Basic Properties of Heights of Subvarieties

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Chapter 1

Introduction

1.1 Preface

In diophantine geometry, one of the most important problems is to control the number of rational points. A basic tool to prove finiteness statements or to describe the distribution of infinitely many points is the height. It measures the arithmetic complexity of the coordinates of a rational point. For example, let P be a $\mathbb{Q}$-rational point of projective space $\mathbb{P}^n$, i.e. it is given by homogeneous coordinates $[x_0 : \cdots : x_n]$. Clearly, we may assume that the coordinates x_i are integers and coprime. Then the height of P is given by

$$h(P) := \log \max_{i=0,\ldots,n} |x_i|.$$

As an illustration, we note the following properties:

(P) The height is non-negative.

(N) There are only finitely many points of bounded height.

(W) If we change the coordinates, then the height changes by a bounded function on $\mathbb{P}^n$.

A non-trivial absolute value on $\mathbb{Q}$ is up to a power equal to the usual absolute value $|\ |_\infty = |\ |$ or equal to a p-adic absolute value $|\ |_p$ for a prime number p. By the prime factorization theorem, we have, for a non-zero rational number α, the important product formula

$$\prod_v |\alpha|_v = 1$$

where v ranges over all prime numbers or ∞. It follows easily that

$$h(P) = \sum_v \log \max_{i=0,\ldots,n} |x_i|_v = -\sum_v \log \|\ell(\mathbf{x})\|_v \tag{1.1}$$

for any linear form ℓ with $\ell(\mathbf{x}) \neq 0$, where

$$\|\ell(\mathbf{x})\|_v = \frac{|\ell(\mathbf{x})|_v}{\max_{i=0,\ldots,n} |x_i|_v}. \tag{1.2}$$

Now the formula (1.1) may be generalized to a point of $\mathbb{P}^n$ with coordinates in a number field K, i.e. in a finite dimensional field extension of $\mathbb{Q}$. We just let v vary over all absolute values of K extending the ones of $\mathbb{Q}$ considered above. However, the absolute values on K have to be normalized in such a way that the product formula still holds and that the height of P defined by (1.1) is independent of the choice of K. We get a height of points in $\mathbb{P}^n$ with coordinates in the algebraic closure $\bar{\mathbb{Q}}$.

Property (P) is still true. The generalization of (N) is Northcott's theorem [No] that there are only finitely many $\bar{\mathbb{Q}}$-rational points of $\mathbb{P}^n$ with bounded height and coordinates of bounded degree. For (W), one could ask the following more general question. Let X be a projective variety over $\bar{\mathbb{Q}}$, i.e. X is a zero set of homogeneous polynomials in the coordinates. Thus we may see X as a closed subset of $\mathbb{P}^n$ and we get a height of every $\bar{\mathbb{Q}}$-rational point $P \in X$. What happens with the height if we vary the embedding? The answer to this question was given by Weil [We2] proving that the height up to bounded functions on X depends only on the isomorphism class of the line bundle $O_{\mathbb{P}^n}(1)|_X$. This important generalization of (W) is called Weil's theorem and implies immediately that there is a canonical homomorphism from the Picard group to the space of real functions modulo bounded functions. The Picard group is the set of isomorphism classes of line bundles on X and is an important invariant in algebraic geometry denoted by $\mathrm{Pic}(X)$. Thus any relation in $\mathrm{Pic}(X)$ gives a corresponding relation for heights.

This principle may be used on abelian varieties. They are beautiful objects in algebraic geometry simply defined as projective group varieties. As the name already indicates, one can prove that they are commutative. Moreover, there is an important relation in the Picard group called the theorem of the cube. For details of any result used about abelian varieties, the reader is referred to the fundamental book of Mumford [Mu]. This relation was used by Néron [Ne1] to show that on an abelian variety A, one can associate to every $L \in \mathrm{Pic}(A)$ a canonical height function $\hat{h}_L$ called the Néron-Tate height. The point is that it is really canonical as a function and not only up to bounded functions as predicted by Weil's theorem. It is characterized by the fact that $\hat{h}_L$ is a quadratic function, i.e.

$$\hat{b}(x,y) = \hat{h}_L(x+y) - \hat{h}_L(x) - \hat{h}_L(y)$$

is a bilinear function on A. For definition, one can use Tate's limit argument

$$\hat{h}_L(x) := \lim_{n\to\infty} 2^{-n^2}\hat{h}_L(2^n x)$$

if L is even, and for L odd, one has to replace 2^{-n^2} by 2^{-n}.

As an illustration for the use of heights, we mention Faltings' theorem [Fa1] proving that a projective curve C of genus $g \geq 2$ over a number field K has at most finitely many K-rational points. It was conjectured by Mordell in his paper [Mord] showing that the $\mathbb{Q}$-rational points of an elliptic curve over $\mathbb{Q}$ form a finitely generated group. The latter is called the Mordell-Weil theorem and was generalized by Weil [We1] to abelian varieties over any number field. Already its proof relies on the theory of heights using it for the descent. Faltings' first step was the proof of Tate conjecture that the Tate module of an abelian variety over K with respect to a prime ℓ is a semisimple $\mathrm{Gal}(\bar{\mathrm{K}}/\mathrm{K})$-module and that its endomorphism ring is isomorphic to $\mathrm{End}_{\mathrm{K}}(\mathrm{A}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. The proof is based on heights on the moduli space of abelian varieties using generalizations of Northcott's theorem and Weil's theorem above. As a consequence, Faltings proved that there are only finitely many isogeny classes of abelian varieties over K of given dimension with good reduction outside a given set S of places. An isogeny is a homomorphism of abelian varieties with finite kernel and cokernel. As a consequence, Faltings proved the Shafarevich conjecture. It states that the finiteness holds also for the number of isomorphism classes of polarized abelian varieties of given polarization degree and given dimension with good reduction outside S.

The proof ot this step relies on similar height arguments as for the Tate conjecture. Masser-Wüstholz [MW], in a completely new approach using transcendence methods based on Baker's method of linear forms in logarithms, made it effective. This means that the isogeny $\varphi : A \to B$

may be replaced by an isogeny of degree at most $c\max(1,h)^k$, where h is the minimum of the stable Faltings heights of A, B on the moduli space and c, k are effectively computable constants. Note that heights are indispensable for effectiveness statements in diophantine geometry because to have explicit bounds for the heights just means effectivity. By an argument of Parshin, the Shafarevich conjecture implies the Mordell conjecture finishing Faltings' proof.

Vojta [Vo2] gave a different proof of Faltings' theorem based on diophantine approximation. A crucial result there is Roth's theorem [Ro] given here in Lang's general formulation. For $\kappa > 2$, a finite set S of normalized absolute values of a number field K and $\alpha_v \in \overline{K_v}$, there are only finitely many $\beta \in K$ with

$$\sum_{v \in S} \log \min(1, |\alpha_v - \beta|_v) \leq -\kappa h(\beta) \tag{1.3}$$

where the height of β is equal to the height of the corresponding point $[1:\beta] \in \mathbb{P}^1$ and $\overline{K_v}$ is the completion of K with respect to v. Vojta's main idea was to consider certain divisors $V := d_1 P_0 \times C + d_2 C \times P_0 - d\Delta$ on $C \times C$ where P_0 is a fixed K-rational point, Δ is the diagonal and d, d_1, d_2 are suitable parameters such that the height $h_{O(V)}(P,Q)$ is an indefinite quadratic form on the K-rational points of the Jacobian. Then using local calculations of heights and using the arithmetic Riemann-Roch theorem, he deduced a lower bound for $h_{O(V)}(P,Q)$. Together with Northcott's theorem and Mumford's gap principle (saying that the height of large points on C increase rather quickly), Vojta showed Faltings' theorem. Bombieri [Bo] eliminated the use of the difficult arithmetic Riemann Roch theorem by an application of Siegel's lemma and the classical Riemann-Roch for surfaces from algebraic geometry. If the reader would like to have an impression of the powerful applications of heights, he should consult this elementary proof of Faltings' theorem.

