.10.4. An easy generalization of [Fu], Lemma 1.7.1 to our setting proves (c). Using flatness of the base change, (d) follows immediately and (e) is a consequence of the generalization of [Fu], Proposition 2.3 (d) to noetherian schemes. If the extension K'/K is finite, then (b) follows from Proposition 12. In general, the generic point of an irreducibe component of $X \hat{\otimes}_K K'$ is mapped to a generic point of an irreducible component of X ([EGA IV], Corollaire 3.2). Then we can follow the second part of the proof of Proposition 12 to get (b). □

3. Divisors on admissible formal schemes

Let K be a field with non-trivial non-archimedean complete absolute value $|\,|$ and let $\mathfrak{X}$ be an admissible formal scheme over the valuation ring K°. We denote by $\mathfrak{X}^{f-\mathrm{an}}$ the corresponding formal analytic variety over K (1.10). The rigid analytic variety $\mathfrak{X}^{\mathrm{an}}$ over K (1.10) will be denoted by X. It is called the generic fibre of $\mathfrak{X}$.

Remark 3.1. Since $\mathfrak{X}$ is a locally ringed space, we have the notion of a Cartier divisor on $\mathfrak{X}$ ([EGA IV], §21.1). Using the same notation as in Definition 2.2, a Cartier divisor is a global section of $\mathscr{M}_{\mathfrak{X}}^*/\mathscr{O}_{\mathfrak{X}}^*$. As in Remark 2.3, we have a Cartier divisor $\mathrm{div}(s)$ for any invertible meromorphic section s of an invertible sheaf $\mathscr{L}$ on $\mathfrak{X}$.

Definition 3.2. A horizontal cycle on $\mathfrak{X}$ is a cycle on X. A vertical cycle on $\mathfrak{X}$ is a locally finite sum $\sum \lambda_W W$ where W ranges over all irreducible closed subsets of $\mathfrak{X}$ and $\lambda_W \in \mathbb{R}$. For the range of coefficients, we may use $\log|(K^a)^*|$ instead of $\mathbb{R}$ where K^a is the algebraic closure of K. A cycle on $\mathfrak{X}$ is a sum of a horizontal and a vertical cycle on $\mathfrak{X}$. The dimension of a horizontal cycle is the same as the dimension of the corresponding cycle in X. We define the dimension of a vertical cycle as the dimension of the corresponding closed subset of the special fibre minus 1.

Proposition 3.3. *Let Y be a closed analytic subvariety of X with ideal sheaf $\mathscr{J}$. For an open subset $\mathscr{U}$ of $\mathfrak{X}$, let $\mathscr{K}(\mathscr{U}) := \mathscr{J}(U) \cap \mathscr{O}_{\mathfrak{X}}(\mathscr{U})$ where U is the generic fibre of $\mathscr{U}$. Then $\mathscr{K}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-ideal and the corresponding closed subvariety $\bar{Y}$ of $\mathfrak{X}$ is an admissible formal scheme over K° with generic fibre Y. If $\mathscr{Y}$ is a closed subvariety of $\mathfrak{X}$ with generic fibre Y and if $\mathscr{Y}$ is an admissible formal scheme over K°, then $\mathscr{Y} = \bar{Y}$.*

Proof. Clearly, $\mathscr{K}$ is a sheaf of saturated ideals on $\mathfrak{X}$. Recall that an ideal I of a K°-algebra A is called saturated if and only if A/I has no K°-torsion. We have to prove that $\mathscr{K}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-ideal. This is a local question, so we may assume $\mathfrak{X} = \mathrm{Spf} A$ for an admissible K°-algebra A. Since the ideal $I := \mathscr{K}(\mathfrak{X})$ is saturated, I is finitely generated ([BL3], Lemma 1.2) and A/I is an admissible K°-algebra ([BL3], Proposition 1.1). We have to show that $\mathscr{K}$ is isomorphic to the $\mathscr{O}_{\mathfrak{X}}$-module associated to I (Appendix). It is enough to prove that for any $f \in A$, the natural homomorphism

$$\Phi : I_{\{f\}} \to \mathscr{J}(U) \cap A_{\{f\}}$$

is an isomorphism where $M_{\{f\}}$ is the completion of M_f and U is the generic fibre of $\mathrm{Spf} A_{\{f\}}$. Since $I_{\{f\}}$ may be viewed as a closed ideal of $A_{\{f\}}$ (Appendix, Lemma 4), we have

$$I_{\{f\}} = IA_{\{f\}}.$$

This proves injectivity. On the other hand, we have

$$\mathscr{I}(U) \cong \mathscr{I}(X) A_{\{f\}}[\pi^{-1}] \cong IA_{\{f\}}[\pi^{-1}]$$

for any $\pi \in K^{\circ\circ}$. Hence, for any $a \in \mathscr{I}(U) \cap A_{\{f\}}$, there is $n \in \mathbb{N}$ with $\pi^n a \in IA_{\{f\}}$. Since $I_{\{f\}}$ is the ideal of the closed subvariety $\mathrm{Spf} A/I$ of $\mathrm{Spf} A$ on the formal open subset $\mathrm{Spf} A_{\{f\}}$ (Appendix, Lemma 4), we conclude that $A_{\{f\}}/I_{\{f\}}$ is an admissible K°-algebra ([BL 3], Proposition 1.7). Therefore the ideal $I_{\{f\}}$ of $A_{\{f\}}$ is saturated and it follows that $a \in I_{\{f\}}$. This proves surjectivity of Φ. Hence $\mathscr{K}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-module. Moreover, it follows from the above that $\bar{Y}$ is admissible. Clearly, the generic fibre of $\bar{Y}$ is Y.

To prove the last claim, we may assume $\mathfrak{X} = \mathrm{Spf} A$. Then $\mathscr{Y}$ is given by a saturated ideal I' of A. Since the generic fibre of $\mathscr{Y}$ is Y, we have $I'[\pi^{-1}] = \mathscr{I}(X)$ where X is the generic fibre of $\mathfrak{X}$. It follows easily $I' = \mathscr{K}(\mathfrak{X})$. □

To define multiplicities in the special fibre, we need the following statement ([Be 1], Proposition 2.4.4).

Lemma 3.4. *Let $\mathscr{A}$ be a K-affinoid algebra. For any minimal prime ideal $\tilde{\mathfrak{p}}$ of the reduction $\tilde{\mathscr{A}}$, there is a unique real function p on $\mathscr{A}$ satisfying*

(i) $p(a) \geqq 0$,

(ii) $p(\alpha a) = |\alpha| p(a)$,

(iii) $p(a+b) \leqq \max\{p(a), p(b)\}$,

(iv) $p(ab) = p(a)p(b)$,

(v) $p(a) \leqq |a|_{\sup}$ (*cf.* 1.3)

for all $a, b \in \mathscr{A}$, $\alpha \in K$ and

(vi) $\tilde{\mathfrak{p}} = \{a \in \mathscr{A}^\circ;\ p(a) < 1\}/\mathscr{A}^{\circ\circ}$.

Remark 3.5. Let $X = \mathrm{Sp}\,\mathscr{A}$ and let W be an irreducible component of $\tilde{X} = \mathrm{Spec}\,\tilde{\mathscr{A}}$. Let $\pi: X \to \tilde{X}$ be the reduction map (1.6) and let W' be a non-empty open affine subset of W which intersects no other irreducible component of $\tilde{X}$. For $a \in \mathscr{A}$, we define

$$|a(W)| := \sup\{|a(x)|;\ x \in X,\ \pi(x) \in W'\}.$$

Note that this equals the supremum semi-norm for the formal open affinoid subspace $\pi^{-1}(W')$. Therefore the function $a \mapsto |a(W)|$ satisfies (i)–(iii). Since W' is irreducible, (iv) is a consequence of [BGR], Proposition 6.2.3/5. Obviously, (v) is true. To prove (vi), we have to show that $\{a \in \mathscr{A}^\circ; |a(W)| < 1\}/\mathscr{A}^{\circ\circ} \subset \tilde{\mathfrak{p}}$. Here $\tilde{\mathfrak{p}}$ is the prime ideal corresponding to W. So let $a \in \mathscr{A}^\circ$ with $|a(W)| < 1$. Since π maps the points of X onto the closed points of $\tilde{X}$ ([BGR], Theorem 7.1.5/4), the reduction $\tilde{a}$ of a vanishes on W'. This proves (vi). We have shown the existence of p in the lemma. For uniqueness, see [Be1], p. 37.

Lemma 3.6. *Let $X = \mathrm{Sp}\,\mathscr{A}$ and let W be an irreducible component of $\tilde{X}$. If $a \in \mathscr{A}$ is not a zero-divisor, then $|a(W)| \neq 0$.*

Proof. Let W' be a non-empty open affine subset of W not intersecting any other irreducible component of $\tilde{X}$. Then $\pi^{-1}(W')$ is a formal open affinoid subspace of X and its supremum semi-norm is zero exactly on nilpotent elements ([BGR], Proposition 6.2.1/4). Using Remark 5, we see that $|a(W)| = 0$ if and only if the restriction of a to $\pi^{-1}(W')$ is nilpotent. But this restriction can not be a zero divisor ([BGR], Corollary 7.3.2/6). This proves the claim. □

3.7. Let $\mathfrak{X}$ be an admissible formal scheme over K° and let D be a Cartier divisor on $\mathfrak{X}$. We are going to define a Weil divisor $\mathrm{cyc}(D)$ associated to D which may be viewed as the intersection product of D with the horizontal $\mathrm{cyc}(X)$. The horizontal part of $\mathrm{cyc}(D)$ is the Weil divisor on X associated to $D|_X$ (cf. 2.5). It remains to define the vertical part $\mathrm{cyc}_v(D)$ of $\mathrm{cyc}(D)$.

3.8. Assume that K is algebraically closed. First, we assume that the generic fibre X is an irreducible rigid analytic variety and that the special fibre $\tilde{\mathfrak{X}}$ is reduced. Then $\mathfrak{X}$ is isomorphic to the formal scheme associated to the distinguished formal analytic $\mathfrak{X}^{f-\mathrm{an}}$ (Proposition 1.11). Let W be an irreducible component of $\tilde{\mathfrak{X}}$. Let $\mathscr{U}$ be a formal affine open subset of $\mathfrak{X}$ containing the generic point of W such that D is given on $\mathscr{U}$ by a/b for elements a, b of $\mathcal{O}_{\mathfrak{X}}(\mathscr{U})$ which are not zero-divisors. There is a K-affinoid algebra $\mathscr{A}$ such that the generic fibre of $\mathscr{U}$ is isomorphic to $\mathrm{Sp}\,\mathscr{A}$ and such that $\mathcal{O}_{\mathfrak{X}}(\mathscr{U}) \cong \mathscr{A}^\circ$. In Remark 5, we have introduced a multiplicative semi-norm associated to the irreducible component $W \cap \mathscr{U}$ of $\mathrm{Spec}\,\tilde{\mathscr{A}}$. We define the order of D in W by

$$\mathrm{ord}(D, W) := \log|b(W \cap \mathscr{U})| - \log|a(W \cap \mathscr{U})| .$$

By Lemma 6, this is a real number. Then the vertical part of the Weil divisor associated to D is

$$\mathrm{cyc}_v(D) := \sum_W \mathrm{ord}(D, W)\, W$$

where W is ranging over all irreducible components of $\tilde{\mathfrak{X}}$.

Lemma 3.9. *The order of D in W does not depend on the choice of $\mathscr{U}$ and a/b.*

Proof. Since the semi-norm is multiplicative and bounded by the supremum semi-norm, we conclude $|\gamma(W \cap \mathscr{U})| = 1$ for any unit γ in $\mathcal{O}_{\mathfrak{X}}(\mathscr{U}) \cong \mathscr{A}^\circ$. Therefore the order does not depend on the choice of a/b.

To prove independence of $\mathscr{U}$, we may assume that $\mathscr{U}'$ is a formal affine open subset of $\mathscr{U}$ containing the generic point of W. By the above, it is enough to show

$$|a(W \cap \mathscr{U})| = |a(W \cap \mathscr{U}')|.$$

This is an immediate consequence of Remark 5. □

Definition 3.10. Let D be a Cartier divisor on the admissible formal scheme $\mathfrak{X}$ over K°. First, we assume that K is algebraically closed and that the generic fibre X is irreducible and reduced. By [BGR], Theorem 6.4.3/1, and Proposition 1.11, $\mathfrak{X}' := (\mathfrak{X}^{f-\mathrm{an}})^{f-\mathrm{sch}}$ is an admissible formal scheme with generic fibre X and reduced special fibre. Moreover, we have a canonical morphism $i: \mathfrak{X}' \to \mathfrak{X}$. Using 3.8 and 1.10, we define

$$\mathrm{cyc}_v(D) := i_*\big(\mathrm{cyc}_v(i^*D)\big).$$

Now we assume no longer that X is irreducible and reduced. Let $X = \sum_j m_j X_j$ be the decomposition into prime cycles. Using Proposition 3, let

$$\mathrm{cyc}_v(D) := \sum_j m_j \, \mathrm{cyc}_v(D|_{\bar{X}_j}).$$

Finally, we skip the assumption that K is algebraically closed. Let $\mathfrak{X}^a$ be the base change of $\mathfrak{X}$ to the valuation ring of the completion $\widehat{K^a}$ of the algebraic closure K^a of K. Then the base change of D gives a Cartier divisor D^a on $\mathfrak{X}^a$. Clearly, $\mathrm{cyc}_v(D^a)$ is $\mathrm{Gal}(\tilde{K}^a/\tilde{K})$ invariant. Hence, there is a unique cycle $\mathrm{cyc}_v(D)$ on $\tilde{\mathfrak{X}}$ which is mapped by base change to $\mathrm{cyc}_v(D^a)$. The cycle $\mathrm{cyc}(D) := \mathrm{cyc}(D|_X) + \mathrm{cyc}_v(D)$ is called the Weil divisor associated to the Cartier divisor D.

The vertical part of $\mathrm{cyc}(D)$ may be computed directly over K if $\tilde{\mathfrak{X}}$ is geometrically reduced (cf. Lemma 3.21) or if K is stable (cf. Lemma 5.8).

Remark 3.11. Next we consider the effect of base change. Let K'/K be a field extension and let $|\ |$ be a complete absolute value of K' extending the given one on K. Under base change, a horizontal cycle on $\mathfrak{X}$ is mapped to a horizontal cycle on $\mathfrak{X}' := \mathfrak{X} \hat{\otimes}_{K^\circ} K'^\circ$. Moreover, we have

$$(\mathfrak{X} \hat{\otimes}_{K^\circ} (K')^\circ)^\sim \cong \tilde{\mathfrak{X}} \otimes_{\tilde{K}} \widetilde{K'}$$

and so we have a natural base change of vertical cycles as well. If $\mathfrak{Z}$ is a cycle on $\mathfrak{X}$, we denote the base change of $\mathfrak{Z}$ by $\mathfrak{Z}'$.

Lemma 3.12. *The homomorphism $\mathfrak{Z} \mapsto \mathfrak{Z}'$ is one-to-one. If D is a Cartier divisor on $\mathfrak{X}$, then*

$$\mathrm{cyc}\big(D \hat{\otimes}_{K^\circ} (K')^\circ\big) = \mathrm{cyc}(D)'$$

where $D \hat{\otimes}_{K^\circ} (K')^\circ$ is the Cartier-divisor on $\mathfrak{X}'$ obtained from D by base change.

Proof. The first claim is obvious. For the horizontal parts, the second claim follows from Proposition 2.13. Moreover, it is clear for vertical parts if $K' = \widehat{K^a}$. Together with the first claim, this shows that we may assume K, K' algebraically closed. By linearity and flatness of the base change (cf. proof of Proposition 2.13), we may assume that the generic fibre of $\mathfrak{X}$ is irreducible and reduced. Then $\mathfrak{X}^{f-\mathrm{an}}$ is distinguished ([BGR], Theorem 6.4.3/1).

Let W be an irreducible component of $\mathfrak{X}$ and let $\mathscr{U}$ be a formal affine open subset of $\mathfrak{X}$ containing the generic point of W. Using additivity, we may assume that D is given on $\mathscr{U}$ by $a \in \mathcal{O}_{\mathfrak{X}}(\mathscr{U})$. Note that $W' := W \otimes_{\tilde{K}} \tilde{K}'$ is an irreducible component of $\tilde{\mathfrak{X}}'$ and $\mathscr{U}' := \mathscr{U} \hat{\otimes}_{K^\circ} K'^\circ$ contains the generic point of W'.

We have $\mathscr{U}^{f-\mathrm{an}} \hat{\otimes}_K K' = \mathscr{U}'^{f-\mathrm{an}}$ and

$$(1) \qquad \left((\mathscr{U}')^{f-\mathrm{an}}\right)^\sim = (\mathscr{U}^{f-\mathrm{an}})^\sim \otimes_{\tilde{K}} \widetilde{K'}$$

by [Bo1], Satz 6.4. Moreover, an irreducible component of $\left((\mathscr{U}')^{f-\mathrm{an}}\right)^\sim$ is obtained from an irreducible component of $(\mathscr{U}^{f-\mathrm{an}})^\sim$ by base change. So it is enough to show that

$$(2) \qquad |a(V)| = |a(V')|$$

for $V = W \cap \tilde{\mathscr{U}}$ and $V' = W' \cap \widetilde{\mathscr{U}'}$. Passing to a formal affinoid subdomain of $\mathscr{U}^{f-\mathrm{an}}$, we may assume that V equals the special fibre of $\mathscr{U}^{f-\mathrm{an}}$. Using Remark 5, we see that $|\cdot(V)|$ is the supremum semi-norm on $\mathscr{U}^{f-\mathrm{an}}$. Moreover, V' is the special fibre of $(\mathscr{U}')^{f-\mathrm{an}}$ and so $|\cdot(V')|$ is the supremum semi-norm of $(\mathscr{U}')^{f-\mathrm{an}}$. Using (1), we see that $|a(V)| = 1$ if and only if $|a(V')| = 1$. Since the values of the supremum semi-norm on $\mathscr{A}$ are contained in $|K|$, we get (2). □

Definition 3.13. A morphism $\varphi : \mathfrak{X} \to \mathfrak{X}'$ of admissible formal schemes over K° is called proper if and only if the induced morphisms between the generic fibres and between the special fibres are both proper.

Remark 3.14. Let $\varphi : \mathfrak{X} \to \mathfrak{X}'$ be a morphism of quasi-compact admissible formal schemes. Under the hypothesis that the complete valuation on K is discrete, it is shown in [Lü] that φ is proper if and only if at least one of the induced morphisms $\varphi^{\mathrm{an}} : X \to X'$ (generic fibres) and $\tilde{\varphi} : \tilde{\mathfrak{X}} \to \tilde{\mathfrak{X}}'$ (special fibres) is proper. The easier part is to show that if φ^{an} is proper then $\tilde{\varphi}$ is proper. Using the results of [BL3] and [BL4], the same proof as for discrete valuations shows that this part holds also if the complete non-archimedean absolute value on K is not discrete. In [Lü], it is conjectured that also the converse implication which is the difficult part is true in general.