It was pointed out by Weil that most of the properties of heights are coming from local heights. For example in (1.1), the height of P is the sum of the local heights $-\log \|l(\mathbf{x})\|_v$. Here, local means that we fix an absolute value $|\ |_v$ on K and we work over the corresponding completion K_v. Its best to understand local heights in terms of metrics, that is the point of view of Arakelov theory [Ar] picked up by Faltings in the proof of Mordell conjecture. Let L be a line bundle on a projective variety X over K endowed with a continuous metric $\|\ \|_v$ and let s be a meromorphic section of L. Then the local height of a K_v-rational point of X is given by

$$\lambda_{\mathrm{div}(s)}(P) = -\log \|s(P)\|_v.$$

For example in (1.1), one may consider $\ell(\mathbf{x})$ as a global section of $O_{\mathbb{P}^n}(1)$ and the local height is with respect to the standard metric (1.2) on $O_{\mathbb{P}^n}(1)$. Note that the local height depends on the meromorphic section s. We have seen in (1.1) that the product formula gives independence of $h(P)$ on s. The local height is well-defined and continuous outside of the support of D along which it has a logarithmic singularity. The local height has the following characteristic properties:

a) It is linear in the divisor $D = \mathrm{div}(s)$ (more precisely in the variable $(L, \|\ \|, s)$).

b) If $\varphi : X' \to X$ is a morphism of projective varieties such that $\varphi(P')$ is not contained in the support of D, then
$$\lambda_D(\varphi(P')) = \lambda_{\varphi^* D}(P')$$
using the pull-back metric on the right hand side.

c) If $L = O_X$ with the trivial metric and f is a rational function, then
$$\lambda_{\mathrm{div}(f)}(P) = -\log |f(P)|_v.$$

d) If we change the metric on L, then the difference of local heights is a bounded function on X.

e) On projective space with respect to the standard metric (1.2) on $O_{\mathbb{P}^n}(1)$, the local height is given by $\lambda_{\mathrm{div}(\ell)}(P) = -\log \|\ell(\mathbf{x})\|_{\mathbf{v}}$ in (1.1).

Using this notion, Roth's theorem may be reformulated in the following way: There are only finitely many K-rational points P in $\mathbb{P}^1$ with

$$\sum_{v \in S} \lambda_{\alpha_v}(P) \geq \kappa h(P).$$

This stresses already the use of local heights in diophantine approximation, a local height with respect to an effective divisor may be seen as the inverse of the distance to the divisor. Néron [Ne2] has also noticed that on an abelian variety, there are canonical local heights, determined by the divisor up to an additive constant. By product formula, the ambiguity disappears if the local heights are summed up and he obtained the Néron-Tate height. By a linear extension to all 0-dimensional cycles, Néron's local heights are really canonical functions if they are restricted to cycles of degree 0. By functoriality, Néron also obtained a canonical local height for zero-dimensional cycles of degree 0 with respect to divisors algebraically equivalent to 0 called the Néron symbol.

The height of a polynomial is defined to be the height of the vector of coefficients in suitable projective space. Next, one could ask for the height of a subvariety of $\mathbb{P}^n$ defined over K. It should measure the heights of equations defining Y. However, as the set of equations is not unique, one should give a more intrinsic definition. The first definition was given by Nesterenko and Philippon (cf. [Ph1]). They define the height of Y as the height of the Chow form. The latter is a multihomogeneous polynomial of multidegree $(\deg(Y), \dots, \deg(Y))$ associated to Y. This height of subvarieties was first applied to prove transcendence results.

Faltings [Fa3] picked up Vojta's approach to prove a higher dimensional generalization of his theorem: If Y is a subvariety of an abelian variety over K such that the K-rational points are dense in K, then Y is a translation of an abelian subvariety. In his paper, he introduced a new definition of heights of subvarieties using the arithmetic intersection theory of Gillet-Soulé [GS2]. Faltings' height for a t-dimensional subvariety Y of $\mathbb{P}^n$ is the arithmetic analogue of the degree in algebraic geometry, it is defined as an arithmetic intersection number with $t+1$ arithmetic divisors $\widehat{\mathrm{div}}(s_i)$ for generic global sections s_i of $O_{\mathbb{P}^n}(1)$ and with respect to the Fubiny-Study metric on $O_{\mathbb{P}^n}(1)$. This definition is very fruitful as we may apply it on every arithmetic variety with respect ot $t+1$ hermitian line bundles and it has the right functorial properties.

There are the following generalizations of properties (P),(N),(W) to heights of subvarieties. It was noticed by Faltings that $h(Y) \geq 0$. It is also clear by Philippon's approach that there are only finitely many subvarieties of bounded height and bounded degree defined over a number field of bounded degree. Weil's theorem was generalized to heights of subvarieties in [Gu1]. It says that the height $h(Y)$ is determined by the isomorphism class of $\varphi^* O_{\mathbb{P}^n}(1)|_Y$ up to functions bounded by a multiple of the degree. It was proved by Soulé [So] that the heights of Faltings and Philippon are equivalent in this sense. Using the generalization of Weil's theorem and Tate's limit argument, Néron-Tate heights for subvarieties of abelian varieties were obtained in [Gu1] with respect to even line bundles. In the case of one very ample even line bundle giving rise to a projective normal embedding, they were already introduced by Philippon [Ph2]. For an even ample line bundle, a moduli theoretic argument was given by Kramer (cf. appendix of [Gu1]).

The analogue to the degree is really striking, Gillet-Soulé [GS1] proved an arithmetic Hilbert-Samuel formula and Bost-Gillet-Soulé [BoGS] gave an arithmetic Bézout theorem for the height of subvarieties. The former is a special case of the arithmetic Riemann-Roch theorem ([Fa4], [GS3] already used in Vojta's proof of Faltings' theorem. Zhang [Zh3] used the arithmetic Hilbert-Samuel formula to prove the Bogomolov conjecture. The theorem of Zhang says that for a subvariety Y of an abelian variety A which is not a translate of an abelian subvariety by a torsion point, there is a constant $\epsilon > 0$ such that the $\bar{K}$-rational points of Y of Néron-Tate height smaller than ϵ doesn't form a dense subset of X. This was originally conjectured by Bogomolov for a curve embedded in an abelian variety. This special case was proved by Ullmo [Ullm]. The proof of Zhang is an outstanding application of heights of subvarieties. In particular, he characterizes the translates of abelian subvarieties by torsion points as the subvarieties of A of Néron-Tate height 0. The theorem of Zhang was generalized by Moriwaki [Mori] from number fields to finitely generated fields over $\mathbb{Q}$.

Using Faltings' definition, it becomes clear how to define the local height $\lambda(Y)$ of a t-dimensional subvariety Y. In the archimedean case, $\lambda(Y)$ is given by integrating the $*$-product of $t+1$ Green currents of the form $[\log \|s_j\|^{-2}]$ over Y. In the finite case, the local height is an intersection number of Y and $t+1$ Cartier divisors $\mathrm{div}(s_j)$ on a model over the corresponding discrete valuation ring. In [Gu2], a systematic study of local heights of subvarieties was done. It was shown that they are characterized by 5 properties generalizing a)-e) above. For a non-archimedean absolute value on any field K, their existence was shown in [Gu3]. Note that the valuation ring K° in this general case hasn't to be a noetherian ring. In fact, noetherian is equivalent to have a discrete valuation. Hence the intersection theory on models over K° from algebraic geometry is not available, the multiplicities of Cartier divisors in irreducible components of the special fibre would be infinite. To bypass this problem, one has to work on admissible formal K°-models $\mathfrak{X}$ using the theory of Bosch-Lütkebohmert (cf. [BL3], [BL4]) initiated by Raynaud. Using valuations, an intersection theory with Cartier divisors on $\mathfrak{X}$ was given in [Gu3] leading to the desired local heights in the non-archimedean case.

Classically, one has considered heights of points over number fields or function fields. In the basic book of Lang [La] about diophantine geometry, he considers fields with families of absolute values $|\ |_v$ such that for non-zero x, we have $|x|_v \neq 1$ only for finitely many places v and such that the product formula holds. It was pointed out by Vojta [Vo1] that there are striking similarities of the height of points to the characteristic function in Nevanlinna theory, for example Weil's theorem corresponds to the first main theorem in Nevanlinna theory and Roth's theorem is similar to the second main theorem. This leads to far reaching conjectures for number fields and in Nevanlinna theory.

In [Gu2], the notion of M-field was introduced. It is a field K with a positive measure μ on a set M and a family $(|\ |_v)_{v\in M}$ satisfying μ-ae the axioms of absolute values such that $\log|x|_v$ is integrable on M for non-zero $x \in K$. M-fields include the fields considered by Lang, in particular number fields, function fields and even their algebraic closures, the examples arising from Nevanlinna theory and also the finitely generated fields over $\mathbb{Q}$ considered by Moriwaki in his generalization of Zhang's theorem. It was shown in [Gu2] how to construct global heights of subvarieties by integrating local heights over M. Moreover, Weil's theorem was deduced for heights of subvarieties over M-fields leading to a generalization of the first main theorem of Nevanlinna theory to higher dimensions.