3.15. If φ is proper, then we have push-forward maps for horizontal cycles and for vertical cycles. If $\mathfrak{Z}$ is a cycle on $\mathfrak{X}$ with horizontal part Z and vertical part $\mathfrak{Z}_v$, then the push-forward of $\mathfrak{Z}$ is defined by

$$\varphi_*(\mathfrak{Z}) := \varphi_*^{\mathrm{an}}(Z) + \tilde{\varphi}_*(\mathfrak{Z}_v)\,.$$

3.16. Now assume that $\varphi : \mathfrak{X} \to \mathfrak{X}'$ is a flat morphism of admissible formal schemes over K°, i.e. if locally the map looks like $\mathrm{Spf}\,\mathscr{A} \to \mathrm{Spf}\,\mathscr{A}'$, then $\mathscr{A}$ is a flat $\mathscr{A}'$-algebra. Then φ^{an} and $\tilde{\varphi}$ are flat and we can define a pull-back map φ^* similarly as above.

The next 3 lemmata are needed for the proof of the projection formula in section 4. A field is called stable if the degree of a finite extension is the sum of the products of residue degrees and ramification indices (cf. [BGR], 3.6).

Lemma 3.17. *Assume that K is a complete stable field and let $\mathscr{A}$ be a reduced K-affinoid algebra such that $\tilde{\mathscr{A}}$ is an integral domain. Then $\mathscr{A}$ is an integral domain and its field of fractions $Q(\mathscr{A})$ is stable with respect to the absolute value induced by the supremum norm.*

Proof. Using [BGR], Proposition 6.2.3/5, it follows that $\mathscr{A}$ is an integral domain and the supremum norm induces an absolute value $|\ |_{\mathscr{A}}$ on $Q(\mathscr{A})$. By noetherian normalization ([BGR], Corollary 6.1.2/2), there is a finite monomorphism $\varphi : K\langle x_1, \ldots, x_d\rangle \to \mathscr{A}$. Let Q be the field of fractions of $K\langle x_1, \ldots, x_d\rangle$. On Q, the Gauss norm yields a canonical absolute value $|\ |_Q$. Using [BGR], Proposition 6.2.2/2, we see that the supremum norm on $\mathscr{A}$ is the restriction of the spectral norm ([BGR], p. 44, p. 134) of the extension $Q(\mathscr{A})/Q$. As both the spectral norm and $|\ |_{\mathscr{A}}$ are faithful Q-algebra norms on $Q(\mathscr{A})$, they are the same. Therefore the spectral norm is an absolute value and hence it is the only extension of the Gauss norm to an absolute value on $Q(\mathscr{A})$ ([BGR], 3.3.3).

Let $L/Q(\mathscr{A})$ be a finite field extension and let $\{|\ |_j;\ j = 1, \ldots, n\}$ be the absolute values on L extending $|\ |_{\mathscr{A}}$. By e_j (resp. e_j'), we denote the ramification index of $|\ |_j$ over $|\ |_{\mathscr{A}}$ (resp. over $|\ |_Q$). Similarly, we have the residue degree f_j (resp. f_j'). Note that Q is stable [BGR], Theorem 5.3.2/1. By [BGR], Proposition 3.6.2/6, we have

$$\sum_{j=1}^{n} e_j' f_j' = [L : Q] .$$

If e (resp. f) is the ramification index (resp. residue degree) of $|\ |_{\mathscr{A}}$ over $|\ |_Q$, then we have $e_j' = e e_j$ and $f_j' = f f_j$. Again by stability of Q, we have $ef = [Q(\mathscr{A}) : Q]$. Finally, this gives

$$\sum_{j=1}^{n} e_j f_j = [L : Q(\mathscr{A})] .$$

This proves the stability of $Q(\mathscr{A})$ ([BGR], Proposition 3.6.2/6). □

The next lemma is included in [Be1], Proposition 2.4.4. For convenience of the reader, we give a proof here.

Lemma 3.18. *Let $\mathscr{A}$ be a K-affinoid algebra which is an integral domain. If W is an irreducible component of* $\operatorname{Spec}\tilde{\mathscr{A}}$, *then $|\cdot(W)|$ extends uniquely to an absolute value $|\ |_W$ on the field of fractions $Q(\mathscr{A})$. Let* $\operatorname{Spf}\mathscr{A}'$ *be a non-empty formal open affinoid subspace of* $\operatorname{Spf}\mathscr{A}$ *such that* $\operatorname{Spec}\tilde{\mathscr{A}}'$ *intersects no other irreducible component of* $\operatorname{Spec}\tilde{\mathscr{A}}$ *than W. Then $\mathscr{A}'$ is an integral domain and the supremum semi-norm on $\mathscr{A}'$ induces an absolute value on $Q(\mathscr{A}')$ whose restriction to the dense subfield $Q(\mathscr{A})$ equals $|\ |_W$. If the value group of K is divisible, then $\tilde{K}(W)$ is isomorphic to the residue field of $|\ |_W$.*

Proof. Lemma 4 and Lemma 6 show that $|\cdot(W)|$ extends to an absolute value $|\ |_W$ of $Q(\mathscr{A})$. By [BGR], Corollary 7.3.2/10, $\mathscr{A}'$ is reduced. As the reduction $\operatorname{Spec}\widetilde{\mathscr{A}'}$ is irreducible, we conclude similarly as in Lemma 17 that $\mathscr{A}'$ is an integral domain. The supremum semi-norm $|\ |_{\mathscr{A}'}$ of $\mathscr{A}'$ extends uniquely to an absolute value on $Q(\mathscr{A}')$ ([BGR], Proposition 6.2.3/5). As $\mathscr{A}'$ is a flat $\mathscr{A}$-algebra ([BGR], Corollary 7.3.2/6), the canonical homomorphism $\mathscr{A} \to \mathscr{A}'$ is injective. It follows from Remark 5 that $|\ |_{\mathscr{A}'}$ extends $|\cdot(W)|$.

There is $f \in \mathscr{A}^\circ$ such that $\{\tilde{f} \neq 0\}$ is a dense open subset of $\operatorname{Spec}\widetilde{\mathscr{A}'}$. As $Q(\mathscr{A})$ is dense in $Q(\mathscr{A}\langle f^{-1}\rangle)$, it follows that $Q(\mathscr{A})$ is dense in $Q(\mathscr{A}')$.

To prove that the residue field k of $|\ |_W$ is isomorphic to $\tilde{K}(W)$, we may assume that $\mathscr{A} = \mathscr{A}'$, i.e. $\operatorname{Spec}\tilde{\mathscr{A}} = W$. This follows from the density of $Q(\mathscr{A})$ in $Q(\mathscr{A}')$. We have a natural monomorphism $\tilde{K}(W) \to k$. Let $\alpha \in k \backslash \{0\}$. There is $a, b \in \mathscr{A}$, $|a|_W = |b|_W$, such that α is the residue class of a/b. If the value group of K is divisible, then we may assume $|a|_W = |b|_W = 1$. This proves surjectivity of $\tilde{K}(W) \to k$. □

Lemma 3.19. *Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a finite monomorphism of K-affinoid algebras. If $\tilde{\mathscr{A}}$ is an integral domain, then $V \mapsto |\cdot(V)|$ is a one-to-one correspondence between irreducible components of* $\operatorname{Spec}\tilde{\mathscr{B}}$ *mapping onto* $\operatorname{Spec}\tilde{\mathscr{A}}$ *and multiplicative semi-norms on $\mathscr{B}$ extending the supremum semi-norm of $\mathscr{A}$.* (*Multiplicative means that* (iv) *of Lemma* 3.4 *is fulfilled.*)

Proof. Note that the reduction $\tilde{\varphi}$ is finite (Theorem 1.9). By Lemma 4 and Remark 5, $|\cdot(V)|$ is a multiplicative semi-norm on $\mathscr{B}$ for any irreducible component V of $\operatorname{Spec}\tilde{\mathscr{B}}$ and V is uniquely determined by $|\cdot(V)|$. If V maps onto $\operatorname{Spec}\tilde{\mathscr{A}}$, then it's an immediate consequence of uniqueness in Lemma 4 that $|\cdot(V)|$ extends the supremum semi-norm $|\ |_{\mathscr{A}}$ of $\mathscr{A}$.

It remains to check that any multiplicative semi-norm p on $\mathscr{B}$ extending $|\ |_{\mathscr{A}}$ has the form $|\cdot(V)|$ for V as above. First, we assume that $\mathscr{A}$ is the Tate algebra $K\langle x_1, \ldots, x_d\rangle$ and φ is torsion-free. Then we may assume that $\mathscr{B}$ is reduced. Let Q be the field of fractions of $\mathscr{A}$. Then $\mathscr{B}_Q := \mathscr{B} \otimes_{\mathscr{A}} Q$ is a reduced finite dimensional Q-algebra and $\mathscr{B} \subset \mathscr{B}_Q$. By [BGR], Proposition 6.2.2/2, the spectral norm of $\mathscr{B}_Q/Q$ ([BGR], Definition 3.2.2/1) restricts to the supremum semi-norm of $\mathscr{B}$. By Dedekind's Lemma ([BGR], Proposition 3.1.4/1), $\mathscr{B}_Q$ is a direct product of finitely many finite field extensions L_i of Q. It follows easily from [BGR], 3.2.2 that any multiplicative semi-norm on $\mathscr{B}$ extending $|\ |_{\mathscr{A}}$ is induced by an absolute value on some L_i. By [BGR], Theorem 3.2.2/2, Proposition 3.3.3/1, the spectral norm of $b \in \mathscr{B}_Q$ equals

$$|b|_{sp} = \max_i \max_{|\ |_i} |b_i|_i$$

where b_i is the component of b in L_i and $|\ |_i$ ranges over all absolute values of L_i extending the Gauss norm. As φ is torsion-free, any irreducible component of $\operatorname{Spec}\tilde{\mathscr{B}}$ maps onto $\operatorname{Spec}\tilde{\mathscr{A}}$. Otherwise, there is $\tilde{a} \in \tilde{\mathscr{A}} \backslash \{0\}$ and $\tilde{b} \in \tilde{\mathscr{B}} \backslash \{0\}$ such that $\tilde{a}$ is zero on the image of this component and $\tilde{b}$ is zero outside of this component. In particular, we have $|ab|_{\mathscr{B}} < 1$ and $|a|_{\mathscr{A}} = |b|_{\mathscr{B}} = 1$ which is impossible ([BGR], Proposition 6.2.2/2). So any irreducible component V of $\operatorname{Spec}\tilde{\mathscr{B}}$ induces a multiplicative semi-norm $|\ |_V$ on $\mathscr{B}_Q$ extending the Gauss norm. By [Be1], Proposition 2.4.4, we have $|b|_{sp} = \max_V |b|_V$. Using the approximation theorem ([BGR], 3.3.2), it follows that every $|\ |_i$ is induced by some $|\ |_V$. In particular, p is of the form $|\cdot(V)|$ for some irreducible component V of $\operatorname{Spec}\tilde{\mathscr{B}}$. This proves the claim for torsion-free φ and $\mathscr{A} = K\langle x_1, \ldots, x_d\rangle$.

Next, we assume only that $\mathscr{A}$ is a Tate-algebra. Let I be the ideal of $\mathscr{A}$-torsion in $\mathscr{B}$. Let $\mathscr{B}' := \mathscr{B}/I$. Then $\varphi' : \mathscr{A} \to \mathscr{B}'$ is a finite torsion-free monomorphism. As p is multiplicative, it induces a multiplicative semi-norm p' on $\mathscr{B}'$. By the consideration above, there is an irreducible component V' of $\operatorname{Spec}\tilde{\mathscr{B}}'$ such that $p' = |\cdot(V')|$. By dimensionality reasons,

V' is mapped onto an irreducible component V under the finite morphism $\operatorname{Spec}\widetilde{\mathscr{B}/I} \to \operatorname{Spec}\tilde{\mathscr{B}}$. Again by uniqueness in Lemma 4, we conclude $p = |\cdot(V)|$.

Finally, we prove the claim in general. By noetherian normalization ([BGR], Corollary 6.1.2/2), there is a finite monomorphism $\varphi: K\langle x_1, \ldots, x_d\rangle \to \mathscr{A}$. As we have seen above, there is a component V of $\operatorname{Spec}\tilde{\mathscr{B}}$ mapping onto $\operatorname{Spec}\tilde{K}[x_1, \ldots, x_d]$ such that $p = |\cdot(V)|$. Obviously, V maps onto $\operatorname{Spec}\tilde{\mathscr{A}}$. This proves the claim in general. □

By noetherian normalization, an affinoid variety is a finite covering of a unit ball. We can use this morphism to reduce some problems to the zero-dimensional situation. The next corollary describes the resulting zero-dimensional variety and makes it clear why it is useful in the study of Weil divisors.

Corollary 3.20. *Let X be an irreducible and reduced K-affinoid variety. We denote the field of fractions of $K\langle x_1, \ldots, x_d\rangle$ by Q. Suppose that $\varphi: X \to \operatorname{Sp} K\langle \mathbf{x}\rangle$ is a finite surjective morphism. For an irreducible component V of $\tilde{X}$, let $\widehat{K(X)}^V$ be the completion of $K(X)$ with respect to $|\ |_V$ (cf. Lemma 18). Then*

$$\mathscr{A} \otimes_{K\langle\mathbf{x}\rangle} \hat{Q} \cong \prod_V \widehat{K(X)}^V$$

where V is ranging over all irreducible components of $\tilde{X}$.

Proof. Let $\mathscr{A}$ be the K-affinoid algebra of X. Note that $K(X) \cong \mathscr{A} \otimes_{K\langle\mathbf{x}\rangle} Q$. We have seen in the proof of Lemma 19 that there is a one-to-one correspondence between irreducible components of $\operatorname{Spec}\tilde{\mathscr{A}}$ and absolute values on $K(X)$ extending the Gauss norm of Q. From valuation theory, we know that $K(X) \otimes_Q \hat{Q}$ is isomorphic to the product of the $\widehat{K(X)}^V$. This proves the claim. □

Lemma 3.21. *Let $\mathfrak{X}$ be an admissible formal affine scheme over K° with geometrically reduced special fibre $\tilde{\mathfrak{X}}$. Then there is a distinguished K-affinoid algebra $\mathscr{A}$ with $\mathfrak{X} = \operatorname{Spf}\mathscr{A}^\circ$. Suppose that a is not a zero-divisor in $\mathscr{A}^\circ$. Then the multiplicity of $\operatorname{cyc}(\operatorname{div}(a))$ in an irreducible component W of $\tilde{\mathfrak{X}}$ equals $-\log|a(W)|$.*

Proof. It follows from Proposition 1.11 that $\mathfrak{X}$ is the formal scheme $\operatorname{Spf}\mathscr{A}^\circ$ for a distinguished K-affinoid algebra $\mathscr{A}$. In particular, $\mathscr{A}$ is reduced. In fact, $\mathscr{A}$ is geometrically reduced since the same argument shows that $\mathscr{A}' := \mathscr{A} \hat{\otimes}_K \widehat{K^a}$ is a distinguished $\widehat{K^a}$-algebra and $(\mathscr{A}')^\circ \cong \mathscr{A}^\circ \hat{\otimes}_{K^\circ} (K^a)^\circ$. Here we have used that the special fibre of $\mathfrak{X}$ is geometrically reduced. Let W' be an irreducible component of $W \hat{\otimes}_{\tilde{K}} \widetilde{K^a}$. By Lemma 4 and Remark 5, we have $|a(W')| = |a(W)|$. So we may assume K algebraically closed.

Let $\tilde{U}$ be a non-empty open affine subset of W not intersecting any other component of $\tilde{\mathfrak{X}}$. We denote the corresponding subdomain of $\operatorname{Sp}\mathscr{A}$ by $U = \operatorname{Sp}\mathscr{B}$. Then $\mathscr{B}$ is an integral domain (as in Lemma 17). There is a unique irreducible component Y of $\operatorname{Sp}\mathscr{A}$ containing U. Let m be the multiplicity of the Weil divisor associated to $\operatorname{div}(a)$ in W. Then Y is the only irreducible component of $\operatorname{Sp}\mathscr{A}$ contributing to m. Note that U is a formal affinoid subdomain of Y. We conclude that there is a unique irreducible component V of $\tilde{Y}$ lying over W. Then $\tilde{U}$ is a dense open subset of V. By Remark 5, we have $|a(V)| = |a(W)|$. Since $m = -\log|a(V)|$, we get the claim. $\square$

4. Intersection with divisors

Let K be a field with non-trivial non-archimedean complete absolute value $|\ |$ and let $\mathfrak{X}$ be an admissible formal scheme over K° with generic fibre X as in section 3.

Let D be a Cartier divisor on $\mathfrak{X}$ and let Y be a horizontal prime cycle of dimension d. We assume that $D|_X$ intersects Y properly. By Proposition 3.3, there is a canonical closed formal subscheme $\bar{Y}$ of $\mathfrak{X}$ with generic fibre Y. Then $D|_{\bar{Y}}$ is a Cartier divisor on $\bar{Y}$ and its associated Weil divisor (Definition 3.10) may be viewed as a cycle on $\mathfrak{X}$.

Definition 4.1. The intersection product $D.Y$ of D and Y is the cycle on $\mathfrak{X}$ induced by $\mathrm{cyc}(D|_{\bar{Y}})$.

Remark 4.2. Note that $D.Y$ is a $d-1$ dimensional cycle on $\mathfrak{X}$. Now we assume that Y is a vertical prime cycle on $\mathfrak{X}$ of dimension d. If the support of D does not contain Y, this is called a proper intersection, then the restriction of D to Y is a well-defined Cartier divisor on Y and we define $D.Y$ as the $d-1$ dimensional cycle on $\mathfrak{X}$ induced by $\mathrm{cyc}(D|_Y)$.

Let $Z(\mathfrak{X})$ be the set of cycles on $\mathfrak{X}$. It is a group with respect to addition. For $\mathfrak{X}^a := \mathfrak{X} \mathbin{\hat{\otimes}}_{K^\circ} (\widehat{K^a})^\circ$, we may consider $Z(\mathfrak{X})$ as a subgroup of $Z(\mathfrak{X}^a)$ (Lemma 3.12). Let R_v^a be the subgroup of rational divisors on $\tilde{\mathfrak{X}}^a$. We define

$$CH(\mathfrak{X}, v) := Z(\mathfrak{X}) / \big(R_v^a \cap Z(\mathfrak{X})\big).$$

If the vertical prime cycle Y is contained in the support of D, then we cannot define $D.Y$ as a cycle. For our purposes, it is enough to define it as a class in $CH(\mathfrak{X}, v)$: Let $\mathcal{O}(D)$ be the invertible sheaf on $\mathfrak{X}$ induced by D. Then we define $D.Y$ as the class in $CH(\mathfrak{X}, v)$ induced by

$$c_1\big(\mathcal{O}(D)|_Y\big) \in CH^1(Y).$$

Let $\mathfrak{Z}$ be a cycle on $\mathfrak{X}$. We say that D intersects $\mathfrak{Z}$ properly in the generic fibre if the support of $D|_X$ intersects the horizontal part of $\mathfrak{Z}$ properly in X. If the intersection of the support of D with the vertical part of $\mathfrak{Z}$ is also proper, then we say that D intersects $\mathfrak{Z}$ properly.

Definition 4.3. Let $\mathfrak{Z}$ be a cycle on $\mathfrak{X}$ intersecting the Cartier divisor D properly in the generic fibre. By linearity, we extend the above definitions to get a class $D.\mathfrak{Z} \in CH(\mathfrak{X}, v)$ called the intersection product of D and $\mathfrak{Z}$. If D intersects $\mathfrak{Z}$ properly, then $D.\mathfrak{Z}$ is a well-defined cycle.

Proposition 4.4. *Let D, D' be Cartier divisors on $\mathfrak{X}$ and let $\mathfrak{Z}$, $\mathfrak{Z}'$ be cycles on $\mathfrak{X}$.*

(i) *If D and D' intersect $\mathfrak{Z}$ properly in the generic fibre X, then*

$$(D + D').\mathfrak{Z} = D.\mathfrak{Z} + D'.\mathfrak{Z} \in CH(\mathfrak{X}, v).$$

(ii) *If D intersects $\mathfrak{Z}$ and $\mathfrak{Z}'$ properly in the generic fibre X, then*

$$D.(\mathfrak{Z}+\mathfrak{Z}') = D.\mathfrak{Z} + D.\mathfrak{Z}' \in CH(\mathfrak{X}, v).$$

If the intersections are proper, then we have identities of cycles.

Proof. The first claim follows from the additivity of $\operatorname{ord}(D, W)$ in D and the second claim is nearly by definition. □

Next we have the projection formula:

Proposition 4.5. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a proper morphism of admissible formal schemes over K°. Assume that D is a Cartier divisor on $\mathfrak{X}$ and let $\mathfrak{Z}'$ be a cycle on $\mathfrak{X}'$. If $\varphi^*(D)$ intersects $\mathfrak{Z}'$ properly in the generic fibre, then*

$$\varphi_*(\varphi^* D.\mathfrak{Z}') = D.\varphi_* \mathfrak{Z}' \in CH(\mathfrak{X}, v).$$

If $\varphi^*(D)$ intersects $\mathfrak{Z}'$ properly, then we have an identity of cycles.

Proof. It is enough to check that for a prime cycle $\mathfrak{Z}'$. If $\mathfrak{Z}'$ is vertical, then the claim follows from the projection formula for varieties over $\tilde{K}$. So we may assume that $\mathfrak{Z}'$ is horizontal. Moreover, we may assume that the generic fibre X' of $\mathfrak{X}'$ is an irreducible reduced rigid analytic variety with $\mathfrak{Z}' = \operatorname{cyc}(X')$. By the proper mapping theorem ([Ki1], Satz 4.1), the image of X' is a closed subvariety of X, so we may assume $X = \varphi^{\mathrm{an}}(X')$. If $\dim X' > \dim X$, then both sides of the projection formula are zero. So let us assume that X and X' are equi-dimensional. By Lemma 3.12 and Proposition 2.13, it is enough to prove the claim for K algebraically closed. So under the hypothesis above, we have to prove

$$\varphi_*(\operatorname{cyc}(\varphi^* D)) = D.\varphi_*(\mathfrak{Z}'). \tag{1}$$

The horizontal parts of both sides agree by Proposition 2.10. It remains to show that the vertical parts are equal.

First we assume that the special fibres of $\mathfrak{X}$ and $\mathfrak{X}'$ are reduced. By Proposition 1.11, we have

$$\mathfrak{X} = (\mathfrak{X}^{f-\mathrm{an}})^{f-\mathrm{sch}}, \quad \tilde{\mathfrak{X}} = (\mathfrak{X}^{f-\mathrm{an}})^{\sim}$$

and similarly for $\mathfrak{X}'$. The direct image theorem ([Ki1], Theorem 3.3) tells us that $\varphi_*^{\mathrm{an}} \mathcal{O}_{X'}$ is a coherent sheaf on X. We can form the rigid analytic variety $X'' := \operatorname{Sp} \varphi_*^{\mathrm{an}} \mathcal{O}_{X'}$ which is finite over X. Over a K-affinoid admissible open subspace U of X, it is given by $\operatorname{Sp} \mathcal{O}_{X'}((\varphi^{\mathrm{an}})^{-1} U)$. Then $\varphi^{\mathrm{an}} = f'' \circ f'$ with natural morphisms $f' : X' \to X''$ and $f'' : X'' \to X$. Moreover, f' is a proper surjective morphism with connected fibres and f'' is finite and surjective (for details, see [BGR], 9.6.3). This canonical decomposition of φ^{an} is called the Stein factorization of φ. Using the formal analytic structure of X, we get a natural formal analytic structure on X''. Let $\mathfrak{X}''$ be the corresponding formal scheme over K°.

By construction, f'' is continuous with respect to the formal analytic topologies and it is easy to see that the same is true for f'. Therefore f' and f'' may be viewed as morphisms of formal analytic varieties over K. The supremum norm of a regular function g on an admissible open part U of X'' equals the supremum norm of $g \circ f'$ on $f'^{-1}(U)$. Therefore we have $f'_* \mathcal{O}^\circ_{X'} = \mathcal{O}^\circ_{X''}$. Note that X'' is a reduced rigid analytic variety. Since K is algebraically closed, it follows from [BGR], Theorem 6.4.3/1, that X'' is distinguished. By Proposition 1.11, $\mathfrak{X}''$ is an admissible formal scheme with reduced special fibre. Let $\varphi' := (f')^{f-\mathrm{sch}}$ and $\varphi'' := (f'')^{f-\mathrm{sch}}$ be the corresponding morphisms of formal schemes. It follows that

$$\widetilde{\mathfrak{X}''} = \operatorname{Spec} \tilde{\varphi}_* \mathcal{O}_{\widetilde{\mathfrak{X}'}} . \tag{2}$$

In particular, $\tilde{\varphi} = \widetilde{\varphi''} \circ \tilde{\varphi}'$ is the Stein factorization of $\tilde{\varphi}$. We conclude that $\widetilde{\varphi'}$ is a proper surjective morphism with connected fibres and that $\widetilde{\varphi''}$ is finite. By Definition 3.13, φ' is a proper morphism of admissible formal schemes.

First we prove (1) for φ'. Outside the inverse image of a lower dimensional closed subset S of $\tilde{\mathfrak{X}}$, $\widetilde{\varphi'}$ is finite. Here we have used that X, X', X'' and all irreducible components of the reductions have the same dimension. From (2) it follows that $\widetilde{\varphi'}$ is an isomorphism outside of $(\tilde{\varphi})^{-1}(S)$. Let $\pi : X \to \tilde{\mathfrak{X}}$ be the reduction map. By [BGR], Corollary 6.4.2/2, we conclude that φ' induces an isomorphism

$$\psi' : \mathfrak{X}' \backslash (\tilde{\varphi})^{-1}(S) \to \mathfrak{X}'' \backslash (\widetilde{\varphi''})^{-1}(S) .$$

For dimensionality reasons, the components of $\tilde{\mathfrak{X}}'$ lying over S are mapped by $\widetilde{\varphi'_*}$ to 0. If we replace φ' by ψ', then the vertical parts of both sides of (1) will not change up to passing to closures of components. But for an isomorphism, (1) is clearly satisfied. This proves (1) for φ'.

Next we prove (1) for φ''. Together with the claim for φ', formula (1) for $\varphi = \varphi'' \circ \varphi'$ will follow easily. The claim for φ'' is an immediate consequence of the following local statement:

Let $f : X' \to X$ be a finite surjective morphism of irreducible K-affinoid varieties with reduced special fibres and let γ be a non-zero regular function on X. If $\varphi : \mathfrak{X}' \to \mathfrak{X}$ is the corresponding morphism of formal schemes, then we have

$$\varphi_*\big(\operatorname{cyc}\big(\varphi^* \operatorname{div}(\gamma)\big)\big) = \operatorname{div}(\gamma) . \varphi_* \operatorname{cyc}(X') . \tag{1'}$$

Note that we still assume that K is algebraically closed. We have to check that the multiplicities of both sides of (1′) in an irreducible component W of $\tilde{\mathfrak{X}}$ agree. Passing to formal subdomains, we may assume that $W = \tilde{\mathfrak{X}}$. The vertical part of the Weil divisor associated to $\operatorname{div}(\gamma)$ equals

$$-\log|\gamma(W)| W .$$

On the other hand, the vertical part of the Weil divisor associated to $\operatorname{div}(\varphi^* \gamma)$ is equal to

$$-\log|\gamma(W)| \sum_V V$$

where V is ranging over all irreducible components of $\tilde{\mathfrak{X}}'$. To prove (1′), it is enough to show that

$$(3) \qquad [K(X'):K(X)] = \sum_V [\tilde{K}(V):\tilde{K}(W)]\,.$$

By Lemma 3.19, there is a one-to-one correspondence between irreducible components of $\tilde{\mathfrak{X}}'$ and absolute values on $K(X')$ extending the supremum norm on X. As K is algebraically closed, its value group is divisible. It follows from Lemma 3.18 that $[\tilde{K}(V):\tilde{K}(W)]$ equals the residue degree of $|\ |_V$ over $|\ |_W$. Moreover, all ramification degrees are 1. Now (3) follows from the stability of $K(X)$ (Lemma 3.17) and [BGR], Proposition 3.6.2/6.

We have proved (1) for admissible formal schemes with reduced special fibres. Now let us consider the general case. By assumption, $\mathfrak{X}$ and $\mathfrak{X}'$ are reduced. As K is algebraically closed, $\mathfrak{X}^{f-\mathrm{an}}$ and $(\mathfrak{X}')^{f-\mathrm{an}}$ are distinguished ([BGR], Theorem 6.4.3/1). By Proposition 1.11, $\mathscr{Y} := (\mathfrak{X}^{f-\mathrm{an}})^{f-\mathrm{sch}}$ and $\mathscr{Y}' := (\mathfrak{X}'^{f-\mathrm{an}})^{f-\mathrm{sch}}$ are admissible formal schemes over K° with reduced special fibres. Moreover, we have a natural commutative diagram

$$\begin{array}{ccc} \mathscr{Y}' & \xrightarrow{i'} & \mathfrak{X}' \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\ \mathscr{Y} & \xrightarrow{i} & \mathfrak{X} \end{array}$$

where i, i' are finite morphisms ([BGR], Corollary 6.4.1/5) and $\psi = (\varphi^{f-\mathrm{an}})^{f-\mathrm{sch}}$. Note that ψ equals φ on generic fibres. Since $i \circ \tilde{\psi}$ is proper, we conclude that $\tilde{\psi}$ is proper, i.e. ψ is proper. Let $\mathscr{Z}$ (resp. $\mathscr{Z}'$) be the horizontal cycle on $\mathfrak{X}$ (resp. $\mathfrak{X}'$) associated to X (resp. X'). By definition, we have

$$D.\mathscr{Z} = i_*(i^*D.\mathscr{Z})$$

and

$$\varphi^*D.\mathscr{Z}' = i'_*(\psi^*i^*D.\mathscr{Z}')\,.$$

Using (1) for ψ, we get

$$\varphi_*(\varphi^*D.\mathscr{Z}') = i_*(i^*D.\psi_*\mathscr{Z}')\,.$$

Note that $\psi_*\mathscr{Z}'$ is a multiple n of $\mathscr{Z}$. We conclude that

$$\varphi_*(\varphi^*D.\mathscr{Z}') = nD.\mathscr{Z} = D.\varphi_*\mathscr{Z}'\,.$$

This proves (1). □

5. Commutativity

Let K be a field with a non-trivial non-archimedean complete absolute value $|\ |$.

5.1. Consider a Cartier divisor D on the admissible formal scheme $\mathfrak{X}$ over K° with reduced special fibre. Let W be an irreducible component of $\tilde{\mathfrak{X}}$. We are going to construct a canonical Cartier divisor on W equivalent to the divisor of any non-zero meromorphic section of $\mathcal{O}(D)|_W$. Let $\mathcal{U}$ be a formal affine open subset of $\mathfrak{X}$ such that D is given on $\mathcal{U}$ by $\gamma = a/b$ for elements a, b of $\mathcal{O}_{\mathfrak{X}}(\mathcal{U})$ which are not zero-divisors. We suppose that $\tilde{\mathcal{U}} \cap W \neq \emptyset$. Let W' be a non-empty affine open subscheme of $\tilde{\mathcal{U}}$ intersecting no other irreducible component of $\tilde{\mathcal{U}}$ than W. Since $\mathfrak{X}^{f-\mathrm{an}}$ is distinguished (Proposition 1.11), there is $\alpha_W \in K^*$ with

$$-\log|\alpha_W| = \log \sup_{\pi(x)\in W'} |b(x)| - \log \sup_{\pi(x)\in W'} |a(x)|$$

where $\pi : X \to \tilde{\mathfrak{X}}$ is the reduction map. Here we have used that $\pi^{-1}(W')$ is a formal open affinoid subspace of $\mathfrak{X}^{f-\mathrm{an}}$ whose supremum semi-norm p is multiplicative (Remark 3.5). Moreover $p(\gamma)$ does not depend on the choice of γ and $p(\gamma) = |\alpha_W|$. Therefore γ/α_W may be reduced to a well-defined non-zero rational function on W'. Obviously, the latter extends uniquely to a rational function $f_{\mathcal{U}}$ on W. The formal scheme $\mathfrak{X}$ is covered by subsets $\mathcal{U}$ as above. Therefore the corresponding reductions $\tilde{\mathcal{U}}$ cover the scheme $\tilde{\mathfrak{X}}$. We get a Cartier divisor D_W on W, given by $f_{\mathcal{U}}$ on $\tilde{\mathcal{U}} \cap W$, such that $\mathcal{O}(D_W) \cong \mathcal{O}(D)|_W$. To prove this, let $\mathcal{U}_1$, $\mathcal{U}_2$ be formal affine open subsets of $\mathfrak{X}$ with $\tilde{\mathcal{U}}_1 \cap W \neq \emptyset$ and $\tilde{\mathcal{U}}_2 \cap W \neq \emptyset$. Moreover, we assume that D is given on $\mathcal{U}_i$ by γ_i $(i = 1, 2)$. Let W' be a non-empty affine open subset of $\tilde{\mathfrak{X}}$ contained in $\tilde{\mathcal{U}}_1 \cap \tilde{\mathcal{U}}_2$. Then $f_{\mathcal{U}_1}/f_{\mathcal{U}_2}$ equals $(\gamma_1/\gamma_2)^\sim$ on W'. But $(\gamma_1/\gamma_2)^\sim$ is the transition function of $\mathcal{O}(D)|_W$ on $\tilde{\mathcal{U}}_1 \cap \tilde{\mathcal{U}}_2 \cap W$. This shows that D_W is a Cartier divisor with $\mathcal{O}(D_W) \cong \mathcal{O}(D)|_W$.

5.2. Now we assume no longer that the special fibre of $\mathfrak{X}$ is reduced. Let D and D' be Cartier divisors on the admissible formal scheme $\mathfrak{X}$ over K° which intersect properly in the generic fibre. If D and D' does not intersect properly, then the vertical part of $D.\mathrm{cyc}(D')$ is only an equivalence class. In the following, we define a cycle $D.D'$ on $\mathfrak{X}$ which is a canonical representative of $D.\mathrm{cyc}(D')$. Let $\widehat{K^a}$ be the completion of the algebraic closure of K and let $\mathfrak{X}^a := \mathfrak{X} \mathbin{\hat{\otimes}}_{K^\circ} (\widehat{K^a})^\circ$. We may assume that the generic fibre of $\mathfrak{X}^a$ is irreducible and reduced, otherwise we proceed by linearity. Then $\mathfrak{X}^a$ is an admissible formal scheme over $(\widehat{K^a})^\circ$ and the formal scheme $\mathfrak{X}'$ associated to $(\mathfrak{X}^a)^{f-\mathrm{an}}$ is an admissible formal scheme over $(\widehat{K^a})^\circ$ with reduced special fibre (Proposition 1.11). Using the canonical morphism $i : \mathfrak{X}' \to \mathfrak{X}^a$ and base change, we get Cartier divisors D^a and $(D')^a$ on $\mathfrak{X}'$ induced by D and D', respectively. First, we define the cycle $D^a.(D')^a$ on $\mathfrak{X}'$. Its horizontal part is defined as usual. By 5.1, $D^a.W'$ may be defined as a canonical Weil divisor on W' for any irreducible component W' of $\tilde{\mathfrak{X}}'$. It is the Weil divisor associated to $D^a_{W'}$. By this procedure and using linearity, we can define $D^a.(D^a)'$ as a cycle on $\mathfrak{X}'$. Then $i_*\big(D^a.(D^a)'\big)$ is defined over K equal to the base change of a unique cycle $D.D'$ on $\mathfrak{X}$. This is clear for the horizontal part (Proposition 2.13) and is a consequence of the invariance of the vertical part of $D^a.(D')^a$ under $\mathrm{Gal}(\widetilde{K^a}/\tilde{K})$.

Obviously, the class of $D.D'$ in $CH(\mathfrak{X}, v)$ equals $D.\mathrm{cyc}(D')$. If the special fibre of $\mathfrak{X}$ is geometrically reduced, it is seen as in Lemma 3.21 that the vertical part of $D.D'$ is equal to

$$\sum_W m(D', W)\,\mathrm{cyc}(D_W)$$

where W is ranging over all irreducible components of $\tilde{\mathfrak{X}}$ and $m(D', W)$ is the multiplicity of $\mathrm{cyc}(D')$ in W.

Lemma 5.3. *Let $\mathscr{A}$ be a K-affinoid algebra and let $\varphi : K\langle x_1, \dots, x_n\rangle \to \mathscr{A}$ be a homomorphism. We denote the field of fractions of $K\langle x_1, \dots, x_n\rangle$ by Q and its completion with respect to the Gauss norm by $\hat{Q}$. Then $\mathscr{A}' := \mathscr{A} \mathbin{\hat{\otimes}}_{K\langle \mathbf{x}\rangle} \hat{Q}$ is a $\hat{Q}$-affinoid algebra and for any ideal I of $\mathscr{A}$, we have*

$$(\mathscr{A}/I) \mathbin{\hat{\otimes}}_{K\langle \mathbf{x}\rangle} \hat{Q} \cong \mathscr{A}'/(I\mathscr{A}') .$$

If Z is a prime cycle on $\operatorname{Sp}\mathscr{A}$ with induced reduced structure, then $Z \mathbin{\hat{\otimes}}_{K\langle \mathbf{x}\rangle} \hat{Q}$ is a closed subvariety of $\operatorname{Sp}\mathscr{A}'$ with cycle Z'. By linearity, we extend the map $Z \to Z'$ to all horizontal cycles. Then the kernel of the base change consists of those cycles which have no component mapping Zariski densly into $\mathbb{B}^n$ by the morphism corresponding to φ.

Proof. Adding some more variables $x_{n+1}, \dots, x_m$, we get a strict epimorphism

$$\psi : K\langle x_1, \dots, x_m\rangle \to \mathscr{A}$$

extending φ. By [BGR], Corollary 6.1.1/8, we get

$$K\langle x_1, \dots, x_m\rangle \mathbin{\hat{\otimes}}_{K\langle x_1, \dots, x_n\rangle} \hat{Q} \cong \hat{Q}\langle x_{n+1}, \dots, x_m\rangle .$$

By [BGR], Proposition 2.1.8/6, we get

$$(K\langle x_1, \dots, x_m\rangle / \operatorname{Ker}\psi) \mathbin{\hat{\otimes}}_{K\langle x_1, \dots, x_n\rangle} \hat{Q} \cong \hat{Q}\langle x_{n+1}, \dots, x_m\rangle / \langle \operatorname{Ker}(\psi)\rangle .$$

This proves the first two claims.