In the present article, a first goal is to define local heights in a more general situation than before. In the complex case, the $*$-product of Green currents is only well-defined if the underlying cycles intersect properly, thus the local heights were defined for subvarieties Y such that all possible partial intersections formed with subsets of $\{\mathrm{div}(s_0), \dots, \mathrm{div}(s_t), Y\}$ are proper. Developping a refined $*$-product with currents of the form $[\log \|s_j\|^{-2}]$, a local height for Y is obtained under the less restrictive assumption on the supports

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap Y = \emptyset. \tag{1.4}$$

A glance at the Chow form shows that this is the correct assumption. The refined $*$-product in the style of Fulton does not require any regularity assumption on the underlying proper algebraic variety X. It may be seen as a first step towards a refined intersection theory of arithmetic Chow groups which is currently not available.

For a non-trivial non-archimedean complete absolute value on a complete algebraically closed field K with valuation ring K°, generically proper intersections with Cartier divisors on admissible formal K°-models were studied in [Gu3]. Here, it is shown that if the generic fibre is algebraic then one obtains a refined intersection theory leading to the same conclusions for local heights as in the complex case. In either case, the algebraicity assumption on X is just made for simplicity. The methods for the refined $*$-product (resp. refined intersection product) work on any complex space (resp. rigid analytic variety) X using line bundles with enough meromorphic sections.

The main part of the article is dedicated to develop a theory of canonical local heigths having similar properties as in the zero-dimensional case. There are two cases where they occur, first if one line bundle is algebraically equivalent to 0 and second if one deals with abelian varieties. For an archimedean absolute value, the corresponding canonical metrics are the smooth hermitian metrics with harmonic curvature form (zero in the first case because the bundle is homological trivial), thus nothing new happens and the canonical local heights are just given by the $*$-product.

The delicate point is for a non-archimedean complete absolute value v. If the valuation ring K° is discrete, then the first case is a special case of the height pairing of Beilinson [Bei] and Bloch [Bl]. The height pairing was studied under the assumption of standard conjectures and relations to the Birch-Swinnerton-Dyer conjecture were given, but for divisors these assumptions are not necessary. However, to extend the divisors correspondingly to the canonical metrics, a regular model has to be required. But here, we don't want to suppose that the valuation is discrete and no regularity is assumed either. The idea is to develop a local Chow cohomology theory on admissible formal models over K°. These groups formalize the refined intersection theoretic properties of Chern classes. To get rid of particular K°-models, the projective limit over all admissible formal K°-models is considered giving rise to a local Arakelov-Chow group in the style of non-archimedean Arakelov theory of Bloch-Gillet-Soulé [BlGS] and corresponding local Arakelov-Chow cohomology groups. Then every line bundle L on X takes admissible first Chern classes in the Arakelov-Chow cohomology groups which may be seen as cohomological generalizations of K°-models of L. They give rise to an associated metric on L called a cohomological metric. A theory of local heights with respect to such admissible first Arakelov-Chern classes is given and it is shown that the dependence of local heights on the admissible first Arakelov-Chern classes is given by the associated metrics. Since a canonical metric on a line bundle algebraically equivalent to 0 is a cohomological metric (but one may not expect that it is associated to a formal K°-model of L if the valuation is not discrete), we get canonical local heights in the first case. It doesn't depend on the metrics of the other line bundles and is closely related to the Néron symbol.

In the second case of an abelian variety, we use Tate's limit argument to define canonical local heights of subvarieties with respect to all line bundles. They agree with the local heights considered in the first case if one line bundle is algebraically equivalent to 0 (i.e. odd).

Finally, global heights of subvarieties Y over an M-field K are defined under the condition (1.4). They are obtained by integrating local heights of Y over M and properties a)-e) carry over. If the product formula is satisfied, then the global heights do not depend on the choice of meromorphic sections $s_0, \ldots, s_t$. If one line bundle is algebraically equivalent to zero or if the underlying space is an abelian variety, then we get global canonical heights including all previously considered cases.

The article is organized as follows: First, we recall the basics of refined intersection theory with divisors on a scheme of finite type over K. This is well-known and borrowed from the book of Fulton [Fu], but it fixes notation and may be seen as a guide line for the constructions in the following chapters.

In chapter one, we handle the complex part of the theory. In section 1, the $*$-product of Green forms from Burgos [Bu] is extended to arbitrary complex spaces. The difference to Gillet-Soulé's original approach [GS2] is that Burgos was working completely inside the world of differential forms with logarithmic singularities. His underlying spaces were smooth projective varieties suitable to apply Hodge theory. As we are only interested in the case of divisors, we need not to bother about logarithmic forms or existence of Green forms.

This $*$-product is used in section 2 to define local heights for compact analytic cycles on complex spaces and it is a special case of arbitrary intersection numbers with respect to the $*$-product.

In section 3, we restrict to the case of a complex compact algebraic variety X. Now we pick up Gillet-Soulé's point of view of Green currents. We develop a refined $*$-product in the case $-\log \|s\|^{-2} * g_Z$ for an invertible meromorphic section s of a hermitian line bundle and for a Green current g_Z of a cycle Z on X. Since we consider only divisoral operations, no smoothness assumption is required.

In section 4, the refined $*$-product of section 3 is used to deduce a theory of local heights of subvarieties on a compact complex algebraic variety independently of sections 1 and 2. The Burgos approach is more general working on arbitrary complex spaces and the proofs are formally easier, but the classical approach has the advantage that it leads to a refined $*$-product in the case of divisors.

In chapter 2, we pass to the non-archimedean case of a complete absolute value on our ground field K with valuation ring K°. In sections 1-3, we recall the properties of generically proper intersection theory with Cartier divisors on rigid analytic varieties, formal analytic varieties and admissible formal schemes over K°. For proofs, we refer to [Gu3], new is only the flat pull-back rule at the end of section 3. For simplicity, we will always assume that K is algebraically closed to get a bijective correspondence between reduced formal analytic varieties and admissible formal schemes over K° with reduced special fibres. This is important for defining the multiplicities of a Cartier divisor in the components of the special fibre. The general case may be deduced by base change.

In section 4, we take the point of view of non-archimedean Arakelov theory of Bloch-Gillet-Soulé. For given rigid analytic generic fibre X over K, we consider all possible admissible formal K°-models, X is always assumed to be quasi-compact and quasi-separated. They form a partially ordered set where $\mathfrak{X}' \geq \mathfrak{X}$ if the identity on generic fibres extends to a morphism $\mathfrak{X}' \to \mathfrak{X}$. Using the theory of admissible formal schemes of Bosch, Lütkebohmert and Raynaud, which is basic for all our considerations in the non-archimedean case, one knows that the K°-models form a directed set and that many properties of rigid analytic varieties may be extended to sufficiently large K°-models. In the style of non-archimedean Arakelov theory of Bloch-Gillet-Soulé, we use the projective limit over all K°-models to define the Arakelov-cycle group of X getting a proper intersection theory with Arakelov divisors from section 3.

For a proper algebraic variety over K, we define in section 4 local Arakelov-Chow groups in the same spirit and we deduce a refined intersection theory with Arakelov-Cartier divisors. Local means with support, a notion which is basic for our purposes.

Obviously, we get also a refined intersection theory with Cartier divisors on individual K°-models. This is the content of section 5. In contrast to the local Arakelov-Chow groups, we have flat pull-back inside the local Chow groups of the K°-models.

To study canonical local heights, we have to introduce local Chow cohomology groups. This is the content of chapter 4. In section 1, we recall Fulton's construction [Fu] of Chow cohomology in algebraic geometry but we restrict to the case of proper varieties which are the main objects of interest for us.

In section 2, we introduce local Chow cohomology groups on admissible formal K°-models of a proper algebraic variety over K. They formalize the properties of Chern classes developped in section 3.6. Moreover, we get also local Arakelov-Chow cohomology groups acting on the corresponding local Arakelov-Chow groups.