Let $\mathscr{P}$ be a prime ideal of $\mathscr{A}$. Then

$$(\mathscr{A}/\mathscr{P}) \mathbin{\hat{\otimes}}_{K\langle x_1, \dots, x_n\rangle} \hat{Q}$$

is non-zero if and only if $K\langle x_1, \dots, x_n\rangle \to \mathscr{A}/\mathscr{P}$ is injective. This proves the last claim. □

Lemma 5.4. *Let $\mathscr{A}$ be a reduced K-affinoid algebra and let*

$$\varphi : K\langle x_1, \dots, x_n\rangle \to \mathscr{A}$$

be a homomorphism such that all minimal primes of $\operatorname{Spec}\tilde{\mathscr{A}}$ are mapped by $\tilde{\varphi}^{-1}$ to the zero ideal in $\tilde{K}[x_1, \dots, x_n]$. Let Q be the field of fractions of $K\langle x_1, \dots, x_n\rangle$. Then we have

$$\mathscr{A}^\circ \mathbin{\hat{\otimes}}_{K^\circ\langle \mathbf{x}\rangle} Q^\circ \cong (\mathscr{A} \mathbin{\hat{\otimes}}_{K\langle \mathbf{x}\rangle} Q)^\circ$$

and

$$\tilde{A} \otimes_{\tilde{K}[\mathbf{x}]} \tilde{Q} \cong (\mathscr{A} \mathbin{\hat{\otimes}}_{K\langle \mathbf{x}\rangle} Q)^\sim .$$

Moreover $\mathscr{A} \mathbin{\hat{\otimes}}_{K\langle \mathbf{x}\rangle} \hat{Q}$ is reduced.

Proof. Let W be an irreducible component of $\operatorname{Spec}\tilde{\mathscr{A}}$ and let $|\,|_W$ be the corresponding multiplicative semi-norm on $\mathscr{A}$. Since the prime ideal in $\tilde{\mathscr{A}}$ corresponding to W is mapped to the zero ideal in $\tilde{K}[x_1, \dots, x_n]$, it follows from uniqueness in Lemma 3.4

that the composition of $|\ |_W$ and φ equals the Gauss norm. Note that the supremum semi-norm $|\ |_{\mathscr{A}}$ on $\mathscr{A}$ is the maximum of all $|\ |_W$ with W ranging over the irreducible components of $\operatorname{Spec}\tilde{\mathscr{A}}$ (use Corollary 3.20 or [Be1], Proposition 2.4.4). Therefore φ is an isometry with respect to the Gauss norm $|\ |$ on $K\langle\mathbf{x}\rangle$ and $|\ |_{\mathscr{A}}$ on $\mathscr{A}$. Moreover, we have

$$|\varphi(f)\,a|_{\mathscr{A}} = |f|\,|a|_{\mathscr{A}}$$

for any $f \in K\langle\mathbf{x}\rangle$ and $a \in \mathscr{A}$.

We have a natural homomorphism

$$\psi : \mathscr{A}^\circ \,\hat{\otimes}_{K^\circ\langle\mathbf{x}\rangle} Q^\circ \to (\mathscr{A}\,\hat{\otimes}_{K\langle\mathbf{x}\rangle} Q)^\circ\,.$$

Using the formula above, it follows that the induced homomorphism

$$\tilde{\psi} : \tilde{\mathscr{A}} \otimes_{\tilde{K}[\mathbf{x}]} \tilde{Q} \to (\mathscr{A}\,\hat{\otimes}_{K\langle\mathbf{x}\rangle} Q)^\sim$$

is one-to-one. As the reduction of $\mathscr{A}\,\hat{\otimes}_{K\langle\mathbf{x}\rangle} Q$ is the same as the one of $\mathscr{A} \otimes_{K\langle\mathbf{x}\rangle} Q$, it is clear that the map is surjective as well. This proves the second claim.

Note that $\mathscr{A}^\circ \,\hat{\otimes}_{K^\circ\langle\mathbf{x}\rangle} Q^\circ$ is the completion of $\mathscr{A}^\circ \otimes_{K^\circ\langle\mathbf{x}\rangle} Q^\circ$ with respect to the tensor norm. Then ψ is an isometry of complete normed $\hat{Q}^\circ$-algebras. Clearly, the image of $\mathscr{A}^\circ \otimes_{K^\circ\langle\mathbf{x}\rangle} Q^\circ$ in $(\mathscr{A}\,\hat{\otimes}_{K\langle\mathbf{x}\rangle} Q)^\circ$ is dense. By definition of completion, ψ is an isomorphism.

Obviously, the tensor norm on $\mathscr{A}\,\hat{\otimes}_{K\langle\mathbf{x}\rangle} Q$ is power multiplicative. Therefore it is the supremum norm and $\mathscr{A}\,\hat{\otimes}_{K\langle\mathbf{x}\rangle} Q$ is reduced ([BGR], Proposition 6.2.1/4). □

Lemma 5.5. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a proper morphism of admissible formal schemes over K° and let D_1, D_2 be Cartier divisors on $\mathfrak{X}$ intersecting properly in the generic fibre. We assume that the restriction of φ to the generic fibre is a surjective morphism of irreducible rigid analytic varieties over K of the same dimension. If the following statement* (b) *is true, then* (a) *is also true*:

(a) $D_1 . D_2 = D_2 . D_1$;

(b) $\varphi^* D_1 . \varphi^* D_2 = \varphi^* D_2 . \varphi^* D_1$.

Proof. By projection formula, we get

$$\begin{aligned}\varphi_*\big(\varphi^* D_1 . \varphi^* D_2 . \operatorname{cyc}(X')\big) &= D_1 . D_2 . \varphi_* \operatorname{cyc}(X')\\ &= [X' : X]\, D_1 . D_2 . \operatorname{cyc}(X) \in CH(\mathfrak{X}, v)\end{aligned}$$

and similarly

$$\begin{aligned}\varphi_*\big(\varphi^* D_2 . \operatorname{cyc}(\varphi^* D_1)\big) &= D_2 . D_1 . \varphi_* \operatorname{cyc}(X')\\ &= [X' : X]\, D_2 . D_1 . \operatorname{cyc}(X) \in CH(\mathfrak{X}, v)\,.\end{aligned}$$

The construction of D_W in 5.1 is clearly compatible with pull-back, i.e. if W' is an irreducible component of $\tilde{\mathfrak{X}}'$ mapping onto the irreducible component W of $\tilde{\mathfrak{X}}$, then we have

$\tilde{\varphi}^* D_W = D_{W'}$. Then we easily deduce that the projection formulas used above yield an equality of cycles and we get

$$\varphi_*(\varphi^* D_1 . \varphi^* D_2) = [X' : X] \, D_1 . D_2$$

and

$$\varphi_*(\varphi^* D_2 . \varphi^* D_1) = [X' : X] \, D_2 . D_1 \, .$$

This proves the claim. □

Lemma 5.6. *Let $\mathfrak{X} = \operatorname{Spf} A$ be an admissible formal affine scheme over K° with generic fibre X. Suppose that $\varphi : \mathfrak{X} \to \operatorname{Spf} K^\circ\langle x_1, \ldots, x_d \rangle$ is a morphism such that $\tilde{\varphi}$ maps the irreducible components of $\tilde{\mathfrak{X}}$ densely into* $\operatorname{Spec} \tilde{K}[\mathbf{x}]$. *We denote the field of fractions of $K\langle \mathbf{x} \rangle$ by Q and the completion of Q with respect to the Gauss norm by $\hat{Q}$. As usual, we denote the generic fibres of $\mathfrak{X}$, $\mathfrak{X}'$ by X and X', respectively.*

(a) *$\mathfrak{X}' := \mathfrak{X} \, \hat{\otimes}_{K^\circ\langle \mathbf{x} \rangle} Q^\circ$ is an admissible formal scheme over $\hat{Q}^\circ$ with generic fibre $X' = X \, \hat{\otimes}_{K\langle \mathbf{x} \rangle} Q$.*

(b) *In the notation of Lemma* 3, *we have* $\operatorname{cyc}(X)' = \operatorname{cyc}(X')$.

(c) *Let $\operatorname{Sp} \mathscr{B}$ be an irreducible component of X. Then $\mathscr{B}' := \mathscr{B} \, \hat{\otimes}_{K\langle \mathbf{x} \rangle} Q$ is a non-zero reduced $\hat{Q}$-affinoid algebra.*

(d) *We have $\tilde{\mathscr{B}}' \cong \tilde{\mathscr{B}} \otimes_{\tilde{K}[\mathbf{x}]} \tilde{Q}$. If V is an irreducible component of* $\operatorname{Spec} \tilde{\mathscr{B}}$, *then $V' := V \otimes_{\tilde{K}[\mathbf{x}]} \tilde{Q}$ is an irreducible component of* $\operatorname{Spec} \tilde{\mathscr{B}}'$. *This gives a one-to-one correspondence between irreducible components of* $\operatorname{Spec} \tilde{\mathscr{B}}$ *and irreducible components of* $\operatorname{Spec} \tilde{\mathscr{B}}'$.

(e) *If $b \in \mathscr{B}$, then $|b(V)| = |b(V')|$.*

(f) *The fields of rational functions on V and V' are isomorphic.*

(g) *If W is an irreducible component of $\tilde{\mathfrak{X}}$, then $W' := W \otimes_{\tilde{K}[\mathbf{x}]} \tilde{Q}$ is an irreducible component of $\tilde{\mathfrak{X}}'$. The map $W \to W'$ gives also a one-to-one correspondence between irreducible components of $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{X}}'$. Moreover, $\tilde{K}(W)$ is isomorphic to $\tilde{Q}(W')$.*

Proof. By adding some more variables $x_{d+1}, \ldots, x_m$, we may assume that A is the quotient of $K^\circ\langle x_1, \ldots, x_m \rangle$ by the ideal J. From Lemma 4 and the proof of Lemma 3, we get

$$K^\circ\langle x_1, \ldots, x_m \rangle \, \hat{\otimes}_{K^\circ\langle x_1, \ldots, x_d \rangle} Q^\circ \cong \hat{Q}^\circ\langle x_{d+1}, \ldots, x_m \rangle \, .$$

Hence $A' := A \, \hat{\otimes}_{K^\circ\langle \mathbf{x} \rangle} Q^\circ$ is the quotient of $\hat{Q}^\circ\langle x_{d+1}, \ldots, x_m \rangle$ by the ideal generated by J. To prove (a), it remains to show that A' has no $\hat{Q}^\circ$-torsion. Equivalently, we may prove that A' has no K°-torsion.

We claim that A' is a flat A-algebra. This is clear modulo any finitely generated ideal I contained in $K^{\circ\circ}$. Similarly as in the proof of Proposition 2.13, it follows that A' is flat over A. In particular, A' has no K°-torsion. This proves (a). Moreover, the $\mathscr{A} := A \otimes_{K^\circ} K$-algebra $\mathscr{A} \hat{\otimes}_{K\langle \mathbf{x}\rangle} Q$ is flat and so (b) follows similarly as the last statement in 2.8.

By Theorem 1.9 and 1.10, all irreducible components of $\operatorname{Spec}\tilde{\mathscr{B}}$ are mapped densely into $\operatorname{Spec}\tilde{K}[\mathbf{x}]$. Then (c) and the first claim of (d) are consequences of Lemmata 3 and 4. Moreover, $\tilde{\mathscr{B}}'$ is a torsion-free $\tilde{K}[\mathbf{x}]$-algebra. Note that $\tilde{\mathscr{B}}'$ is the localization of $\tilde{\mathscr{B}}$ in the zero-ideal of $\tilde{K}[\mathbf{x}]$. Hence $\tilde{\mathscr{B}}$ may be viewed as a subalgebra of $\tilde{\mathscr{B}}'$. By clearing denominators, it is clear that V' is a prime cycle on $\operatorname{Spec}\tilde{\mathscr{B}}'$ with $\tilde{Q}(V') \cong \tilde{K}(V)$. Note that $\tilde{\mathscr{B}}'$ is a flat $\tilde{\mathscr{B}}$-algebra. Therefore V' is an irreducible component of $\operatorname{Spec}\tilde{\mathscr{B}}'$ and any irreducible component has this form. This proves (d) and (f). Similarly, we prove (g). Finally, (e) follows easily from uniqueness in Lemma 3.4 and Remark 3.5. □

To compute the multiplicity of a Weil divisor in a component of the special fibre, it is sometimes inconvenient to do first base change to $\widehat{K^a}$ (cf. 3.10). The next lemma gives a direct formula if the valued field K is stable.

Lemma 5.7. *Assume that the complete non-archimedean absolute value $|\ |$ on K is stable. Let a be an element of the admissible K°-algebra A which is not a zero-divisor. Let $(X_j)_{j=1,\dots,r}$ be the irreducible components of the K-affinoid variety $X := \operatorname{Sp} A \otimes_{K^\circ} K$ and let m_j be the multiplicity of X in X_j. We consider the Weil divisor on* $\operatorname{Spf} A$ *associated to* $\operatorname{div}(a)$. *Then its multiplicity in the irreducible component W of the special fibre* $\operatorname{Spec}\tilde{A}$ *is equal to*

$$-\sum_{j=1}^{r} m_j \sum_{V_j} e(V_j)[\tilde{K}(V_j) : \tilde{K}(W)] \log|a(V_j)|$$

where V_j is ranging over all irreducible components of $\tilde{X}_j$ lying over W with respect to the canonical finite morphism $\tilde{X}_j \to \operatorname{Spec}\tilde{A}$ and where $e(V_j)$ is the index of $|K^*|$ in the value group of the absolute value $|\ |_{V_j}$ on $K(X_j)$. The ramification index $e(V_j)$ is always finite.

Proof. It follows from Theorem 1.9 and 1.10 that $\tilde{X}_j \to \operatorname{Spec}\tilde{A}$ is finite and that $\tilde{X}_j$ is mapped onto an irreducible component. Let K'/K be an extension of fields such that the absolute values coincide on K. Using the proof of Proposition 2.13, we see that $A' := A \hat{\otimes}_{K^\circ} (K')^\circ$ is an admissible $(\widehat{K'})^\circ$-algebra. Let X' be the generic fibre of $\operatorname{Spf} A'$ and let m_j' be the multiplicity of X' in its irreducible component X_j'. We denote by $\operatorname{div}(a; \widetilde{K'})$ the vertical Weil divisor on $\operatorname{Spf} A'$ with multiplicity

$$(1) \qquad -\sum_j m_j' \sum_{V_j'} e(V_j')[\widetilde{K'}(V_j') : \widetilde{K'}(W')] \log|a(V_j')|$$

in the irreducible component W' of $\operatorname{Spec}\widetilde{A'}$ where j is ranging over the indices of the irreducible components of X' and V_j' is ranging over all irreducible components of $\widetilde{X_j'}$ lying over W'. We have to prove

$$(2) \qquad \operatorname{div}(a; \widetilde{K^a}) = \operatorname{div}(a; \tilde{K}) \hat{\otimes}_{\tilde{K}} \widetilde{K^a}.$$

Passing to a formal open subset, we may assume that W is the only irreducible component of $\operatorname{Spec}\tilde{A}$. Note that the ramification indices do not change (Lemma 3.18). By noetherian normalization, there is a finite morphism from $\operatorname{Spec}\tilde{A}$ onto $\operatorname{Spec}\tilde{K}[x_1, \dots, x_d]$ where d is the dimension of W. Let $\varphi : \mathrm{Spf} A \to \mathrm{Spf} K^\circ\langle \mathbf{x}\rangle$ be any lift of this morphism and let Q be the field of fractions of $K\langle \mathbf{x}\rangle$. Two applications of Lemma 6, once to the $K^\circ\langle \mathbf{x}\rangle$-algebra A and once to the $(\widehat{K^a})^\circ\langle \mathbf{x}\rangle$-algebra $A \hat{\otimes}_{K^\circ} (\widehat{K^a})^\circ$, show that it is enough to prove the claim for $A \hat{\otimes}_{K^\circ\langle \mathbf{x}\rangle} Q^\circ$. Note that $\hat{Q}$ is stable ([BGR], Theorem 5.3.2/1, Proposition 3.6.2/3).

So we may assume from the beginning that $\tilde{A}$ is finite over $\tilde{K}$. By Theorem 1.9 and 1.10, X is finite over K. There is a finite extension L/K such that the irreducible components of $\mathrm{Sp} A \otimes_{K^\circ} L$ are L-rational points and such that the irreducible components of $\mathrm{Sp} A \otimes_{\tilde{K}} \tilde{L}$ are $\tilde{L}$-rational points. Easily, we get

$$\mathrm{div}(a; \widetilde{K^a}) = \mathrm{div}(a; \tilde{L}) \otimes_{\tilde{L}} \widetilde{K^a}$$

and so it is enough to prove

$$\mathrm{div}(a; \tilde{L}) = \mathrm{div}(a; \tilde{K}) \otimes_{\tilde{K}} \tilde{L}\,. \tag{3}$$

Let $\tilde{K}^s$ be the separable closure of $\tilde{K}$ in $\tilde{L}$. Then $\tilde{K}^s$ has a primitive element $\tilde{\alpha}$ with minimum polynomial $\tilde{f}$ over $\tilde{K}$. Let $f \in K^\circ[x]$ be a lift of $\tilde{f}$. By Hensel's Lemma, $\tilde{\alpha}$ may be lifted to a simple zero $\alpha \in L^\circ$ of f. Then $K(\alpha)$ is the maximal unramified extension of K contained in L and $\big(K(\alpha)\big)^\sim = K^s$. Therefore it is enough to prove (3) for finite purely inseparable extensions $\tilde{L}/\tilde{K}$ and for finite unramified extension L/K, respectively.

There is a finite field extension K_j/K such that $X_j = \mathrm{Sp} K_j$. Let m_j be the multiplicity of X in X_j. Note that $K_j \otimes_K L$ is isomorphic to a finite product of finite dimensional local L-algebras R_{jk} with residue fields L_{jk}. Then the irreducible components of $X' := \mathrm{Sp} A \otimes_{K^\circ} L^\circ$ are equal to the full list $X'_{jk} = \mathrm{Sp} L_{jk}$. The decomposition of $\mathrm{cyc}(X')$ into prime cycles is equal to

$$\mathrm{cyc}(X') = \sum_{j,k} m_j \ell(R_{jk})\,\mathrm{cyc}(X'_{jk})$$

where ℓ denotes the length of a ring. Let us fix an irreducible component W' of $\mathrm{Spec}\,\tilde{A} \otimes_{\tilde{K}} \tilde{L}$ lying over the irreducible component W of $\mathrm{Spec}\,\tilde{A}$. Then the multiplicity of $\mathrm{div}(a; \tilde{K})$ in W is equal to

$$-\sum_j m_j e(K_j/K)[\tilde{K}_j : \tilde{K}(W)] \log|a|_j \tag{4}$$

where j is ranging over all irreducible components X_j with $\tilde{X}_j$ lying over W and where $|\ |_j$ is the continuation of $|\ |$ to K_j with ramification index $e(K_j/K)$. Similarly, the multiplicity of $\mathrm{div}(a; \tilde{L})$ in W' equals

$$-\sum_{j,k} m_j \ell(R_{jk}) e(L_{jk}/L)[\tilde{L}_{jk} : \tilde{L}(W')] \log|a|_j \tag{5}$$

where j, k is ranging over all irreducible components X'_{jk} of X' with $\widetilde{X'_{jk}}$ lying over W'. Let us fix j with $\tilde{X}_j$ lying over W. As above, $\tilde{K}_j \otimes_{\tilde{K}} \tilde{L}$ is a product of finite dimensional local $\tilde{L}$-algebras S_{jl} with residue fields κ_{jl}. By considering the cartesian diagram

$$\begin{array}{ccc}
\operatorname{Spec} \tilde{K}_j \otimes_{\tilde{K}} \tilde{L} & \longrightarrow & W \otimes_{\tilde{K}} \tilde{L} \\
\downarrow & & \downarrow \\
\operatorname{Spec} \tilde{K}_j & \longrightarrow & W
\end{array}$$

and using [Fu], Proposition 1.7, the multiplicity of $W \otimes_{\tilde{K}} \tilde{L}$ in W' is equal to

$$\frac{1}{[\tilde{K}_j : \tilde{K}(W)]} \sum_l \ell(S_{jl})[\kappa_{jl} : \tilde{L}(W')] \tag{6}$$

where l is ranging over all irreducible components of $\operatorname{Spec} \tilde{K}_j \otimes_{\tilde{K}} \tilde{L}$ lying over W'. Suppose that $\widetilde{X'_{jk}}$ is lying over W'. Then it is lying over an irreducible component $\operatorname{Spec} \kappa_{jl(k)}$ of $\operatorname{Spec} \tilde{K}_j \otimes_{\tilde{K}} \tilde{L}$. Then $\kappa_{jl(k)}/\tilde{L}(W')$ is a subextension of $\widetilde{L_{jk}}/\tilde{L}(W')$. If we can prove

$$e(K_j/K)\ell(S_{jl}) = \sum_{l(k)=l} e(L_{jk}/L)\ell(R_{jk})[\tilde{L}_{jk} : \kappa_{jl}]\,, \tag{7}$$

then (3) follows from (4)–(6). By [Fu], Lemma A.1.3, we have

$$\ell(S_{jl}) = [S_{jl} : \tilde{L}]/[\kappa_{jl} : \tilde{L}] \tag{8}$$

and

$$\ell(R_{jk}) = [R_{jk} : L]/[L_{jk} : L]\,. \tag{9}$$

By stability of L ([BGR], Corollary 3.6.2/7), we have

$$e(L_{jk}/L)[\tilde{L}_{jk} : \tilde{L}] = [L_{jk} : L]\,. \tag{10}$$

Using (8)–(10), (7) is equivalent to

$$e(K_j/K)[S_{jl} : \tilde{L}] = \sum_{l(k)=l} [R_{jk} : L]\,. \tag{11}$$

First, we suppose that $\tilde{L}/\tilde{K}$ is purely inseparable. From the theory of tensor products of fields, we know that $\tilde{K}_j \otimes_{\tilde{K}} \tilde{L}$ is a local $\tilde{L}$-algebra, i.e. there is only one index l. Hence (11) reads as

$$e(K_j/K)[\tilde{K}_j : \tilde{K}] = \sum_k [R_{jk} : L]\,. \tag{12}$$

The right hand side equals $[K_j : K]$ and (12) follows by stability.

Finally, we have to consider a finite unramified extension L/K. As we have seen above, L is generated by an element α of absolute value one with $\tilde{L} = \tilde{K}(\tilde{\alpha})$. We have denoted the minimum polynomial of α by $f(x)$. Since $[L : K] = [\tilde{L} : \tilde{K}]$, we have $e(L/K) = 1$. By [BGR], Proposition 3.6.2/4, we get

$$L^\circ \cong K^\circ[x]/\langle f(x) \rangle\,.$$

It follows that

$$K_j \otimes_K L \cong K_j[x]/\langle f(x) \rangle .$$

Then $f(x)$ is the product of distinct K_j-irreducible polynomials $f_k(x)$ with

$$R_{jk} \cong K_j[x]/\langle f_k(x) \rangle .$$

By Gauss' Lemma, we may assume that all factors $f_k(x)$ have Gauss norm one. By Hensel's Lemma and the separability of $\tilde{f}(x)$, the reductions $\tilde{f}_k(x)$ are $\tilde{K}_j$-irreducible and distinct. We conclude that

$$S_{jl(k)} \cong \tilde{K}_j[x]/\langle \tilde{f}_k(x) \rangle .$$

This implies

$$\sum_{l(k)=l} [R_{jk} : K_j] = [S_{jl} : \tilde{K}_j] . \tag{13}$$

Note that

$$[R_{jk} : K_j] = \frac{[L:K]}{[K_j:K]} [R_{jk} : L]$$

and

$$[S_{jl} : \tilde{K}_j] = \frac{[\tilde{L}:\tilde{K}]}{[\tilde{K}_j:\tilde{K}]} [S_{jl} : \tilde{L}] .$$

By stability, (13) implies (11). This proves the claim. □

Remark 5.8. Assume that K is stable and that the value group is divisible. Let $\mathfrak{X}$ be an admissible formal scheme over K° and let D, D' be Cartier divisors on $\mathfrak{X}$ intersecting properly in the generic fibre X. In 5.2, we have defined a cycle $D.D'$ on $\mathfrak{X}$. For the vertical part of $D.D'$, we have used base change to $(\widehat{K^a})^\circ$. In our special case, we prove that this is not necessary.

By linearity, we may assume that X is irreducible and reduced. Note that $\mathfrak{X}^{f-\mathrm{an}}$ is a distinguished formal analytic variety over K ([BGR], Theorem 6.4.3/1). Let $\mathscr{Y}$ be the associated formal scheme. First, we assume $\mathscr{Y} = \mathfrak{X}$, i.e. the special fibre of $\mathfrak{X}$ is reduced. By 5.1, we have a Cartier divisor D_W on any irreducible component W of $\tilde{\mathfrak{X}}$. The claim is that the vertical part of $D.D'$ equals

$$\sum_W m(D', W)\, D_W . W \tag{14}$$

where $m(D', W)$ is the multiplicity of $\mathrm{cyc}(D')$ in W. Let $\sum_j m_j' X_j'$ be the decomposition of the generic fibre of $\mathfrak{X}' := \mathfrak{X} \mathbin{\hat{\otimes}}_{K^\circ} (\widehat{K^a})^\circ$ into irreducible components X_j'. Let $\mathscr{Y}_j'$ be the formal scheme associated to $\overline{X_j'}^{f-\mathrm{an}}$. Then the vertical part of the base change of $D.D'$ to $(\widehat{K^a})^\circ$ is equal to

$$\sum_j m_j' \sum_k m(D', W_{jk}') (i_{jk})_* (D_{jk} . W_{jk}') \tag{15}$$

where W'_{jk} is ranging over all irreducible components of $\widetilde{\mathscr{Y}'_j}$ and where $i_{jk}: W'_{jk} \to \tilde{\mathfrak{X}}'$ is the canonical finite morphism. Here, $m(D', W'_{jk})$ is the multiplicity of the base change of D' in W'_{jk} and $D_{jk} := D_{W'_{jk}}$ is the canonical Cartier divisor on W_{jk} associated to the base change of D. Let $\pi: \mathfrak{X}' \to \mathfrak{X}$ be the morphism induced by base change. Then we have

$$(\tilde{\pi} \circ i_{jk})^*(D_W) = D_{jk}$$

for any W'_{jk} lying over W. By projection formula, (15) equals

$$\sum_{W'} m(D', W')\, \tilde{\pi}^*(D_W). W'$$

where $m(D', W')$ is the multiplicity of the base change of D' in the irreducible component W' of $\tilde{\mathfrak{X}}'$ lying over W. Applying base change to (14), we get the claim from the obvious

$$\sum_{W'} m(D', W') W' = \sum_{W} m(D', W)\, \tilde{\pi}^*(W).$$

Finally, if $\mathscr{Y} \neq \mathfrak{X}$ then the claim follows from the above applied to $\mathscr{Y}$ and the compatibility of push-forward with base change.

Theorem 5.9. *Let $\mathfrak{X}$ be an admissible formal scheme over K°. If D and D' are Cartier divisors on $\mathfrak{X}$ with proper intersection in the generic fibre X (but not necessarily in $\mathfrak{X}$), then $D.D' = D'.D$ in the sense of cycles.*

Proof. We may assume that K is algebraically closed (Lemma 3.12). The claim is true in the generic fibre (Proposition 2.11). So it is enough to check commutativity in a prime cycle W of codimension 1 in $\tilde{\mathfrak{X}}$. Passing to an open subset containing the generic point of W, we may assume that $\mathfrak{X}$ is formal affine and that the Cartier divisors D, D' are given by single equations on $\mathfrak{X}$.

Let d be the topological dimension of W. By noetherian normalization, there is a morphism $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{K}[y_1, \ldots, y_d]$ whose restriction to W is finite. We choose a lift $\varphi: \mathfrak{X} \to \operatorname{Spf} K^\circ\langle \mathbf{y} \rangle$ of this morphism. Let Q be the field of fractions of $K\langle \mathbf{y} \rangle$. Then Q is stable ([BGR], Theorem 5.3.2/1) and the value group is divisible. The same is true for the completion $\hat{Q}$ ([BGR], Proposition 3.6.2/3). By Lemmata 6, 7 and Remark 8, it is enough to prove the claim on $\mathfrak{X}'$in the closed irreducible subset $W' := W \otimes_{\tilde{K}[\mathbf{y}]} \tilde{Q}$. By construction, W' is zero-dimensional and any irreducible component of $\tilde{\mathfrak{X}}'$ containing W' is a curve. So we may assume that the special fibre and the generic fibre X' of $\mathfrak{X}'$ are of pure dimension one.

Using the above reduction steps and linearity, we may assume that X is an irreducible and reduced affinoid curve over the algebraically closed field K and that the divisors D, D' are given by single equations. By Lemma 5, Remark 3.14 and Theorem 1.13, we may replace X by an irreducible reduced curve X' with a proper morphism $X' \to X$ onto X and we may use any admissible formal model of X' lying over $\mathfrak{X}$. By passing to the normalization, we may assume that X is also non-singular and $\mathfrak{X} = \operatorname{Spf} \mathscr{A}^\circ$ where $\mathscr{A}$ is the K-affinoid algebra of X. Clearly, it is enough to prove the claim for D, D' given by $a, a' \in \mathscr{A}^\circ \setminus \{0\}$.

Next, we want to replace X by an open subset of a non-singular projective curve using the techniques of [BL1]. Let $X_+(W)$ be the formal fibre in X over W with respect to the reduction map. We denote by $\tilde{y}_1, \ldots, \tilde{y}_n$ the points in the normalization $\tilde{X}'$ of $\tilde{X}$ lying over W. If $V_1, \ldots, V_m$ are the irreducible components of $\tilde{X}$, then $\tilde{y}_i$ is in the normalization $V'_{k(i)}$ of $V_{k(i)}$. Let $\alpha_k \in K^*$ with $|\alpha_k| = |a(V_k)|$. Then $(a/\alpha_k)^{\sim}$ may be viewed as a rational function on V'_k. Let $\mathrm{ord}_{\tilde{y}_i}(a)$ be the order of $(a/\alpha_{k(i)})^{\sim}$. By passing to a formal subdomain, we may assume that it exists $f \in \mathscr{A}^\circ$ such that $\tilde{f}$ has an isolated zero in W. By [BL1], Lemma 2.4, for $0 < r < 1$, $r \in |K^*|$ and r sufficiently close to 1, the periphery $\{x \in X_+(W); |f(x)| \geqq r\}$ of the formal fibre $X_+(W)$ decomposes into n connected components $R_1, \ldots, R_n$ where R_i can be identified with the semi-open annuli

$$\{z \in \mathbb{B}^1; r_i \leqq |z| < 1\}$$

where $r_i := r^{1/\mathrm{ord}_{\tilde{y}_i}(f)}$. Moreover, we have

$$f = z^{\mathrm{ord}_{\tilde{y}_i}(f)}$$

and

$$a/\alpha_k = z^{\mathrm{ord}_{\tilde{y}_i}(a)} \tag{16}$$

on R_i up to units in $\mathcal{O}_X^\circ(R_i)$ ([BL1], Lemma 2.5). By pasting $X_+(W)$ with the discs $B_i := \{z \in \mathbb{P}^1; |z| \geqq r_i\}$ along R_i for all $i = 1, \ldots, n$, we get a curve Y. Clearly, Y is proper over K and therefore Y is the rigid analytic variety associated to a non-singular projective curve. On Y, we use the formal analytic structure given by the admissible formal affinoid covering

$$\{z \in B_i; |z| \geqq r_i\}, \quad i = 1, \ldots, n,$$

together with

$$U := Y \setminus \bigcup_{i=1}^{n} \{z \in B_i; |z| > r_i\} = \{x \in X; |f(x)| \leqq r\}$$

(cf. [BL1], Proposition 4.1). Let $\mathscr{Y}$ be the associated admissible formal scheme, then $\tilde{\mathscr{Y}}$ is the union of n copies $L_1, \ldots, L_n$ of $\mathbb{P}^1_{\tilde{K}}$ meeting all in a point q and the i-th copy minus q is $\tilde{B}_i$. Here, we use that formal fibres are connected ([Bo3], Satz 6.1). The formal fibre $Y_+(q)$ over q is equal to $X_+(W) \setminus \bigcup_{i=1}^{n} R_i$. Let q_i be the point in the normalization of $\tilde{\mathscr{Y}}$ lying over q and lying in the component over L_i. Since the periphery of $Y_+(q)$ has boundary $\{|f(x)| = r\}$, it follows from (16) that

$$|a(L_i)| = r_i^{\mathrm{ord}_{\tilde{y}_i}(a)} |a(V_{k(i)})| \tag{17}$$

and

$$\mathrm{ord}_{\tilde{y}_i}(a) = \mathrm{ord}_{q_i}(a). \tag{18}$$

Note that a, a' induce Cartier divisors $\mathrm{div}_{\mathscr{U}}(a)$, $\mathrm{div}_{\mathscr{U}}(a')$ on the admissible formal affine scheme $\mathscr{U} = U^{f-\mathrm{sch}}$. We claim that it is enough to prove

$$(19) \qquad \operatorname{div}_{\mathscr{U}}(a).\operatorname{div}_{\mathscr{U}}(a') = \operatorname{div}_{\mathscr{U}}(a').\operatorname{div}_{\mathscr{U}}(a)$$

in q. Since r is sufficiently close to 1, the horizontal parts of $\operatorname{cyc}(D)$, $\operatorname{cyc}(D')$ intersected with $X_+(W)$ are supported in U and equal to the horizontal parts of the Weil divisors associated to $\operatorname{div}_{\mathscr{U}}(a)$ and $\operatorname{div}_{\mathscr{U}}(a')$, respectively. It follows that the difference of the multiplicities of $D'.D$ in W and of $\operatorname{div}_{\mathscr{U}}(a').\operatorname{div}_{\mathscr{U}}(a)$ in q is equal to

$$-\sum_{i=1}^{n} \operatorname{ord}_{\tilde{y}_i}(a') \log|a(V_{k(i)})|$$

minus

$$-\sum_{i=1}^{n} \operatorname{ord}_{q_i}(a') \log|a(L_i)|\,.$$

Using (17) for a and (18) for a', we see that the above difference is symmetric in a and a', i.e. it is enough to prove (19).

By the semi-stable reduction theorem, Y has a model $\mathscr{Y}'$ with semi-stable reduction. By a careful look at the proof of [BL1], Lemma 7.3, we may assume that the formal topology on Y induced by $\mathscr{Y}'$ refines the formal topology induced by $\mathscr{Y}$. In fact, the construction of $\mathscr{Y}'$ is by blowing up the singularities of $\mathscr{Y}^{f-\mathrm{an}}$. We conclude that U has an admissible formal model with semi-stable reduction.

To summarize the above reduction steps, it is enough to prove the claim for an admissible formal affine scheme $\mathfrak{X} = \operatorname{Spf} A$ over K° with semi-stable reduction such that the generic fibre is an irreducible and non-singular curve X over the algebraically closed field K. Remember that we have to check

$$(20) \qquad D.D' = D'.D$$

in the closed point W of $\tilde{\mathfrak{X}}$ where D, D' are given by $a, a' \in A$.

First, we assume that W is a non-singular point of $\tilde{\mathfrak{X}}$. By [BL1], Proposition 2.2, the formal fibre $X_+(W)$ of W in X may be identified with the open disc $\mathbb{B}^1_+(0)$. On the disc, we use the coordinate function z. Let V be the component of $\tilde{\mathfrak{X}}$ passing through W and let $\alpha_V \in K$ with $|\alpha_V| = |a(V)|$. Let τ be the order of the reduction of a/α_V in W. Consider the power series expression

$$a = \sum_{n=0}^{\infty} a_n z^n$$

on $\mathbb{B}^1_+(0)$. By [BL1], Lemma 2.4, $a_\tau z^\tau$ is dominant at the periphery of $\mathbb{B}^1_+(0)$, i.e. τ is the smallest $k \in \mathbb{N}$ with

$$|a_k| = \max_n |a_n| = |a(V)|\,.$$

By Hensel's Lemma ([Bou], chap. III, §4, Théorème 1), a has exactly τ zeros $\alpha_1, \ldots, \alpha_\tau$ (counted with multiplicities) in $\mathbb{B}^1_+(0)$ and we have

$$a = u(z - \alpha_1) \cdots (z - \alpha_\tau)$$

where the power series u is a unity on $\mathbb{B}^1_+(0)$. Similarly, we have a decomposition

$$a' = u'(z - \alpha'_1) \cdots (z - \alpha'_{\tau'})$$

of a' on $\mathbb{B}^1_+(0)$. Then the multiplicity of $D.D'$ in W is equal to

$$\begin{aligned} &-\tau \log|a'_{\tau'}| - \sum_{k=1}^{\tau'} \log|a(\alpha'_k)| \\ &= -\tau \log|a'_{\tau'}| - \sum_{k=1}^{\tau'} \log|u(\alpha'_k)| - \sum_{j=1}^{\tau} \sum_{k=1}^{\tau'} \log|\alpha_j - \alpha'_k| . \end{aligned}$$

Here, the first summand is obtained by intersecting D with the vertical part of $\operatorname{cyc}(D')$ and the remaining terms are from the intersection of D with the horizontal part of $\operatorname{cyc}(D')$. For the unity u, the constant term of the power series expression is dominant on the periphery of $\mathbb{B}^1_+(0)$, so we have

$$|u(\alpha'_k)| = |u(V)| = |a_\tau|$$

for all k. By symmetry, we get (20).

Now we assume that W is an ordinary double point of $\tilde{\mathfrak{X}}$. Let us denote by W_1 and W_2 the points in the normalization of $\tilde{\mathfrak{X}}$ lying over W. Note that the formal fibre $X_+(W)$ is isomorphic to an open annulus of height r ([BL1], Prop. 2.3). Assume, for the moment, that a and a' are units on $X_+(W)$. Let V_1, V_2 be the components of $\tilde{\mathfrak{X}}$ passing through W (possibly $V_1 = V_2$). We may assume that W_i lies in the normalization of V_i $(i = 1, 2)$. Then the multiplicity of $\operatorname{div} a . \operatorname{div} a'$ in W equals

$$-\log|a'(V_1)| \operatorname{ord}_{W_1}(a) - \log|a'(V_2)| \operatorname{ord}_{W_2}(a) .$$

Note that this is true in the case $V_1 = V_2$ because of the identity $\operatorname{ord}_{W_1}(a) = \operatorname{ord}_{W_2}(a) = 0$ ([BL1], Proposition 3.2). Using $\operatorname{ord}_{W_1}(a) = -\operatorname{ord}_{W_2}(a)$ ([BL1], Proposition 3.1), the multiplicity above equals

$$\operatorname{ord}_{W_1}(a) \log\big(|a'(V_2)|/|a'(V_1)|\big) .$$

By [BL1], Proposition 3.2, this equals

$$-\operatorname{ord}_{W_1}(a) \operatorname{ord}_{W_2}(a) \log r .$$

By symmetry, this proves the claim for units a, a' on $X_+(W)$. Now we assume no longer that a and a' are units on $X_+(W)$. We choose little balls B_j around the zeros and poles $x_1, \ldots, x_k$ of a and a' contained in $X_+(W)$. By [BL1], Proposition 4.1, these balls together with the complement of $\bigcup_j (B_j)_+(\tilde{x}_j)$ form an admissible formal affinoid covering of X inducing a finer formal analytic topology. Let $\mathfrak{X}' \to \mathfrak{X}$ be the corresponding admissible formal blowing up. By Lemma 5, it is enough to prove (20) for a closed point W' of $\tilde{\mathfrak{X}}'$ lying over W. But either W' is a regular point of $\tilde{\mathfrak{X}}'$ or a and a' are units on the formal fibre over W'. The first case is already settled. In the second case, we use another admissible formal blowing up to get double points lying over W'. As a and a' are units on the fibres lying over these double points, we get the claim. □

6. Comparison with usual intersection theory

Let K be a field with a non trivial non-archimedean complete absolute value $|\ |$. We discuss first the relation between divisors on schemes and on its associated rigid analytic varieties.