In section 3, we take the point of view of metrics. For coefficients of vertical cycles, we allow a subfield R of the reals containing the value group. Every formal K°-model of a line bundle L gives rise to a so-called formal metric on L determing completely the divisoral operation on the local Arakelov-Chow groups. An admissible metric is locally equal to a tensor product of R-powers of a formal metric. We describe admissible metrics explicitly on semistable curves over K°. They correspond to continuous, piecewise linear functions on the intersection graph completely analogous to Chinburg-Rumely [CR] and Zhang [Zh1]. For formal-analytic varieties, we can associate a Weil-divisor to an invertible meromorphic section s of L endowed with an admissible metric and we get similar results as in section 3.3. However, we can not extend the operation of $\widehat{\mathrm{div}(s)}$ to vertical cycles. This is only possible for a semistable K°-model of a smooth projective curve C and this operation gives rise to a local Chow-cohomology class. This theorem is the heart of this section. As a consequence, we can prove a local Hodge index theorem on the semistable model. Moreover, we conclude that the a canonical metric on a line bundle of degree 0 on C is admissible.

In section 4, we define admissible first Arakelov-Chern classes for line bundles in Arakelov-Chow cohomology. They generalize the divisoral operations of formally metrized line bundles on any proper scheme over K and also of admissibly metrized line bundles on curves considered in the previous section. They give rise to associated metrics on the line bundles. The main result is that for every line bundle L algebraically equivalent to 0, there is an admissible first Arakelov-Chern class which is orthogonal to vertical cycles. The associated metric is a canonical metric on L. Using a correspondence of L to a curve, this is deduced from the corresponding result on curves proved in the previous section.

We prove in section 5 that intersection numbers of admissible Arakelov-Chern classes give rise to a theory of local heights of subvarieties depending only on the associated metrics and with similar properties as in the archimedean case. Moreover, they include the canonical local heights if one line bundle is algebraically equivalent to 0.

In chapter 5, we are interested in global heights of subvarieties. However, in section 1, we first summarize the properties of local heights of subvarieties stressing the similarities between the archimedean and the non-archimedean case. Then we extend the theory of local heights allowing uniform limits of the semipositive metrics considered before. This has the advantage that on an abelian variety, all canonical metrics of line bundles are included and we get canonical local heights of subvarieties with respect to any line bundles. They are deduced as a special case from a dynamic set-up.

In section 2, we recall the notion of an M-field from [Gu2]. For a proper scheme over an M-field K, we define global height of subvarieties in section 3 by integrating the corresponding local heights over M. To get integrability, some uniformity in the choice of metrics is needed. This is certainly the case if we have a line bundle generated by global sections by choosing the pull-back of standard metrics (resp. Fubini-Study-metrics) from some $O_{\mathbb{P}^n}(1)$. It is also shown that the canonical local heights considered above are integrable over M and thus give rise to global heights. The properties of global heights are studied in detail. If the product formula is satisfied, then they do not depend on the choice of meromorphic sections and the canonical heights even depend only on the isomorphism class of the line bundles. On an abelian variety, they can be characterized by a homogenity property with respect to multiplication by $m \in \mathbb{Z}, |m| \geq 2$.

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1.2 Refined Intersection Theory for Cartier Divisors

In this section, we recall the basic definitions and results for intersection with divisors on schemes of finite type over a field K. For proofs, we refer to [Fu], §1-2. This section may be seen as a program for the following sections to generalize actions of Cartier divisors in several situations. It also fixes definitions and notions. We denote by X, X' schemes of finite type over K.

1.2.1 The group of *cycles* on X is the free abelian group generated by closed irreducible subsets of X. The elements of this canonical basis are called prime cycles. We write a cycle in the form

$$Z = \sum_Y n_Y Y$$

where Y ranges over all closed irreducible subsets of X and where the *multiplicity* $n_Y \in \mathbb{Z}$ is non-zero only for finitely many Y. The Y with non-zero multiplicity are called the *components* of Z and their union is the *support* of Z denoted by $|Z|$. A cycle has dimension r (resp. codimension p) if all components have dimension r (resp. codimension p). A cycle of codimension 1 is called a *Weil divisor* on X.

1.2.2 A *meromorphic function* f on X is locally given by $f = a/b$ where a, b are regular functions and b is not a zero-divisor. The meromorphic functions on X form a sheaf $\mathcal{M}_X$ of $\mathcal{O}_X$-algebras and the invertible elements are called *invertible meromorphic functions*. They are locally given by $f = a/b$ with both a, b not zero-divisors. If X is reduced, then this means that f does not vanish identically on any irreducible component of X. If X is integral, we write $K(X)$ for the field of meromorphic functions and the latter are called rational functions.

1.2.3 A line bundle L on X may be given by a trivialization $\{U_i\}_{i\in I}$ and transition functions g_{ij} on $U_j \cap U_j$. A section of L is given on the trivialization U_i by a regular function f_i and we have the characteristic rule

$$f_i = g_{ij} f_j \tag{1.5}$$

on $U_j \cap U_j$. A *meromorphic section* of L is given by meromorphic functions f_i on U_i satisfying (1.5). They may be seen as sections of L defined on open dense subsets of X. A meromorphic section is called *invertible* if all f_i are invertible meromorphic functions. If s is an invertible meromorphic section given by $\{U_i, f_i\}$, then $\{U_i, f_i^{-1}\}$ defines a meromorphic section of L^{-1} denoted by s^{-1}. If s and s' are meromorphic sections of line bundles L and L', we choose a common trivialization $\{U_i\}_{i\in I}$ such that s and s' are given by f_i, f_i' on U_i. Then $s \otimes s'$ is the meromorphic section on $L \otimes L'$ given by $f_i f_i'$ on U_i.

1.2.4 A *Cartier divisor* D on X is a global section of the sheaf $\mathcal{M}_X^\times / \mathcal{O}_X^\times$, i.e. D is given by the data $\{U_i, f_i\}_{i\in I}$ where $\{U_i\}_{i\in I}$ is an open covering of X, f_i is an invertible meromorphic function on U_i and f_i/f_j is an invertible regular function on $U_i \cap U_j$. To fix notation, $A^\times$ denotes always the ring of invertible elements in a ring A. It gives rise to a line bundle $O(D)$ on X given by the trivialization $\{U_i\}_{i\in I}$ and the transition functions $g_{ij} := f_i/f_j$ on $U_i \cap U_j$. Moreover, $O(D)$ has a canonical invertible meromorphic section s_D given by $\{U_i, f_i\}_{i\in I}$ with respect to this trivialization.

Conversely, every invertible meromorphic section s of a line bundle L is given by the data $\{U_i, f_i\}_{i\in I}$ with respect to this trivialization and it may be seen as a Cartier divisor on X

denoted by $\operatorname{div}(s)$. Obviously, we have $s = s_{\operatorname{div}(s)}$ and $\operatorname{div}(s_D) = D$. The tensor product of invertible meromorphic sections induces an addition on Cartier divisors.

Let $X \setminus |D|$ be the set of $x \in X$ where D is given in a neighbourhood of x by an invertible regular function. Then $|D|$ is a closed subset of X called the *support* of D.

1.2.5 Remark. The pull-back of a Cartier divisor D on X with respect to a morphism $\varphi : X' \to X$ is a subtle point which is handled in [EGA IV], 21.4. We say that the *pull-back* $\varphi^*(D)$ is well-defined if D may be given by data $\{U_i, a_i/b_i\}_{i\in I}$ where a_i, b_i are not zero-divisors on U_i and $a_i \circ \varphi, b_i \circ \varphi$ are not zero-divisors on $\varphi^{-1}U_i$ for all $i \in I$. Then we set $\varphi^*(D) = \{\varphi^{-1}U_i, a_i \circ \varphi / b_i \circ \varphi\}_{i\in I}$. If X is reduced, then $\varphi^* D$ is well-defined if and only if no irreducible component of X' is mapped into the support of D. If φ is flat, then the pull-back of Cartier divisors is always well-defined.

Similarly, we define the pull-back $s \circ \varphi$ of an invertible meromorphic section s of a line bundle L on X. We say that $\varphi^* s = s \circ \varphi$ is a well-defined invertible meromorphic section of $\varphi^* L$ if $\varphi^* D$ is well-defined. If s is given by local data $\{U_i, f_i\}_{i\in I}$, then $s \circ \varphi$ is given by local data $\{U_i, f_i \circ \varphi\}_{i\in I}$.