6.1. Let X be a scheme locally of finite type over K. Naturally, there is a rigid analytic variety X^{an} over K associated to X given by the following construction. Locally, X is given by a closed immersion into affine space. Using the same set of equations, we get a K-affinoid variety embedded in the closed ball with center O and radius $r \in |K^*|$. By a gluing process with respect to varying r and varying affine open subsets of X, we get X^{an}. There is a natural flat morphism $X^{\mathrm{an}} \to X$ of locally G-ringed spaces inducing a bijection between $\bar{K}$-rational points ([Be1], Theorem 3.4.1). We have a pull-back homomorphism $Z \mapsto Z^{\mathrm{an}}$ between cycles of X and X^{an}. From flatness, we get

Proposition 6.2. *If D is a Cartier divisor on X with pull-back D^{an} to X^{an}, then*

$$\operatorname{cyc}(D)^{\mathrm{an}} = \operatorname{cyc}(D^{\mathrm{an}})\,.$$

Note that $X \to X^{\mathrm{an}}$ is functorial. Let $\varphi : X' \to X$ be a morphism of finite type. Then we have a morphism $\varphi^{\mathrm{an}} : (X')^{\mathrm{an}} \to X^{\mathrm{an}}$. The morphism φ is finite (resp. proper, resp. flat) if and only if the corresponding property is true for φ^{an} ([Be1], Proposition 3.4.7).

Proposition 6.3. *Under the hypothesis above, we have*:

(a) *If φ is proper and Z' is a cycle on X', then* $\big(\varphi_*(Z')\big)^{\mathrm{an}} = (\varphi^{\mathrm{an}})_*\big((Z')^{\mathrm{an}}\big)$.

(b) *If φ is flat and Z is a cycle on X, then* $\varphi^*(Z)^{\mathrm{an}} = (\varphi^{\mathrm{an}})^*(Z^{\mathrm{an}})$.

(c) *If Y is a closed subspace of X, then* $\operatorname{cyc}(Y)^{\mathrm{an}} = \operatorname{cyc}(Y^{\mathrm{an}})$.

Proof. By a local consideration, (c) follows from the following fact for noetherian schemes: Flat pull-back commutes with forming the cycle associated to a closed subspace (cf. 2.8).

To prove (a), we may assume that Z' is a prime cycle and that $Z' = \operatorname{cyc}(X')$. Moreover, we may assume that φ is surjective. Note that Y^{an} is of pure dimension n if Y is of pure dimension n. So we may assume that X and X' have the same dimension. Then φ is finite over an open dense subset U of X. It follows that φ^{an} is finite over U^{an}. By definition of $(\varphi^{\mathrm{an}})_*$, we may suppose that $U = X$, i.e. φ is finite. Moreover, we may assume $X = \operatorname{Spec} A$. There is a K-algebra and finite A-module A' such that X' is isomorphic to $\operatorname{Spec} A'$. Let $a_1', \dots, a_n'$ be a set of generators of the A-module A'. The K-algebra A is a quotient of $K[x_1, \dots, x_m]$ given by the relations $f_1(\mathbf{x}), \dots, f_M(\mathbf{x})$. Then A' is given in

$$K[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

by the relations $f_1(\mathbf{x}), \ldots, f_M(\mathbf{x}), g_1(\mathbf{x}, \mathbf{y}), \ldots, g_N(\mathbf{x}, \mathbf{y})$ where a'_j corresponds to y_j. Using the multiplication of A', we get relations

$$y_i y_j = \sum_{k=1}^{n} a_{ijk}(\mathbf{x}) y_k . \tag{1}$$

Let r, $R \in |K^*|$. Locally, X^{an} is given by the K-affinoid algebra

$$K\langle r^{-1}\mathbf{x}\rangle / \langle f_1(\mathbf{x}), \ldots, f_M(\mathbf{x})\rangle$$

and $(X')^{\mathrm{an}}$ is given by the K-affinoid algebra

$$K\langle r^{-1}\mathbf{x}, R^{-1}\mathbf{y}\rangle / \langle f_1(\mathbf{x}), \ldots, f_M(\mathbf{x}), g_1(\mathbf{x}, \mathbf{y}), \ldots, g_N(\mathbf{x}, \mathbf{y})\rangle .$$

If R is sufficiently large, then the latter is isomorphic to

$$K\langle r^{-1}\mathbf{x}\rangle [\mathbf{y}] / \langle f_1(\mathbf{x}), \ldots, f_M(\mathbf{x}), g_1(\mathbf{x}, \mathbf{y}), \ldots, g_N(\mathbf{x}, \mathbf{y})\rangle$$

because of the relations (1). We conclude that, locally in X^{an}, the diagram

$$\begin{tikzcd} (X')^{\mathrm{an}} \arrow[r, "\varphi^{\mathrm{an}}"] \arrow[d] & X^{\mathrm{an}} \arrow[d] \\ X' \arrow[r, "\varphi"] & X \end{tikzcd}$$

is Cartesian. Then (a) follows from the corresponding statement for noetherian schemes as in Proposition 2.12. Finally, (b) is an immediate consequence of flatness of $X^{\mathrm{an}} \to X$. □

6.4. Now we suppose that $|\ |$ is a complete discrete absolute value on K. Let K° be the discrete valuation ring of K with uniform parameter π. We assume that the absolute value is normalized by $-\log|\pi| = 1$. Let $\mathfrak{X}$ be a flat scheme locally of finite type over K° and let D be a Cartier divisor on $\mathfrak{X}$. The maximal ideal of the discrete valuation ring K° is denoted by $K^{\circ\circ}$.

The formal completion $\hat{\mathfrak{X}}$ of $\mathfrak{X}$ is given by the following construction. Let $\operatorname{Spec} A$ be an affine open subset of $\mathfrak{X}$. Then the formal completion of $\operatorname{Spec} A$ is $\operatorname{Spf} \hat{A}$ where $\hat{A}$ is the completion of A with respect to the $K^{\circ\circ}$-adic topology. By a gluing process, we get $\hat{\mathfrak{X}}$. Note that $\hat{A}$ is a flat A-algebra ([BGR], Corollary 7.3.1/6). It follows that $\hat{\mathfrak{X}}$ is an admissible formal scheme over K°. Naturally, the special fibre of $\hat{\mathfrak{X}}$ is isomorphic to the special fibre $\tilde{\mathfrak{X}} = \mathfrak{X} \otimes_{K^\circ} \tilde{K}$ of $\mathfrak{X}$. Using the local construction above, D induces a Cartier divisor $\hat{D}$ on $\hat{\mathfrak{X}}$.

Proposition 6.5. *Under the isomorphism above, the vertical part of the Weil divisor associated to D on $\mathfrak{X}$ corresponds to the vertical part of the Weil divisor associated to $\hat{D}$ on $\hat{\mathfrak{X}}$.*

Proof. We may assume $\mathfrak{X} = \operatorname{Spec} A$ and $D = \operatorname{div}(a)$ for an element $\gamma \in A$ which is not a zero-divisor. Since $\operatorname{Spec} \hat{A}$ is a flat noetherian scheme over $\operatorname{Spec} A$, it is enough to show that the vertical part of $\mathrm{div}(\gamma)$ on $\mathrm{Spec}\,\hat{A}$ (in the sense of algebraic geometry) is equal to $\mathrm{cyc}_v(\hat{D})$. By localizing, we may assume that $\tilde{\mathfrak{X}}$ has only one component. Using noetherian normalization, there is a morphism $\varphi : \mathfrak{X} \to \mathbb{A}^d_{K^\circ}$ such that the restriction to special fibres is a finite surjective morphism. Then φ induces a morphism $\hat{\varphi} : \hat{\mathfrak{X}} \to \mathrm{Spf}\,K^\circ\langle x_1, \ldots, x_d\rangle$ with the same property. Let Q be the field of fractions of $K^\circ\langle \mathbf{x}\rangle$. By Lemma 5.6, it is enough to compute $\mathrm{cyc}_v(\hat{D})$ on $\mathfrak{X}' := \hat{\mathfrak{X}} \hat{\otimes}_{K^\circ\langle \mathbf{x}\rangle} Q$. In its proof, we have seen that $A' := \hat{A} \hat{\otimes}_{K^\circ\langle \mathbf{x}\rangle} Q^\circ$ is a flat $\hat{A}$-algebra. Therefore, the vertical part of $\mathrm{div}(\gamma)$ on $\mathrm{Spec}\,A'$ is equal to the pull-back of the vertical part of $\mathrm{div}(\gamma)$ on $\mathrm{Spec}\,\hat{A}$. Using Lemma 5.6, it is enough to prove the claim on $\mathfrak{X}'$. By linearity, we may assume that A' is an integral domain.

We denote the $\hat{Q}$-affinoid algebra $A' \otimes_{\hat{Q}^\circ} \hat{Q}$ by $\mathscr{A}'$. By 1.10, $\widetilde{\mathscr{A}'}$ is is finite over $\widetilde{A'}$ and therefore $\widetilde{\mathscr{A}'}$ is finite over $\tilde{Q}$. It follows from Theorem 1.9 that $\mathscr{A}'$ is a finite $\hat{Q}$-module. By [BGR], Corollary 6.4.1/6, $(\mathscr{A}')^\circ$ is a finite $\hat{Q}^\circ$-module and so it is a finite A'-module. By projection formula, it is enough to prove the claim on $\mathrm{Spf}(\mathscr{A}')^\circ$. Since A' is an integral domain, $\mathscr{A}'$ is a finite field extension of $\hat{Q}$. An application of Lemma 5.7 proves the claim. □

Remark 6.6. Under the assumptions of 6.4, the generic fibre $\hat{X}$ of $\hat{\mathfrak{X}}$ is a subdomain of the rigid analytic variety X^{an} associated to the generic fibre X of $\mathfrak{X}$. The K^a-rational points of $\hat{X}$ are equal to the $(K^a)^\circ$-integral points of $\mathfrak{X}$ (note that $X(K^a) = X^{\mathrm{an}}(K^a)$). Since the morphism $\hat{X} \to X^{\mathrm{an}}$ is flat, the horizontal part of $\mathrm{cyc}(\hat{D})$ is equal to the pull-back of the horizontal part of $\mathrm{cyc}(D)$ under the morphism $\hat{X} \to X$. Using pull-back with respect to this morphism for horizontal cycles and $(\hat{\mathfrak{X}})^\sim \cong \tilde{\mathfrak{X}}$ for vertical cycles, we get a homomorphism $\mathscr{Z} \to \hat{\mathscr{Z}}$ from cycles on $\mathfrak{X}$ to cycles on $\hat{\mathfrak{X}}$. By Proposition 2, this map commutes with flat pull-back and with proper push-forward. Moreover, if $\mathscr{Z}$ is a cycle on $\mathfrak{X}$ intersecting the Cartier divisor D properly in the generic fibre, then Propositions 2 and 5 give

$$(D.\mathscr{Z})^\wedge = \hat{D}.\hat{\mathscr{Z}}\,.$$

7. Formal and approximable metrics

In this section, K is an algebraically closed field with a non-trivial non-archimedean complete absolute value $|\ |$. Let L be a line bundle on the rigid analytic variety X over K.

7.1. The line bundle L is given by an admissible open covering $\{U_i\}$ of X and transition functions $g_{ij} \in \mathcal{O}(U_i \cap U_j)$ satisfying $g_{ii} = 1$ and $g_{ij} g_{jk} = g_{ik}$. Then $\{U_i, g_{ij}\}$ is called a trivialization of L. A metric on L is given by functions ϱ_i on U_i with values in the value group $|K^*|$ satisfying

$$\varrho_i(x)\,|g_{ij}(x)| = \varrho_j(x)$$

for all $x \in U_i$. If s is a regular section of L on U_i, then s corresponds to a regular function γ_i on U_i and we have

$$\|s(x)\| = |\gamma_i(x)|\,\varrho_i(x)\,.$$

As usual, we define dual metrics, tensor metrics and pull-back metrics.

Definition 7.2. If $\mathfrak{X}$ is an admissible formal scheme over K° with generic fibre X, then $\mathfrak{X}$ is said to be a K°-model of X. Similarly, if $\mathscr{L}$ is a line bundle on $\mathfrak{X}$ inducing L on X, then $\mathscr{L}$ is said to be a K°-model of L.

Definition 7.3. A metric $\| \ \|$ on L is said to be formal if there is a trivialization $\{U_i, g_{ij}\}$ such that the metric is given by the functions $\varrho_i = 1$.

Obviously, the dual, the tensor product and the pull-back of formal metrics are again formal metrics.

Lemma 7.4. *Suppose that $\mathscr{L}$ is a K°-model of L on the K°-model $\mathfrak{X}$ of X. Then we have a canonical formal metric $\| \ \|_{\mathscr{L}}$ on L with the following property. If U is the generic fibre of a formal open subset $\mathscr{U}$ such that $\mathscr{L}|_{\mathscr{U}}$ is trivial and if $s \in L(U)$ corresponds to a function γ under this trivialization, then we have*

$$\|s(x)\|_{\mathscr{L}} = |\gamma(x)|$$

for all $x \in U$.

Proof. Let $\{\mathscr{U}_i, g_{ij}\}$ be a trivialization of $\mathscr{L}$. Then the functions g_{ij} are units in $\mathscr{O}_{\mathfrak{X}}(\mathscr{U}_i \cap \mathscr{U}_j)$. Therefore g_{ij} and the inverse g_{ji} have supremum norm 1 on $U_i \cap U_j$ where U_i denotes the generic fibre of $\mathscr{U}_i$. In other words, we have $|g_{ij}(x)| = 1$ for all $x \in U_i \cap U_j$. We conclude that the functions $\varrho_i = 1$ on U_i describe a formal metric on L. Obviously, it has the required property. $\square$

We call $\| \ \|_{\mathscr{L}}$ the formal metric associated to $\mathscr{L}$. Next, we show that any formal metric is of this form, at least locally. Let X_{red} be the subvariety of X with the same underlying space and the induced reduced structure. Then L induces a line bundle L_{red} on X_{red}. There is a one-to-one correspondence between metrics on L and on L_{red}.

Proposition 7.5. *Suppose that X is quasi-compact, quasi-separated and reduced. Then any formal metric $\| \ \|$ on L is the formal metric associated to a K°-model $\mathscr{L}$ of L on a quasi-compact K°-model $\mathfrak{X}$ with reduced special fibre. Moreover, $\mathscr{L}$ is canonically isomorphic to the sheaf $(\hat{L})^\circ$ given by*

$$(\hat{L})^\circ(\mathscr{U}) := \{s \in L(U); \|s(x)\| \leqq 1 \quad \forall x \in U\}$$

on a formal open subset $\mathscr{U}$ of $\mathfrak{X}$ with generic fibre U.

Proof. Let $\{U_i, g_{ij}\}$ be a trivialization of L such that $\| \ \|$ is given by the function $\varrho_i = 1$ on U_i. We may assume that the open subsets U_i are K-affinoid. By [BL 4], Theorem 5.5, there is a quasi-compact K°-model $\mathfrak{X}$ of X and a formal open covering $\{\mathscr{U}_i\}$ of $\mathfrak{X}$ such that the generic fibre of $\mathscr{U}_i$ is U_i. Replacing $\mathfrak{X}$ by the formal scheme associated to $\mathfrak{X}^{f-\mathrm{an}}$, we may assume that $\mathfrak{X}$ has reduced special fibre. Clearly, $(\hat{L})^\circ$ is a sheaf on $\mathfrak{X}$. On $\mathscr{U}_i$, it is isomorphic to $\mathscr{O}_{\mathfrak{X}}^\circ$. Therefore $(\hat{L})^\circ$ is a line bundle on $\mathfrak{X}$ with associated formal metric $\| \ \|$. If $\mathscr{L}$ is another line bundle on $\mathfrak{X}$ with associated formal metric $\| \ \|$, then $\mathscr{L}$ is canonically isomorphic to $(\hat{L})^\circ$ since we may identify $\mathscr{L}(\mathscr{U})$ with the subset $(\hat{L})^\circ(\mathscr{U})$ of $L(U)$ by flatness of $\mathfrak{X}$. $\square$

Lemma 7.6. *Suppose that X is quasi-compact and quasi-separated. Then there is a K°-model $\mathscr{L}$ of L on a quasi-compact K°-model $\mathfrak{X}$ of X.*

Proof. By [BL3], Proposition 5.6, there is a quasi-compact K°-model $\mathfrak{X}$ of X and a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathscr{F}$ on $\mathfrak{X}$ such that the restriction of $\mathscr{F}$ to X is equal to L. Let $\{\mathscr{U}_i\}_{i\in I}$ be a formal affine open covering of $\mathfrak{X}$. By [BL4], Theorem 5.5, we may assume that L is trivial on the generic fibre U_i of $\mathscr{U}_i$ for all $i\in I$ and that I is finite. The $\mathcal{O}_{\mathfrak{X}}(\mathscr{U}_i)$-module $\mathscr{F}(\mathscr{U}_i)$ is generated by finitely many f_{ij}. There is a non-vanishing section γ of L on U_i with $f_{ij}=g_{ij}\gamma$ for suitable $g_{ij}\in\mathcal{O}_{\mathfrak{X}}(\mathscr{U}_i)$. Then the elements $g_{i1}, g_{i2}, \ldots$ generate an open ideal in $\mathcal{O}_{\mathfrak{X}}(\mathscr{U}_i)$. Using admissible formal blowing ups, we may assume that these ideals are principal (cf. [BL3], section 2), i.e. $\mathscr{F}(\mathscr{U}_i)$ is generated by one element for all i. By [BL3], Lemma 1.4, $\mathscr{L}:=\mathscr{F}/\mathscr{F}_{\mathrm{tor}}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Since $\mathscr{L}$ is locally monogeneous, it is an invertible sheaf. We conclude that $\mathscr{L}$ gives rise to a K°-model of L.

Corollary 7.7. *Any line bundle L on a quasi-compact and quasi-separated rigid analytic variety X over K has a formal metric. If s is a local section of L defined and non-vanishing in different points x, y, then there is a formal metric $\| \ \|$ on L with $\|s(x)\| \neq \|s(y)\|$.*

Proof. By Lemmata 4 and 6, there is a formal metric on L. Hence, to prove the last claim, we may assume $L=O_X$ and $s=1$. Using [BL4], Theorem 5.5, we get a quasi-compact K°-model $\mathfrak{X}$ of X such that the reductions $\tilde{x}$, $\tilde{y}$ of x, y are different. Let us consider an admissible formal blowing up $\pi:\mathfrak{X}'\to\mathfrak{X}$ in an ideal $\mathscr{H}$ in $\mathcal{O}_{\mathfrak{X}}$ with support in $\tilde{y}$. Then the ideal $\mathscr{H}\mathcal{O}_{\mathfrak{X}'}$ is invertible with an inverse $\mathscr{L}$. We conclude that $\mathscr{L}$ is a K°-model of L and the canonical extension $\bar{s}$ of s vanishes in $\tilde{y}$. Let $\| \ \|$ be the formal metric associated to $\mathscr{L}$, then we have $\|s(y)\|<1$ and $\|s(x)\|=1$. □

Lemma 7.8. *If $\| \ \|_1$, $\| \ \|_2$ are formal metrics on L, then* $\max\{\| \ \|_1, \| \ \|_2\}$ *and* $\min\{\| \ \|_1, \| \ \|_2\}$ *are formal metrics on L.*

Proof. A trivialization of L is given by an admissible open covering $\{U_i\}_{i\in I}$ of X and non-vanishing $s_i\in L(U_i)$. Then the transition functions g_{ij} are equal to s_i/s_j and the metric is given by $\varrho_i=\|s_i\|$ on U_i. Now let $\{U_i, s_{i1}\}$ and $\{U_i, s_{i2}\}$ be such trivializations of L inducing $\varrho_i=1$ for $\| \ \|_1$ and $\| \ \|_2$, respectively. Note that

$$U_{i1}:=\{x\in U_i;\ \| \ \|_1 \geqq \| \ \|_2 \text{ on } L_x\}$$

and

$$U_{i2}:=\{x\in U_i;\ \| \ \|_1 \leqq \| \ \|_2 \text{ on } L_x\}$$

are subdomains of U_i. We get a new admissible open covering $\{U_{ij}\}_{i\in I, j=1,2}$ of X. On U_{ij}, we have

$$\max\{\|s_{ij}\|_1, \|s_{ij}\|_2\}=1\,.$$

Therefore $\{U_{ij}, s_{ij}\}_{i\in I, j=1,2}$ induces a trivialization of L such that the functions ϱ_{ij} of the metric $\max\{\| \ \|_1, \| \ \|_2\}$ are equal to 1. This shows that $\max\{\| \ \|_1, \| \ \|_2\}$ is a formal metric. Similarly, we prove that $\min\{\| \ \|_1, \| \ \|_2\}$ is a formal metric. □

7.9. Suppose that X is quasi-compact and quasi-separated. Then X is a dense subset of the Berkovich-compactification X^B ([Be2], 1.6). If $X=\mathrm{Sp}\mathscr{A}$ is K-affinoid, then X^B is the set of semi-norms p on $\mathscr{A}$ satisfying $p(1)=1$, $p(ab)=p(a)p(b)$ and $p(a) \leqq |a|_{\sup}$ for all $a, b \in \mathscr{A}$. We identify $x \in X$ with the seminorm $a \mapsto |a(x)|$ in X^B. The topology on X^B is the coarsest topology such that the maps $p \mapsto p(a)$ are continuous for all $a \in \mathscr{A}$. If X is not K-affinoid, then X^B is constructed by a gluing process with respect to a finite admissible open affinoid covering of X. Then X^B is a compact topological space.

Definition 7.10. A metric $\| \;\|$ on L is called extendable if and only if, for any non-vanishing section s of L on an open affinoid subvariety U, the function $\log\|s\|$ has an extension to a continuous function on U^B.

Then the continuous extension is unique and it is also denoted by $\log\|s\|$. There is a trivialization $\{U_i, g_{ij}\}$ of L where $\{U_i\}$ is an admissible open covering of X by finitely many open affinoid subvarieties. If the metric $\| \;\|$ on L is given by the positive function ϱ_i on U_i, then $\| \;\|$ is extendable if and only if all functions ϱ_i have continuous strictly positive extensions to U_i^B. Note that the formal metrics on L are extendable.

Definition 7.11. If $\| \;\|$ and $\| \;\|'$ are metrics on L, then we get a real function $\| \;\|'/\| \;\|$ on X by mapping x to $\|s(x)\|'/\|s(x)\|$ where s is any local non-vanishing section of L in x. Clearly, this is independent of the choice of s. A sequence of metrics $(\| \;\|_n)_{n \in \mathbb{N}}$ on L is said to converge to the metric $\| \;\|$ if and only if, for any open affinoid subvariety U of X, the sequence

$$\sup_{x \in U} (\| \;\|_n / \| \;\|)$$

converges to 1. A metric $\| \;\|$ on L is called approximable if and only if there is a sequence $(\| \;\|_n)_{n \in \mathbb{N}}$ of metrics on L converging to $\| \;\|$ such that a non-trivial power of $\| \;\|_n$ is a formal metric for all $n \in \mathbb{N}$.

Obviously, the tensor product, the dual, the pull-back and the limit of approximable metrics are again approximable metrics.

Theorem 7.12. *Let L be a line bundle on the quasi-compact and quasi-separated rigid analytic variety X over K. Then a metric on L is approximable if and only if it is extendable.*

Proof. First, we assume that the metric is approximable. Then there is a sequence $(\| \;\|_n)_{n \in \mathbb{N}}$ of metrics on L such that a non-trivial power of $\| \;\|_n$ is a formal metric for all $n \in \mathbb{N}$. Let s be a non-vanishing section of L on the open affinoid subvariety U. The metrics $\| \;\|_n$ are extendable. Since U is dense in U^B, $(\log\|s\|_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the space $C(U^B)$ of continuous real functions on the compact set U^B. It converges to a continuous real function extending $\log\|s\|$ to U^B. Therefore $\| \;\|$ is extendable.

Conversely, we have to show that any extendable metric on L is approximable. By Corollary 7, there is a formal metric on L. So we may assume that $L = O_X$. Let us consider the $\mathbb{Q}$-subspace

$$F := \{-\log\|1\|^{1/m};\ \| \;\| \text{ formal metric on } L,\ m \in \mathbb{N} \setminus \{0\}\}$$

of $C(X^B)$. By Lemma 8, F is closed under forming maximum and minimum. By Corollary 7, F separates points of X. Indeed, the same proof shows that F separates points of

X^B. As a consequence of the Stone-Weierstrass theorem, F is dense in $C(X^A)$. This proves that any extendable metric on L is approximable. □

Definition 7.13. Let L be a line bundle on X. A formal metric $\| \ \|$ is called positive if there is a K°-model $\mathscr{L}$ of L with $\| \ \| = \| \ \|_{\mathscr{L}}$ such that some power of $\mathscr{L}$ is base-point-free.

Remark 7.14. Clearly, the tensor product and the pull-back of positive formal metrics remain positive (use Theorem 1.13).

8. Formal metrics and intersection theory

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $| \ |$. We consider a line bundle L on a quasi-compact and quasi-separated rigid analytic variety X over K.

8.1. Let $\mathscr{M}_X$ be the set of K°-models of X. Then $\mathscr{M}_X$ is non-empty (Theorem 1.13). There is a (partial) order relation on $\mathscr{M}_X$ by defining $\mathfrak{X}' \geqq \mathfrak{X}'$ if there is a K°-morphism $\pi : \mathfrak{X}' \to \mathfrak{X}$ of formal schemes inducing the identity on X. Note that π is the only K°-morphism extending the identity ([BL3], proof of Theorem 4.4). If $\mathfrak{X}, \mathfrak{X}' \in \mathscr{M}_X$, then there is $\mathfrak{X}'' \in \mathscr{M}_X$ and admissible formal blowing ups $\mathfrak{X}' \to \mathfrak{X}'$ and $\mathfrak{X}'' \to \mathfrak{X}$ (Theorem 1.13), i.e. $\mathscr{M}_X$ is a directed set.

In fact, we use $\mathscr{M}_X$ only for reduced X. Then X is distinguished ([BGR], Theorem 6.4.3/1), and for any $\mathfrak{X} \in \mathscr{M}_X$, there is $\mathfrak{X}' \in \mathscr{M}_X$ with reduced special fibre and $\mathfrak{X}' \geqq \mathfrak{X}$. So we may restrict our attention to K°-models with reduced special fibres or, equivalently, to formal schemes associated to formal analytic structures on X (Proposition 1.11).

In the following, we denote the group of cycles on an algebraic variety, on a rigid analytic variety or on an admissible formal scheme V by $Z(V)$.

Definition 8.2. The group of vertical cycles on X is defined by

$$Z(X, v) := \varprojlim_{\mathfrak{X} \in \mathscr{M}_{X_{\mathrm{red}}}} Z(\tilde{\mathfrak{X}})$$

where the inverse limit is with respect to $\tilde{\pi}_*$ for morphisms $\pi : \mathfrak{X}' \to \mathfrak{X}$ of K°-models. Moreover, let

$$\mathscr{Z}(X) := Z(X) \oplus Z(X, v)$$

and

$$CH(X, v) := \varprojlim_{\mathfrak{X} \in \mathscr{M}_{X_{\mathrm{red}}}} CH(\mathfrak{X}, v)$$

where the inverse limit is again with respect to push-forward. For elements of $\mathscr{Z}(X)$ or $CH(X, v)$, the component in $Z(X)$ is called the horizontal part and the second component is called the vertical part.

Lemma 8.3. *If $\varphi : X' \to X$ is a proper morphism of quasi-compact and quasi-separated rigid analytic varieties over K, then, by componentwise push-forward, we get a canonical push-forward $\varphi_* : Z(X', v) \to Z(X, v)$. In particular, if Y is an analytic subset of X, then $Z(Y, v)$ may be viewed as a subgroup of $Z(X, v)$.*

Proof. For $a \in Z(X', v)$, the push-forward $b := \varphi_*(a) \in Z(X, v)$ is defined in the following way: For $\mathfrak{X} \in \mathcal{M}_{X_{\mathrm{red}}}$, there is $\mathfrak{X}' \in \mathcal{M}_{X'_{\mathrm{red}}}$ such that φ extends to a morphism $\bar{\varphi} : \mathfrak{X}' \to \mathfrak{X}$ (Theorem 1.13). Then the reduction φ of $\bar{\varphi}$ is proper (Remark 3.14) and we define $b_{\mathfrak{X}} := \tilde{\varphi}_*(a_{\mathfrak{X}'})$. Clearly, this gives a well-defined homomorphism $\varphi_* : Z(X', v) \to Z(X, v)$.

If $\mathcal{Y} \in \mathcal{M}_{Y_{\mathrm{red}}}$, then there is $\mathcal{Y}' \geqq \mathcal{Y}$ and $\mathfrak{X} \in \mathcal{M}_{X_{\mathrm{red}}}$ such that $\mathcal{Y}'$ is a closed subscheme of $\mathfrak{X}$ ([BL4], Corollary 5.4). Since $Z(\mathcal{Y}') \subset Z(\mathfrak{X})$, we get the last claim. □

Using the usual push-forward on horizontal components (2.6), we get a group homomorphism $\varphi_* : \mathcal{Z}(X') \to \mathcal{Z}(X)$. Clearly, it induces a push-forward map

$$\varphi_* : CH(X', v) \to CH(X, v) .$$

Proposition 8.4. *Let s be an invertible meromorphic section of L and let $\| \ \|$ be a metric on L such that $\| \ \|^m$ is a formal metric on $L^{\otimes m}$ for some $m \geqq 1$. If* $\mathrm{div}(s)$ *intersects the horizontal part of $\alpha \in CH(X, v)$ properly, then there is an element $\widehat{\mathrm{div}}(s).\alpha \in CH(X, v)$, uniquely determined by the following condition: If $(\mathfrak{X}, \mathcal{L})$ is a K°-model of $(X_{\mathrm{red}}, L_{\mathrm{red}}^{\otimes m})$ with $\| \ \|^m = \| \ \|_{\mathcal{L}}$ on $L_{\mathrm{red}}^{\otimes m}$ and if $\bar{s}$ denotes the unique extension of $s_{\mathrm{red}}^{\otimes m}$ to a meromorphic section of $\mathcal{L}$, then*

$$\big(\widehat{\mathrm{div}}(s).\alpha\big)_{\mathfrak{X}} = \frac{1}{m}\,\mathrm{div}(\bar{s}).\alpha_{\mathfrak{X}} \in CH(\mathfrak{X}, v) .$$

The class $\widehat{\mathrm{div}}(s).\alpha$ is called the intersection product of $\mathrm{div}(s)$ *and α.*

Proof. We choose a K°-model $\mathfrak{X}$ of X_{red} with reduced special fibre. If the line bundle $\mathcal{L}$ on $\mathfrak{X}$ is a K°-model of $L_{\mathrm{red}}^{\otimes m}$ inducing the right metric on $L_{\mathrm{red}}^{\otimes m}$, then $\mathcal{L}$ is determined by the formal metric up to isomorphism (Proposition 7.5). We define the component of $\widehat{\mathrm{div}}(s).\alpha$ on $\mathfrak{X}$ by the formula above. If $\mathfrak{X}' \geqq \mathfrak{X}$ is a K°-model of X with reduced special fibre, then the definition is compatible with push-forward. By Proposition 7.5, we get a well-defined element $\widehat{\mathrm{div}}(s).\alpha \in CH(\mathfrak{X}, v)$. Using the definition of intersection product on admissible formal schemes, we see that the required property is satisfied. □

A metric as in Proposition 8.4 is called a root of a formal metric. The following two results follow directly from the corresponding properties of intersection product on K°-models.

Proposition 8.5 (Projection formula). *Let $\varphi : X' \to X$ be a proper morphism of quasi-compact and quasi-separated rigid analytic varieties over K. Suppose that the horizontal part of $\alpha' \in CH(X', v)$ intersects $\varphi^*\mathrm{div}(s)$ properly where s is an invertible meromorphic section of L on X. If the metric $\| \ \|$ on L is a root of a formal metric, then*

$$\varphi_*\big(\widehat{\mathrm{div}}(\varphi^* s).\alpha'\big) = \widehat{\mathrm{div}}(s).\varphi_*(\alpha')$$

in $CH(X, v)$.

Theorem 8.6. *If s, s' are invertible meromorphic sections of the line bundles L, L' on X and if* $\operatorname{div}(s)$, $\operatorname{div}(s')$ *and the horizontal part of $\alpha \in CH(X,v)$ intersect properly, then we have*

$$\widehat{\operatorname{div}}(s).\widehat{\operatorname{div}}(s').\alpha = \widehat{\operatorname{div}}(s').\widehat{\operatorname{div}}(s).\alpha$$

in $CH(X,v)$ for all metrics on L and L' which are roots of formal metrics.

Lemma 8.7. *Let $\mathfrak{X}' \geqq \mathfrak{X}$ be K°-models of X both with reduced special fibres. For any irreducible component W of $\tilde{\mathfrak{X}}$, there is exactly one irreducible component W' which is mapped onto W. Moreover, the induced morphism $W' \to W$ is birational.*

Proof. By passing to formal subdomains, we may assume that $\tilde{\mathfrak{X}} = W$. Let us choose $\alpha \in K^{\circ\circ}$. Applying projection formula to the morphism $\mathfrak{X}' \to \mathfrak{X}$ and $\operatorname{div}(\alpha)$ on $\mathfrak{X}$, it follows that there is exactly one irreducible component W' of $\tilde{\mathfrak{X}}'$ mapping onto W and $\tilde{K}(W) \cong \tilde{K}(W')$. This proves the claim. □

Proposition 8.8. *Let s be a global section of L. On L, we consider a root $\| \ \|$ of a formal metric. Suppose that X has a K°-model $\mathfrak{X}$ with reduced special fibre $\tilde{\mathfrak{X}}$ (i.e. X is reduced). Let $m(s,W)$ be the multiplicity of $\big(\widehat{\operatorname{div}}(s).\operatorname{cyc}(X)\big)_{\mathfrak{X}}$ in the irreducible component W of $\tilde{\mathfrak{X}}$. Then there is a non-empty open subset U of $\tilde{\mathfrak{X}}$ contained in W such that*

$$m(s,W) = -\log \|s(x)\|$$

for all $x \in X$ with reduction in U.

Proof. First, we assume that $\| \ \|^m = \| \ \|_{\mathscr{L}}$ for a K°-model $\mathscr{L}$ of L living on $\mathfrak{X}$. Then the claim is a consequence of Lemma 7.4, Remark 3.5 and the definition of $m(s,W)$. In general, there is $m \geqq 1$ and a K°-model $\mathscr{L}$ of $L^{\otimes m}$ with $\| \ \|^m = \| \ \|_{\mathscr{L}}$ (Proposition 7.5). We may assume that $\mathscr{L}$ is a line bundle on $\mathfrak{X}' \geqq \mathfrak{X}$ and that $\mathfrak{X}'$ has reduced special fibre (8.1). Then the claim follows from the above applied to $\mathfrak{X}'$ and from Lemma 8.7. □

9. Non-archimedean local heights

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $| \ |$. We fix $t \in \mathbb{N}$. In this section, we define local heights of cycles of pure dimension t on a complete algebraic variety X over K. There is no loss of generality by restricting our attention to algebraically closed complete fields because we can always achieve this situation by base extension. Without any additional effort, the whole section is more generally true for proper rigid analytic varieties over K instead of complete algebraic varieties. However, then one has to assume that all intersections are proper.

9.1. On $\operatorname{Spf} K^\circ$, we denote by $[1]$ (resp. $[v]$) the cycle induced by the generic (resp. special) fibre. Then we have the following identities for the various groups of cycles introduced in section 8:

$$Z(\operatorname{Sp} K^\circ) = \mathscr{Z}(\operatorname{Sp} K) = CH(\operatorname{Sp} K, v) = \mathbb{Z}[1] \oplus \mathbb{R}[v] .$$

Let $p: X \to \operatorname{Spec} K$ be the morphism of structure. As in 6.1, we denote by X^{an} the rigid analytic variety associated to X. Then we have a pull-back homomorphism

$$Z(X) \to Z(X^{\mathrm{an}}), \quad Z \mapsto Z^{\mathrm{an}}.$$

For $j = 0, \ldots, t$, let s_j be an invertible meromorphic section of the line bundle L_j on X. Note that L_j induces a line bundle L_j^{an} on X^{an}. There is a one-to-one correspondence between metrics on L_j and L_j^{an} and we identify them. On L_j, let us fix a root of a formal metric. We denote the corresponding metrized line bundle by $\hat{L}_j$. Note that s_j induces a meromorphic section s_j^{an} of L_j^{an}. We denote the support of the Cartier divisor $\operatorname{div}(s_j)$ (resp. of a cycle Z) by $|\operatorname{div}(s_j)|$ (resp. $|Z|$).

Definition 9.2. Let Z be a t-dimensional cycle on X. Then $\deg_{L_1, \ldots, L_t}(Z)$ denotes the degree of Z with respect to $L_1, \ldots, L_t$. Suppose that the intersection of $|Z|$, $|\operatorname{div}(s_0)|, \ldots, |\operatorname{div}(s_t)|$ is empty in X. Consider

$$p_*^{\mathrm{an}}\left(\widehat{\operatorname{div}}(s_0^{\mathrm{an}}) \ldots \widehat{\operatorname{div}}(s_t^{\mathrm{an}}).Z^{\mathrm{an}}\right) \in CH^1(\operatorname{Sp} K, v).$$

It is a real multiple $\lambda(Z) = \lambda_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(Z)$ of $[v]$. The number $\lambda(Z)$ is called the local height of Z. (Note that for $k > 0$, the intersection of $|\operatorname{div}(s_k)|, \ldots, |\operatorname{div}(s_t)|$, $|Z|$ has not to be proper. On algebraic varieties, the intersection product introduced in §8 may be refined similarly as in [Fu], §2. Then the above definition makes sense.)