1.2.6 Let Y be a prime divisor on X. We denote by $\mathcal{O}_{X,Y}$ the local ring of $\mathcal{O}_X$ in the generic point of Y. If A is the coordinate of an affine open subvariety U of X with $U \cap Y \neq \emptyset$, then the ideal of vanishing of $Y \cap U$ in U is a prime ideal $\wp$ in A and $\mathcal{O}_{X,Y} \cong A_\wp$. The *order* of $a \in \mathcal{O}_{X,Y}^\times$ is the length of the artinian $\mathcal{O}_{X,Y}$-module $\mathcal{O}_{X,Y}/\mathcal{O}_{X,Y} a$ and is denoted by $\operatorname{ord}(a, Y)$. We have

$$\operatorname{ord}(ab, Y) = \operatorname{ord}(a, Y) + \operatorname{ord}(b, Y)$$

for $a, b \in \mathcal{O}_{X,Y}$. Let f be an invertible meromorphic function on U (or X) given by $f = a/b$ on U for non-zero divisors $a, b \in A$, then the order of f in Y is

$$\operatorname{ord}(f, Y) := \operatorname{ord}(a, Y) - \operatorname{ord}(b, Y).$$

For a Cartier divisor D on X, the order of D in Y is defined by $\operatorname{ord}(D, Y) := \operatorname{ord}(f, Y)$ if D is given on U by f. This definition does not depend on the choice of U and f. The *Weil divisor associated to* D is given by

$$\operatorname{cyc}(D) := \sum_Y \operatorname{ord}(D, Y) Y$$

where Y ranges over all prime divisors.

1.2.7 Remark. If the underlying scheme is not regular, then it is important to distinguish the notions of Cartier divisors and Weil divisors. For us, the point of view of Cartier divisors is much more important because it is equivalent to the concept of invertible meromorphic sections of line bundles. If X is normal, then the map $D \mapsto \operatorname{cyc}(D)$ from Cartier divisors to Weil divisors is injective and if X is local factorial, then the map is bijective ([EGA IV], Corollaire 21.6.10). If X is regular, then this is satisfied and we freely identify Cartier- and Weil divisors.

1.2.8 If f is an invertible meromorphic function on X, then f may be seen as an invertible meromorphic section of the trivial bundle O_X and hence gives rise to a Cartier divisor $\operatorname{div}(f)$. The group of K_1-*chains* on X is the direct sum of the multiplicative groups $K(W)^\times$ where W ranges over all irreducible closed subsets of X. We write a K_1-chain as a family $\mathbf{f} = (f_W)$ where f_W is a non-zero rational function and f_W is not constant 1 only for finitely many W. The divisor of a K_1-chain $\mathbf{f}$ is the cycle on X given by

$$\operatorname{div}(\mathbf{f}) := \sum_W \operatorname{cyc}\left(\operatorname{div}(F_W)\right).$$

The *Chow group* of X is the group of cycles on X divided by the subgroup

$$\{\operatorname{div}(\mathbf{f}) \mid \mathbf{f}\ K_1\text{-chain on } X\ \}.$$

Two cycles are called *rationally equivalent* if their difference is equal to $\mathrm{div}(\mathbf{f})$ for a K_1-chain $\mathbf{f}$.

1.2.9 Let $\varphi : X' \to X$ be a proper morphism of K-varieties. Let V be a prime cycle on X' with image $W := \varphi(V)$. If $\dim(V) = \dim(W)$, then $K(V)$ is a finite dimensional extension of $K(W)$ and we define

$$\varphi_*(V) := [K(V) : K(W)]W.$$

If $\dim(V) > \dim(W)$, then we set $\varphi_*(V) = 0$. For $f'_V \in K(V)^\times$, the norm gives

$$\varphi_*(f'_V) := N_{K(V)/K(W)}(f'_V) \in K(W)^\times$$

in the equi-dimensional case and $\varphi_*(f_V) := 1 \in K(W)^\times$ otherwise. By linearity, we extend the *push-forward* $\varphi_*(Z')$ (resp. $\varphi_*(\mathbf{f}')$) to all cycles Z' and K_1-chains $\mathbf{f}'$ on X'. We have the compatibility

$$\varphi_*\left(\mathrm{div}(\mathbf{f}')\right) = \mathrm{div}\left(\varphi_*(\mathbf{f}')\right).$$

Note that push-forward keeps the grading of cycles by dimension.

1.2.10 The *cycle associated to a closed subscheme* Y of X is

$$\mathrm{cyc}(Y) := \sum m_i Y_i$$

where Y_i ranges over all irreducible components of Y and the multiplicity m_i in Y_i is the length of the local artinian ring $\mathcal{O}_{Y,Y_i}$. Let $\varphi : X' \to X$ be a flat morphism of relative dimension $d \in \mathbb{N}$. Let Y be an irreducible closed subvariety of X. We endow $\varphi^{-1}(Y)$ with the inverse image structure, i.e. it is the closed subscheme of X' given by the sheaf of ideals $\varphi^{-1}(\mathcal{J}_Y)\mathcal{O}_{X'}$. Then

$$\varphi^*(Y) := \mathrm{cyc}\left(\varphi^{-1}(Y)\right).$$

By linearity, we extend the *pull-back* $\varphi^*(Z)$ to all cycles Z on X. If $\mathbf{f}$ is a K_1-chain on X, then $\varphi^*(\mathrm{div}(\mathbf{f})) = \mathrm{div}(\mathbf{f}')$ for some K_1-chain on X'. Note that pull-back keeps the grading of cycles by codimension. The dimension increases by d.

1.2.11 Proposition. *Let us consider the Cartesian diagram*

$$\begin{tikzcd} X_2' \arrow[r, "\psi'"] \arrow[d, "\varphi'"] & X_2 \arrow[d, "\varphi"] \\ X_1' \arrow[r, "\psi"] & X_1 \end{tikzcd}$$

of schemes of finite type over K with φ proper and ψ flat. Then φ' is proper, ψ' is flat and the fibre-square rule

$$\psi^* \circ \varphi_* = \varphi'_* \circ (\psi')^*$$

holds on $Z(X_2)$.

1.2.12 Let $\varphi : X' \to X$ be a morphism over K. Let D be a Cartier divisor on X and let V be a prime cycle on X'. If $\varphi(V)$ is not contained in $|D|$, then $\varphi|_V^*(D)$ is a well-defined Cartier divisor on V equal to $\mathrm{div}(s'_V)$ where $s'_V := \varphi|_V^*(s_D)$. If $\varphi(V) \subset |D|$, then let s'_V be any invertible meromorphic section of the line bundle $\varphi|_V^* O(D)$ on V. In any case, we define the *refined intersection product* $D.{}_\varphi V$ of D and V as the Weil divisor associated to $\mathrm{div}(s'_V)$ considered as a cycle on X'. By linearity, we define $D.{}_\varphi V$ for all cycles on X'. Note that the refined intersection product is well-defined up to $\mathrm{div}(\mathbf{f}')$ for K_1-chains $\mathbf{f}'$ on $\varphi^{-1}|D| \cap |Z'|$. If Z' has dimension t, then $D.{}_\varphi Z'$ has dimension $t-1$.

1.2.13 If D is a Cartier divisor and Y is a prime cycle on X, then the *refined intersection product* is

$$D.Y := i_*(D.{}_i Y)$$

where i is the closed embedding of Y in X. By linearity, we extend $D.Z$ to all cycles Z on X, it is well-defined up to $\mathrm{div}(\mathbf{f})$ for K_1-chains $\mathbf{f}$ on $|D| \cap |Z|$. If D and Z *intersect properly*, i.e. no component of Z is contained in $|D|$, then $D.Z$ is well-defined as a cycle. This follows directly from the definitions. This shows the importance to have an intersection product well-defined up to K_1-chains on $|D| \cap |Z|$ and not only up to K_1-chains on X.

1.2.14 If D is a Cartier divisor on X and Z_1, Z_2 are cycles on X, then

$$D.(Z_1 + Z_2) = D.Z_1 + D.Z_2$$

up to K_1-chain on $|D| \cap (|Z_1| \cup |Z_2|)$.

1.2.15 If D_1, D_2 are Cartier divisors on X and Z is a cycle on X, then

$$(D_1 + D_2).Z = D_1.Z + D_2.Z$$

up to K_1-chains on $(|D_1| \cup |D_2|) \cap |Z|$.

1.2.16 Let $\varphi : X' \to X$ be a proper morphism of K-varieties, let D be a Cartier divisor on X and let Z' be a cycle on X'. Then the *projection formula*

$$\varphi_*(D._{\varphi} Z') = D.\varphi_*(Z')$$

holds up to K_1-chains on $|D| \cap \varphi(|Z'|)$.