Proposition 9.3. *Under the hypothesis above, the local height $\lambda_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(Z)$ is multi-linear and symmetric in the variables $(\hat{L}_j, s_j)_{j=0, \ldots, t}$.*

Proof. Multilinearity follows from bilinearity of intersection product and symmetry is a consequence of commutativity of intersection product (Theorem 8.6). □

Proposition 9.4. *Let $\varphi: X' \to X$ be a morphism of complete algebraic varieties over K. Suppose that Z' is a t-dimensional cycle on X' such that Z', $\operatorname{div}(\varphi^* s_0), \ldots, \operatorname{div}(\varphi^* s_t)$ have empty intersection in X'. Then we have*

$$\lambda_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(\varphi_*(Z')) = \lambda_{\varphi^*(\hat{L}_0, s_0), \ldots, \varphi^*(\hat{L}_t, s_t)}(Z').$$

Proof. This follows from projection formula (Proposition 8.5). □

Proposition 9.5. *Let $L_0, \ldots, L_t$ be line bundles on the complete algebraic variety X over K. Suppose that Z is a t-dimensional effective cycle on X and that s_j is an invertible meromorphic section of L_j such that supports of $\operatorname{div}(s_0), \ldots, \operatorname{div}(s_t)$, Z have empty intersection in X. On L_j ($j = 1, \ldots, t$), we choose a root of a positive formal metric with resulting metrized line bundle $\hat{L}_j$. On L_0, we consider two roots $\| \ \|$, $\| \ \|'$ of formal metrics with metrized line bundles $\hat{L}_0$ and $\hat{L}_0'$, respectively. Let C_+ (resp. C_-) be the supremum (resp. the infimum) of the real function $\log(\| \ \|' / \| \ \|)$ on the support of Z. Then the constants C_+, C_- are finite and we have*

$$\begin{aligned} C_- \deg_{L_1, \ldots, L_t}(Z) &\leqq \lambda_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(Z) - \lambda_{(\hat{L}_0', s_0), \ldots, (\hat{L}_t, s_t)}(Z) \\ &\leqq C_+ \deg_{L_1, \ldots, L_t}(Z). \end{aligned}$$

Proof. By Theorem 7.12, $\log(\| \|'/\| \|)$ extends to a continuous function on $(X^{\mathrm{an}})^B$. Since the latter is compact, the constants C_+, C_- are finite. To prove the inequalities, we may assume Z prime and $Z = \mathrm{cyc}(X)$. For $j = 1, \ldots, t$, there is $m_j \geqq 1$ and a base-point-free K°-model $\mathscr{L}_j$ of $(L_j^{\mathrm{an}})^{\otimes m_j}$ inducing the given formal metric $\| \|$ on L_j^{an}. Clearly, we may assume that all line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_t$ live on the same K°-model $\mathfrak{X}$ of X^{an}. Using the different metrics on L_0, we see that

$$V\colon \quad \big(\widehat{\mathrm{div}}(s_0) - \widehat{\mathrm{div}}'(s_0)\big).\mathrm{cyc}(X^{\mathrm{an}})$$

is vertical, i.e. it is an element of $Z(X^{\mathrm{an}}, v)$. By Proposition 8.8 and 8.1, we have $V \leqq C_+ p^*([v])$, i.e. the inequality holds for the multiplicities in the irreducible components of the special fibre of any K°-model. Note that the difference of local heights $\lambda - \lambda'$ is determined by

$$p_*^{\mathrm{an}}\big(\widehat{\mathrm{div}}(s_1^{\mathrm{an}}) \ldots \widehat{\mathrm{div}}(s_t^{\mathrm{an}}).V\big) = (\lambda - \lambda')[v]\,.$$

To compute the left hand side, we may use the model $\mathfrak{X}$ introduced above with morphism of structure $\bar{p}: \mathfrak{X} \to \mathrm{Sp}\, K^\circ$. Then the left hand side is equal to the intersection number

$$\frac{1}{m_1 \cdots m_t} \tilde{p}_*\big(c_1(\mathscr{L}_1|_{\tilde{\mathfrak{X}}}) \ldots c_1(\mathscr{L}_t|_{\tilde{\mathfrak{X}}}).V_{\tilde{\mathfrak{X}}}\big)$$

on $\tilde{\mathfrak{X}}$ where $\tilde{p}$ is the reduction of $\bar{p}$. Since the line bundles $\mathscr{L}_j|_{\tilde{\mathfrak{X}}}$ are base-point-free, we conclude

$$\begin{aligned}(\lambda - \lambda')[v] &\leqq C_+ p_*^{\mathrm{an}}\big(\widehat{\mathrm{div}}(s_1^{\mathrm{an}}) \ldots \widehat{\mathrm{div}}(s_t^{\mathrm{an}}).p^*[v]\big) \\ &= C_+[v].p_*^{\mathrm{an}}\big(\widehat{\mathrm{div}}(s_1^{\mathrm{an}}) \ldots \widehat{\mathrm{div}}(s_t^{\mathrm{an}})\big)\end{aligned}$$

where the last step is by commutativity (Theorem 8.6) and projection formula (Proposition 8.5). Note that we can handle $[v]$ as a Cartier-divisor, since a non-zero multiple of $[v]$ is the Weil divisor associated to a Cartier divisor on $\mathrm{Sp}\, K^\circ$. By Propositions 6.2 and 6.3, we have

$$p_*^{\mathrm{an}}\big(\widehat{\mathrm{div}}(s_1^{\mathrm{an}}) \ldots \widehat{\mathrm{div}}(s_t^{\mathrm{an}})\big) = \deg_{L_1, \ldots, L_t}(Z) \cdot [1]$$

on $\mathrm{Sp}\, K$. This proves the upper bound and similarly, we get the lower bound. □

Proposition 9.6. *On the complete algebraic K-variety X, let $\hat{L}_0, \ldots, \hat{L}_t$ be line bundles with roots of formal metrics. For $j = 0, \ldots, t$, let s_j be an invertible meromorphic section of L_j and let s_0' be another invertible meromorphic section of L_0. If Z is a t-dimensional cycle on X such that*

$$|\mathrm{div}(s_0)| \cap |\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset$$

and

$$|\mathrm{div}(s_0')| \cap |\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset\,,$$

then we have

$$\begin{aligned}&\lambda_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(Z) - \lambda_{(\hat{L}_0, s_0'), \ldots, (\hat{L}_t, s_t)}(Z) \\ &= \log\left|\left(\frac{s_0'}{s_0}\right)(\mathrm{div}\, s_1 \ldots \mathrm{div}\, s_t.Z)\right|.\end{aligned}$$

(*Here, for the rational function* $f := \frac{s_0'}{s_0}$, *we define* $f(\sum_j n_j P_j) := \prod_j f(P_j)^{n_j}$ *where* $\sum_j n_j P_j$ *is the zero-dimensional cycle* $\operatorname{div} s_1 \ldots \operatorname{div} s_t . Z$ *on* X.)

Proof. The difference $\lambda' - \lambda$ of local heights is determined by

$$(\lambda' - \lambda)[v] = p_*^{\mathrm{an}}\big(\widehat{\operatorname{div}}(f^{\mathrm{an}}).\widehat{\operatorname{div}}(s_1^{\mathrm{an}}) \ldots \widehat{\operatorname{div}}(s_t^{\mathrm{an}}).Z^{\mathrm{an}}\big)$$

where, for $\widehat{\operatorname{div}}(f^{\mathrm{an}})$, we use the trivial metric on O_X. Then the above is equal to

$$p_*^{\mathrm{an}}\big(\operatorname{div}(\widehat{f^{\mathrm{an}}}).\big(\operatorname{div}(s_1^{\mathrm{an}}) \ldots \operatorname{div}(s_t^{\mathrm{an}}).Z^{\mathrm{an}}\big)\big) = -\log|f(\operatorname{div} s_1 \ldots \operatorname{div} s_t . Z)| \cdot [v] \,.$$

The last step follows from Propositions 6.2 and 6.3. This proves the claim. □

Now let Z be a t-dimensional cycle on the multi-projective space $\mathbb{P} := \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_t}$ over K with fixed coordinates on each factor. Let $F_Z(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t)$ be the Chow form of Z. It is a multi-homogeneous polynomial unique up to multiples. The degree d_i of F_Z with respect to the variable $\boldsymbol{\xi}_i := (\xi_{i0}, \ldots, \xi_{in_i})$ is equal to the degree of Z with respect to the line bundles $\big(O_{\mathbb{P}}(e_j)\big)_{j \neq i}$. We view $\boldsymbol{\xi}_i$ as the dual coordinates on $\mathbb{P}^{n_i}$, i.e. the coordinates of hyperplanes on $\mathbb{P}^{n_i}$. For $j = 0, \ldots, t$, let s_j be a global section of $O_{\mathbb{P}}(e_j)$. Its coordinate vector is denoted by $\mathbf{s}_j$. Then $F_Z(\mathbf{s}_0, \ldots, \mathbf{s}_t) = 0$ if and only if $\operatorname{div}(s_0), \ldots, \operatorname{div}(s_t), Z$ have non-empty intersection. This defines the Chow form for prime cycles. By linearity, the definition is extended to all cycles. We denote by $|F_Z|$ the Gauss-norm of F_Z.

The standard metric $\| \ \|$ on $O_{\mathbb{P}^n}(1)$ is defined in the following way. Let $x \in \mathbb{P}^n$ with coordinates $x_0, \ldots, x_n$ and let s be a global section of $O_{\mathbb{P}^n}(1)$. Then we define

$$\|s(x)\| := |s(x)| / \max_j |x_j| \,.$$

By pull back, we get a formal metric on $O_{\mathbb{P}}(e_i)$ called the standard metric. The same proof as for the case of discrete valuations ([Gu 2], Proposition 1.12) shows the following result:

Proposition 9.7. *If* Z, $\operatorname{div}(s_0), \ldots, \operatorname{div}(s_t)$ *have empty intersection in* $\mathbb{P}$, *then the local height of* Z *with respect to* $\big(O_{\mathbb{P}}(e_j), \| \ \|, s_j\big)_{j=0,\ldots,t}$ *is equal to*

$$\log|F_Z| - \log|F_Z(\mathbf{s}_0, \ldots, \mathbf{s}_t)| \,.$$

Remark 9.8. For an irreducible and reduced algebraic variety X over K of dimension t, let $\mathfrak{g}_X^+$ be the set of isometry classes of base-point-free line bundles with roots of positive formal metrics. Together with the local heights introduced in Definition 2, the propositions above show that $(\mathfrak{g}^+, \lambda)$ is a theory of local heights for t-dimensional varieties in the sense of [Gu 2], Definition 1.8.

Appendix. Coherent sheaves

Let K be a field with a non-archimedean non-trivial complete absolute value and valuation ring K°. By using the techniques of [BL 3], section 1, we generalize [EGA I], 10.10 to admissible formal affine schemes over K°. By π, we denote a non-zero element in K° which is not a unit.

A.1. Let A be an admissible K°-algebra and $\mathfrak{X} = \mathrm{Spf} A$. We assume that the reader is familiar with the notion of coherent $\mathcal{O}_{\mathfrak{X}}$-modules ([EGA I], Chap. 0, 5.3). For an A-module M, we have an $\mathcal{O}_{\mathfrak{X}}$-module $M^\triangle$ ([EGA I], 10.10.1), given by completing the $\mathcal{O}_{\mathrm{Spec}\, A}$-module associated to M along $\mathrm{Spec}(A/\pi A)$. A finitely generated A-module M is called coherent if any finitely generated submodule of M is of finite presentation. By [BL 3], Proposition 1.3, A is coherent. Moreover, an A-module is coherent if and only if it is of finite presentation.

Theorem A.2. *The covariant functor $M \to M^\triangle$ is an equivalence of the category of coherent A-modules onto the category of coherent $\mathcal{O}_{\mathfrak{X}}$-modules.*

For the proof, we need the following results:

Lemma A.3. *If N is a submodule of a finitely generated A-module M, then the π-adic topology of M induces the π-adic topology of N.*

Proof. Given $n \in \mathbb{N}$, we have to show the existence of $\lambda \in \mathbb{N}$ with

$$(\pi^\lambda M) \cap N \subset \pi^n N .$$

Let $\varphi : F \to M$ be a surjective homomorphism of a free A-module of finite rank onto M. By [BL 3], Lemma 1.2, there is a $\lambda \in \mathbb{N}$ with

$$(\pi^\lambda F) \cap \varphi^{-1}(N) \subset \pi^n \varphi^{-1}(N) .$$

Applying φ, we get the claim. □

Lemma A.4. *For an A-module M and $f \in A$, let $M_{\{f\}}$ be the completion of the localization M_f with respect to the π-adic topology. Then $M \to M_{\{f\}}$ gives an exact functor from the category of finitely generated A-modules into the category of finitely generated $A_{\{f\}}$-modules.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated A-modules. After localization in f, the sequence remains exact. For $\lambda \in \mathbb{N}$, the sequence

$$0 \to M'_f/(M'_f \cap \pi^\lambda M_f) \to M_f/\pi^\lambda M_f \to M''_f/\pi^\lambda M''_f \to 0$$

is also exact and this remains true if we take projective limits with respect to λ ([EGA I], Chap. 0, Lemma 7.2.8), i.e.

$$0 \to \varprojlim_{\lambda} M'_f/(M'_f \cap \pi^\lambda M_f) \to M_{\{f\}} \to M''_{\{f\}} \to 0$$

is exact. Here, we have used the description of the completion as a projective limit ([EGA I], Chap. 0, Proposition 7.2.7). By Lemma 3, we easily deduce that

$$0 \to M'_{\{f\}} \to M_{\{f\}} \to M''_{\{f\}} \to 0$$

is exact. This proves the claim. □

Corollary A.5. *The functor $M \to M^{\triangle}$ is an exact functor from the category of finitely generated A-modules into the category of $\mathcal{O}_{\mathfrak{X}}$-modules of finite type.*

Lemma A.6. *If M, N are coherent A-modules, then we have natural isomorphisms*:

(i) $\Gamma(\mathfrak{X}, M^{\triangle}) \cong M$,

(ii) $(M \otimes_A N)^{\triangle} \cong M^{\triangle} \otimes_A N^{\triangle}$,

(iii) $(\mathrm{Hom}_A(M,N))^{\triangle} \cong \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(M^{\triangle}, N^{\triangle})$,

(iv) $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{\mathcal{O}_X}(M^{\triangle}, N^{\triangle})$.

Proof. By definition, $\Gamma(\mathfrak{X}, M^{\triangle})$ is the π-adic completion of M. Since M is finitely generated, M is complete. By the proof of [BL3], Lemma 1.6, M is π-adically separated. This proves (i).

For (ii), we have to show

$$M_{\{f\}} \otimes_{A_{\{f\}}} N_{\{f\}} \cong (M \otimes_A N)_{\{f\}}$$

for all $f \in A$. This follows from

$$M_{\{f\}} \cong M \otimes_A A_{\{f\}}.$$

To prove the latter, note that M is of finite presentation. Since the tensor product is right exact and the functor $M \to M_{\{f\}}$ is exact (Lemma 4), it is enough to consider a free A-module M of finite rank. Finally, we reduce to $M = A$ where the claim is obvious. This proves (ii).

Moreover, the above isomorphism and Lemma 4 imply that $A_{\{f\}}$ is a flat A-module. By [EGA I], Chap. 0, 5.7.6, we get (iii). Finally, (iv) follows from (i) and (iii). □

Proof of Theorem A.2. First, we prove that $M^{\triangle}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module if M is a coherent A-module. Let $\mathcal{U}$ be a formal open subset of $\mathfrak{X}$ and let $\psi : \mathcal{O}_{\mathfrak{X}}^n|_{\mathcal{U}} \to M^{\triangle}|_{\mathcal{U}}$ be a homomorphism. We have to show that the kernel is of finite type. We may assume that $\mathcal{U} = \mathrm{Spf} A_{\{f\}}$ for some $f \in A$. By Lemma 6, the homomorphism is induced by a homomorphism $\varphi : A_{\{f\}}^n \to M_{\{f\}}$. By [BL3], Proposition 1.7, $A_{\{f\}}$ is an admissible K°-algebra. Since $M_{\{f\}}$ is an $A_{\{f\}}$-module of finite presentation (use Lemma 4), it is coherent. Therefore the kernel of φ is coherent ([EGA I], Chap. 0, Proposition 5.3.2). By Corollary 5, $\ker \psi$ is of finite type.

It remains to show that any coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is isomorphic to $M^{\triangle}$ for a suitable coherent A-module M. For $n \in \mathbb{N}$, let $\mathfrak{X}_n := \mathrm{Spec}\, A/\pi^{n+1}A$. Using the above, we see that $\mathcal{O}_{X_n}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module. It follows that $\mathcal{F}_n := \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module

([EGA I], Chap. 0, Corollaire 5.3.7), and hence it is a coherent $\mathcal{O}_{\mathfrak{X}_n}$-module ([EGA I], Chap. 0, Proposition 5.3.13). We conclude that $\mathcal{F}_n$ is the coherent sheaf associated to $M_n := \mathcal{F}_n(\mathfrak{X}_n)$ ([EGA I], Théorème 1.4.1). By [EGA I], Corollaire 1.4.3, M_n is a coherent $A_n := A/\pi^{n+1}$ A-module. The modules M_n form a projective system and their projective limit M is a finitely generated A-module with $M \otimes_A A_n \cong M_n$ ([Bou], Chap. III, § 2, no. 11, Proposition 14). Let $F \to M$ be an epimorphism of a free A-module F of finite rank onto M. We have to show that the kernel G is finitely generated. We have an exact sequence

$$0 \to G/(G \cap \pi^{n+1}F) \to F/\pi^{n+1}F \to M_n \to 0\,.$$

Since $F/\pi^{n+1}F$ and M_n are coherent A_n-modules, $G/(G \cap \pi^{n+1}F)$ is a coherent A_n-module. Taking projective limits, we get an exact sequence

$$0 \to \hat{G} \to F \to M \to 0$$

where $\hat{G}$ is the π-adic completion of G (Lemma 3, [EGA I], Chap. 0, Lemma 7.2.8). It follows that $G = \hat{G}$ is finitely generated ([Bou], Chap. III, § 2, no. 11, Proposition 14). This proves that M is coherent. We have a canonical homomorphism $\mathcal{F} \to M^{\triangle}$ since $M^{\triangle}$ is the projective limit of the $\mathcal{F}_n$'s. Locally on an open subset $\mathcal{U} = \operatorname{Spf} A_{\{f\}}$ of $\mathfrak{X}$, $\mathcal{F}$ is the cokernel of a homomorphism $\mathcal{O}_{\mathcal{U}}^m \to \mathcal{O}_{\mathcal{U}}^n$. Let $M_{\mathcal{U}}$ be the cokernel of the corresponding homomorphism $A_{\{f\}}^m \to A_{\{f\}}^n$. By Corollary 5, we have $(M_{\mathcal{U}})^{\triangle} \cong \mathcal{F}|_{\mathcal{U}}$. Modulo powers of π, $M_{\mathcal{U}}$ is isomorphic to $M_{\{f\}}$. Since $M_{\mathcal{U}}$ is complete and separated with respect to the π-adic topology (cf. Lemma 6 (i)), we conclude $M_{\mathcal{U}} \cong M_{\{f\}}$. Therefore $\mathcal{F} \to M^{\triangle}$ is an isomorphism. This proves the claim. □

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