1.2.17 Let $\varphi : X' \to X$ be a flat morphism of relative dimension n, let D be a Cartier divisor and let Z be a cycle on X, then

$$D._{\varphi}\varphi^*(Z) = \varphi^*(D.Z)$$

holds up to K_1-chains on $\varphi^{-1}(|D| \cap |Z|)$.

1.2.18 Let D be the divisor of an invertible meromorphic function on X and let Z be any cycle on X. Then there is a K_1-chain $\mathbf{f}$ on $|Z|$ with

$$D.Z = \mathrm{div}(\mathbf{f}).$$

1.2.19 Let D be a Cartier divisor on X and let $\mathbf{f}$ be a K_1-chain on a closed subset Y of X. Then there is a K_1-chain $\mathbf{g}$ on $|D| \cap Y$ with

$$D.\mathrm{div}(\mathbf{f}) = \mathrm{div}(\mathbf{g}).$$

Now it makes sense to consider several products with Cartier divisors like $D_1 \ldots D_n.Z$. It is always understood that first D_n acts on Z, then D_{n-1} acts on $D_n.Z$ and so on.

1.2.20 Let D, D' be Cartier divisors on X and let Z be a cycle on X. Then

$$D.D'.Z = D'.D.Z$$

up to K_1-chains on $|D| \cap |D'| \cap |Z|$.

1.2.21 Remark. All these results give identities of cycles under suitable assumptions on proper intersections. In 1.2.14, we have to assume that D intersects $|Z_1| \cup |Z_2|$ properly. For 1.2.15, we have to assume that $|D_1| \cup |D_2|$ intersects $|Z|$ properly. In 1.2.16 , we assume Z' prime and that D intersects $\varphi(Z")$ properly. For 1.2.17 , 1.2.18 (resp. 1.2.19), we suppose that D intersects Z (resp. Y) properly. In 1.2.20 , we assume that for every component Y of Z, the codimension of $Y \cap |D| \cap |D'|$ in Y is at least 2. We can deduce it from the above results by using a dimension argument to show that the K_1-chains do not matter.

1.2.22 Not every line bundle L on X has an invertible meromorphic section. Its existence is only guaranteed if X is integral. Nevertheless, we can introduce the following *Chern class* operation on cycles. By linearity, it is enough to consider a prime cycle Z. Then we define

$$c_1(L) \cap Z := \mathrm{div}(s_Z),$$

where s_Z is any invertible meromorphic section of $L|_Z$. Then $c_1(L) \cap Z$ is a cycle on X, well-defined up to K_1-chains on Z. The construction satisfies all properties mentioned above with intersection product replaced by Chern class operations. If s is an invertible meromorphic section of L, then $c_1(L) \cap Z$ is represented by $\mathrm{div}(s).Z$.

1.2.23 Let $L_1, \ldots, L_t$ be line bundles on a proper variety X over K. We consider the morphism of structure $\pi : X \to \mathrm{Spec}K$ and we identify the Chow group of $\mathrm{Spec}K$ with $\mathbb{Z}$. Then the *degree* of a t-dimensional cycle Z on X with respect to $L_1, \ldots, L_t$ is given by

$$\deg_{L_1,\ldots,L_t}(Z) := \pi_* \left(c_1(L_1) \cap \cdots \cap c_1(L_t) \cap Z \right) \in \mathbb{Z}.$$

The degree is multilinear and symmetric in $L_1, \ldots, L_t$, and linear in Z. If the line bundles are generated by global sections (resp. ample), then the degree is non-negative (resp. positive). If one line bundle is algebraically equivalent to 0, then the degree is 0.

Chapter 2

Operations of Divisors in the Complex Case

2.1 Green forms

In this section, we develop the $*$-product of Green forms on a reduced complex space X. We use the approach from Burgos' thesis [Bu]. Because our applications are restricted to hermitian line bundles, we don't need any logarthmicness assumptions on the Green forms and so our presentation is much simpler. This has also the advantage, that no smoothness and projectivity of the ambient space is required. Some easy proofs, which directly carry over from Burgos, are omitted. At the end, we scetch how to extend the $*$-product of logarithmic Green forms to complex manifolds.

2.1.1 First, we introduce differential forms on singular spaces as in Bloom-Herrera [BH]. Locally, the complex space X is a closed analytic subset of an open ball in $\mathbb{C}^n$. A *smooth differential form* ω on an open subset U of X is locally given by the restriction of a smooth differential form defined on an open neighbourhood of U in such a ball. We identify smooth differential forms ω_1, ω_2 on U if $\omega_1 = \omega_2$ on the non-singular part U_{reg} of U. We denote the space of smooth differential forms on U by $A^*(U)$ where the $*$ gives the grading. If U is smooth, then we get the usual differential forms. Let $C^\infty(U) := A^0(U)$ be the C^∞-functions on U. The differential operators $d, d^c, \partial, \bar{\partial}$ and pull-back with respect to morphisms are defined on $A^*(U)$ by extending the forms locally to a ball as above and then using the corresponding constructions for complex manifolds.

We also introduce currents needed only in sections 2.3 and 2.4. For details, we refer to [Kin]. A *current* of dimension r on X is a linear functional T on the space of compactly supported smooth differential forms of degree r satisfying the following property: For every point of X, there is an open neighbourhood U in X which is a closed analytic subset of an open ball $\mathbb{B}^n$ and a current T_U of dimension r on $\mathbb{B}^n$ such that $T_U(\omega) = T(\omega|_U)$ for every compactly supported $\omega \in A^r(\mathbb{B}^n)$. The space of currents is denoted by $D_*(X)$ where the grading is with respect to the dimension. We have also a natural bigrading, i.e. $T \in D_{r,s}(X)$ if T acts non-trivially only on $A^{r,s}(X)$. We define $dT \in D_{r+1}(X)$ by

$$dT(\omega) := (-1)^{r+1}T(d\omega)$$

for every $\omega \in A^r(X)$ with compact support. Similarly, we proceed for $d^c, \partial, \bar{\partial}$. Every continuous differential form η on X gives rise to a current $[\eta]$ by

$$[\eta](\omega) = \int_X \eta \wedge \omega.$$

If η is smooth, then this definition ensures $[d\eta] = d[\eta]$. If Y is an irreducible closed analytic subset of X, then we get $\delta_Y \in D_{\dim(Y),\dim(Y)}(X)$ by

$$\delta_Y(\omega) := \int_Y \omega.$$

If $\varphi : X' \to X$ is a proper morphism of reduced complex spaces and $T' \in D_r(X')$, then $\varphi_*(T') \in D_r(X)$ is defined by

$$\varphi_*(T')(\omega) := T'(\varphi^*\omega)$$

for all $\omega \in A^r(X)$ with compact support.

2.1.2 Definition. For a closed analytic subset Y of X and $p \in \mathbb{N}$, let

$$G_Y^p(X) := \{(\omega, g) \mid \omega \in A^{p+2}(X), g \in A^p(X \setminus Y), d\omega = d^c\omega = 0, \omega|_{X\setminus Y} = dd^c g\}.$$

Such a g is called a *Green form* of degree p with singularities along Y. If Y is nowhere dense in X, then ω is determined by the Green form g. For notational simplicity, we focus usually on this case and we omit the reference to ω. If $Y = X$, then we set formally $g = 0$ and hence ω is any d- and d^c-closed differential form on X. This case is usually easier and we leave it to the reader to complete the results in this case. We set

$$G_Y^*(X) := \bigoplus_p G_Y^p(X)$$

and its elements are called Green forms with singularities along Y.

2.1.3 Let Y and Z be closed analytic subsets of X. Then there is a C^∞-function σ_{YZ} on $X \setminus (Y \cap Z)$ such that $\sigma_{YZ} = 1$ on an open neighbourhood of $Y \setminus Z$ and $\sigma_{YZ} = 0$ in an open neighbourhood of $Z \setminus Y$. Then σ_{YZ} and $\sigma_{ZY} := 1 - \sigma_{YZ}$ form just a partition of unity subordinate to the cover $X \setminus Z$ and $X \setminus Y$ of $X \setminus (Y \cap Z)$.

2.1.4 Definition. The $*$-*product* of $(\omega_Y, g_Y) \in G_Y^p(X)$ and $(\omega_Z, g_Z) \in G_Z^q(X)$ is defined by

$$(\omega_Y, g_Y) * (\omega_Z, g_Z) := (\omega_Y \wedge \omega_Z, g_Y * g_Z)$$

where

$$g_Y * g_Z := dd^c(\sigma_{YZ} g_Y) \wedge g_Z + \sigma_{ZY} g_Y \wedge \omega_Z.$$

2.1.5 It is easy to see that the $*$-product is well-defined in $G_{Y\cap Z}^{p+q+2}(X)$, not depending on the choice of the partition up to boundaries of the form $(0, d\eta + d^c\rho)$ for $\rho, \eta \in A(X \setminus (Y \cap Z))$. So we define

$$\tilde{G}_Y^p(X) := G_Y^p(X) / \left(\partial A^{p-1}(X \setminus Y) + \bar{\partial} A^{p-1}(X \setminus Y)\right)$$

and the class of a Green form $g_Y \in G_Y^p(X)$ in $\tilde{G}_Y^p(X)$ is denoted by $\tilde{g}_Y$. If g_Y and g_Y' have the same class in $\tilde{G}_Y^p(X)$, we write $g_Y \equiv g_Y'$.

2.1.6 Proposition. *The $*$-product is graded commutative, i.e. for $g_Y \in G_Y^p(X)$ and $g_Z \in G_Z^q(X)$, we have*

$$g_Y * g_Z \equiv (-1)^{pq} g_Z * g_Y$$

in $\tilde{G}_{Y\cap Z}^{p+q+2}(X)$.

Proof: Similar as in [Bu], 2.14. □

2.1.7 Lemma. *Let Y, Z and W be closed analytic subsets of X. Then we can choose the partition of unity in 2.1.3 such that*

$$\sigma_{YW}\sigma_{WZ}\sigma_{ZY} = 0$$

holds.

Proof: We choose arbitrary partitions of unity σ_{YW}, σ_{WY} and σ_{WZ}, σ_{ZW} as in 2.1.3. Let S be the closure of $\sigma_{YW}\sigma_{WZ} \neq 0$ in $X \setminus (Y \cap Z)$. Note that $\sigma_{YW}\sigma_{WZ}$ vanishes in an open neighbourhood of $Z \setminus Y \subset (Z \setminus W) \cup (W \setminus Y)$ in $X \setminus (Y \cap Z)$. We conclude that $S \cap (Z \setminus Y) = \emptyset$. Then $X \setminus Z, X \setminus (Y \cup S)$ is a covering of $X \setminus (Y \cap Z)$. If σ_{YZ}, σ_{ZY} is a partition of unity subordinate to this covering, we get the claim. $\square$

2.1.8 Let σ_{YW}, σ_{WZ} and σ_{ZY} be as in Lemma 2.1.7. Then we set

$$\sigma_{Z \cap W, Y} := \sigma_{ZY}\sigma_{WY}$$

and

$$\sigma_{Y \cap Z, W} := \sigma_{YW}\sigma_{ZW}.$$

Elementary topological considerations show that $\sigma_{Z\cap W,Y}$ extends uniquely to a C^∞-function on $X \setminus (Y \cap Z \cap W)$ such that $\sigma_{Z\cap W,Y} = 1$ on an open neighbourhood of $Z \cap W$ in $X \setminus (Y \cap Z \cap W)$ and $\sigma_{Z\cap W,Y} = 0$ on an open neighbourhood of Y in $X \setminus (Y \cap Z \cap W)$. In other words, we may use $\sigma_{Z\cap W,Y}$ and $\sigma_{Y,Z\cap W} := 1 - \sigma_{Z\cap W,Y}$ as partition of unity in 2.1.3. This will make the proof of associativity easier. From Lemma 2.1.7, we deduce easily

$$\sigma_{W,Y\cap Z}\sigma_{ZY} = \sigma_{Z\cap W,Y} \tag{2.1}$$

and this leads to

$$\sigma_{YZ} + \sigma_{Y\cap Z,W}\sigma_{ZY} = \sigma_{Y,Z\cap W}. \tag{2.2}$$

2.1.9 Proposition. *The $*$-product is associative, i.e. for $g_Y \in G_Y(X), g_Z \in G_Z(X)$ and $g_W \in G_W(X)$, we have*

$$(g_Y * g_Z) * g_W \equiv g_Y * (g_Z * g_W)$$

in $\tilde{G}_{Y\cap Z\cap W}(X)$.

Proof: *We may assume that g_Y, g_Z, g_W have degrees p, q, r. We use the partitions constructed in Lemma 2.1.7 and 2.1.8. We have*

$$\begin{aligned} (g_Y * g_Z) * g_W &= dd^c\left(\sigma_{Y\cap Z,W}\, g_Y * g_Z\right) \wedge g_W + \sigma_{W,Y\cap Z}(g_Y * g_Z) \wedge \omega_Z \\ &= \xi_1 + \xi_2 + \xi_3 + \xi_4 \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \xi_1 &= dd^c(\sigma_{YZ}g_Y) \wedge dd^c(\sigma_{Y\cap Z,W}\, g_Z) \wedge g_W \\ \xi_2 &= dd^c(\sigma_{Y\cap Z,W}\,\sigma_{ZY}g_Y) \wedge \omega_Z \wedge g_W \\ \xi_3 &= dd^c(\sigma_{YZ}g_Y) \wedge \sigma_{W,Y\cap Z}\, g_Z \wedge dd^c g_W \\ \xi_4 &= \sigma_{Z\cap W,Y}\, g_Y \wedge \omega_Z \wedge \omega_W \end{aligned}$$

using (2.1) in the last line. Note that

$$\xi_3 = d\rho_1 + \vartheta_1$$

with

$$\rho_1 = (-1)^{p+q} dd^c(\sigma_{YZ}g_Y) \wedge \sigma_{W,Y\cap Z}\, g_Z \wedge d^c g_W$$

and

$$\begin{aligned} \vartheta_1 &= (-1)^{q+1} dd^c(\sigma_{YZ}g_Y) \wedge d(\sigma_{W,Y\cap Z}\, g_Z) \wedge d^c g_W \\ &= d^c\rho_2 + dd^c(\sigma_{YZ}g_Y) \wedge dd^c(\sigma_{W,Y\cap Z}\, g_Z) \wedge g_W \end{aligned}$$

where

$$\rho_2 = (-1)^p dd^c(\sigma_{YZ} g_Y) \wedge d(\sigma_{W,Y\cap Z} g_Z) \wedge g_W.$$

We conclude that

$$\xi_1 + \xi_3 = dd^c(\sigma_{YZ} g_Y) \wedge \omega_Z \wedge g_W + d\rho_1 + d^c\rho_2$$

and hence (2.2) gives

$$\xi_1 + \xi_2 + \xi_3 = dd^c(\sigma_{Y,Z\cap W} g_Y) \wedge \omega_Z \wedge g_W + d\rho_1 + d^c\rho_2. \tag{2.4}$$

Now we compute the right hand side of the associativity law:

$$\begin{aligned} g_Y * (g_Z * g_W) &= dd^c\left(\sigma_{Y,Z\cap W} g_Y\right) \wedge (g_Z * g_W) + \sigma_{Z\cap W,Y} g_Y \wedge \omega_Z \wedge \omega_W \\ &= \xi_5 + \xi_6 + \xi_7 \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \xi_5 &= dd^c\left(\sigma_{Y,Z\cap W} g_Y\right) \wedge dd^c\left(\sigma_{ZW} g_Z\right) \wedge g_W \\ \xi_6 &= dd^c\left(\sigma_{Y,Z\cap W} g_Y\right) \wedge \sigma_{WZ} g_Z \wedge \omega_W \\ \xi_7 &= \sigma_{Z\cap W,Y} g_Y \wedge \omega_Z \wedge \omega_W. \end{aligned}$$

The same computation as above for ξ_3 leads to

$$\xi_6 = dd^c(\sigma_{Y,Z\cap W} g_Y) \wedge dd^c(\sigma_{WZ} g_Z) \wedge g_W + d\rho_3 + d^c\rho_4$$

where

$$\rho_3 = (-1)^{p+q} dd^c(\sigma_{Y,Z\cap W} g_Y) \wedge \sigma_{WZ} g_Z \wedge d^c g_W$$

and

$$\rho_4 = (-1)^p dd^c(\sigma_{Y,Z\cap W} g_Y) \wedge d(\sigma_{WZ} g_Z) \wedge g_W.$$

This proves

$$\xi_5 + \xi_6 = dd^c(\sigma_{Y,Z\cap W} g_Y) \wedge \omega_Z \wedge g_W + d\rho_3 + d^c\rho_4. \tag{2.6}$$

Since $\xi_4 = \xi_7$, we conclude from (2.3)-(2.6) that

$$(g_Y * g_Z) * g_W - g_Y * (g_Z * g_W) = d\rho_1 + d^c\rho_2 - d\rho_3 - d^c\rho_4.$$

An easy calculation shows that the right hand side is equal to

$$(-1)^{p+q} d(\zeta \wedge g_Z \wedge d^c g_W) - (-1)^{p+q} d^c(\zeta \wedge g_Z \wedge dg_W) + d^c d(\zeta \wedge g_Z \wedge g_W)$$

where

$$\zeta = \sigma_{W,Y\cap Z} dd^c(\sigma_{YZ} g_Y) - \sigma_{WZ} dd^c(\sigma_{Y,Z\cap W} g_Y).$$

All these calculations were for smooth forms on $X \setminus (Y \cup Z \cup W)$. To prove associativity, it is now enough to show that ζ extends uniquely to a smooth differential form on $X \setminus (Y \cap Z \cap W)$ which vanishes identically in an open neighbourhood of $(Y \cup Z \cup W) \setminus (Y \cap Z \cap W)$ in $X \setminus (Y \cap Z \cap W)$ whereever it is defined. In an open neighbourhood of $Y \setminus (Z \cup W)$, we have

$$\zeta = (1 - \sigma_{YW}\sigma_{ZW}) dd^c g_Y - \sigma_{WZ} dd^c g_Y = 0.$$

The same holds in open neighbourhoods of $Z \setminus (Y \cup W)$ and $W \setminus (X \cup Y)$. The first is trivial and the second follows by (2.2). Obviously, we have also $\zeta = 0$ in open neighbourhoods of $(Y \cap Z) \setminus W$ and $(Z \cap W) \setminus Y$. Finally, in an open neighbourhood of $(Y \cap W) \setminus Z$, we have

$$\zeta = (1 - \sigma_{YW}\sigma_{ZW}) dd^c g_Y - dd^c\left((1 - \sigma_{ZY}\sigma_{WY}) g_Y\right) = 0.$$

The union of all these open subsets give an open neighbourhood of $(Y \cup Z \cup W) \setminus (Y \cap Z \cap W)$ *in* $X \setminus (Y \cap Z \cap W)$ *where* ζ *vanishes identically. This proves the claim.* □

2.1.10 *Let* Y' *be a closed analytic subset of* X *with* $Y \subset Y'$. *Then it is easy to see that the restriction* $G_Y(X) \to G_{Y'}(X)$ *is compatible with the* $*$*-product.*

If $\varphi : X' \to X$ *is a morphism of reduced complex spaces, then the pull-back of a Green form with singularities along* Y *is a Green form with singularities along* $\varphi^{-1}(Y)$. *If* Z *is also a closed analytic subset of* X *and* g_Z *is a Green form with singularities along* Z, *then it is straightforward to see that*

$$\varphi^*(g_Y * g_Z) \equiv \varphi^*(g_Y) * \varphi^*(g_Z)$$

in $G_{\varphi^{-1}(Y \cap Z)}(X')$. *Especially for pull-back, it is useful to consider elements of* $G_Y(X)$ *as pairs* (ω_Y, g_Y). *Then the pull-back*

$$\varphi^*(\omega_Y, g_Y) = (\varphi^*\omega_Y, \varphi^* g_Y)$$

is defined in $\tilde{G}_{\varphi^{-1}(Y)}(X')$ *even if* $\varphi^{-1}(Y)$ *contains irreducible components of* X' *and all the results remain valid.*

2.1.11 Proposition. *Let* η *be a smooth* q*-form on* X. *Obviously, it is a Green form with singularities along* $\emptyset$. *If* $g_Y \in G_Y^p(X)$, *then we have*

$$(-1)^{pq} g_Y * \eta \equiv \eta * g_Y \equiv \eta \wedge \omega_Y$$

in $G_Y(X)$.

Proof: This follows easily by choosing $\sigma_{\emptyset Y}$ identically 0 and using commutativity. □

2.1.12 Example. Let L be a holomorphic line bundle on X with a smooth hermitian metric $\| \ \|$. Then $\hat{L} = (L, \| \ \|)$ will be called a *hermitian line bundle.* We call a meromorphic section s of L *invertible*, if there is an open dense subset U of X such that s restricts to a nowhere vanishing holomorphic section of L on U. For an invertible meromorphic section s, we have a well-defined Cartier divisor $\mathrm{div}(s)$ on X. It is clear from the definitions that $\log \|s\|^{-2}$ is a Green form with singularities along the support of the Cartier divisor $\mathrm{div}(s)$ and that $dd^c \log \|s\|^{-2}$ is the Chern form $c_1(L, \| \ \|)$ outside this support.

2.1.13 Definition. A *hermitian pseudo-divisor* on X is a triple $(\hat{L}, Y, s)$ where $\hat{L}$ is a hermitian line bundle on X, where Y is a closed analytic subset of X and where s is a nowhere vanishing section of L on $X \setminus Y$. Then $\hat{L}$ is called the *hermitian line bundle*, Y is called the *support* and s is called the *section* of the hermitian pseudo-divisor. Two hermitian pseudo-divisors on X are considered as equal if the support is the same and if there is an isometry of the hermitian line bundles mapping one section to the other.

2.1.14 Note that the section does not necessarily extend to a meromorphic section on X. If s is an invertible meromorphic section of a hermitian line bundle, then $(\hat{L}, Y, s|_{X \setminus Y})$ is a hermitian pseudo-divisor on X for every closed analytic subset Y containing the support of $\mathrm{div}(s)$. If we choose $Y = \mathrm{div}(s)$, then it is called the hermitian pseudo-divisor associated to $(\hat{L}, s)$.

2.1.15 Definition. Let $(\hat{L}, Y, s)$ and $(\hat{L}', Y', s')$ be two hermitian pseudo-divisors on X. Then the *sum* is the hermitian pseudo-divisor

$$(\hat{L}, Y, s) + (\hat{L}', Y', s') := (\hat{L} \otimes \hat{L}', Y \cup Y', s \otimes s')$$

on X. If $\varphi : X' \to X$ is a morphism of reduced complex analytic spaces, then the *pull-back* of $(\hat{L}, Y, s)$ is the hermitian pseudo-divisor

$$\varphi^*(\hat{L}, Y, s) := (\varphi^* \hat{L}, \varphi^{-1}(Y), \varphi^* s)$$

on X'.

2.1.16 If $(\hat{L}, Y, s)$ is a hermitian pseudo-divisor on X, then $\log \|s\|^{-2}$ is a Green form with singularities along the support Y.

Proof: Similarly as in Example 2.1.12, it is clear from the definition that $dd^c \log \|s\|^{-2}$ is the Chern form of $\hat{L}$ on $X \setminus Y$. □

Burgos [Bu] considered the $*$-product of logarithmic Green forms on projective manifolds. This concept can be extended to the case of complex manifolds without problems. This will be scetched in the remaining part of this section.

2.1.17 Definition. A differential form η on the complex manifold X is called *logarithmic* along the closed analytic subset Y if for all $y \in Y$, there is an open neighbourhood U and a proper morphism π from a complex manifold $\tilde{U}$ to U satisfying the following conditions:

a) $E := \pi^{-1}(Y)$ is a divisor with normal crossings.

b) π induces an isomorphism form $\tilde{U} \setminus E$ onto $U \setminus Y$.

c) If V is a local chart on $\tilde{U}$ with coordinates $z_1, \ldots, z_n$ such that E is given on V by the equation $z_1 \cdots z_k = 0$, then the restriction of $\pi^*(\eta)$ to V is in the $A(V)$-subalgebra of $A(V \setminus E)$ generated by

$$\log |z_i|, \frac{dz_i}{z_i}, \frac{d\bar{z}_i}{\bar{z}_i} \qquad (i = 1, \ldots, k).$$

2.1.18 Remark. On a complex manifold X, the logarithmic differential forms along Y build an $A(X)$-subalgebra of $A(X \setminus Y)$ which is closed under d, d^c. Now it is easy to see that for logarithmic Green foms g_Y, g_Z along closed analytic subsets Y, Z, there is a partition $\sigma_{YZ} + \sigma_{ZY} = 1$ as in Remark 2.1.3 with σ_{YZ} logarithmic along $Y \cap Z$. Using always such partitions to define the $*$-product of logarithmic forms, we conclude as in 2.1.5 that $g_Y * g_Z$ is a logarithmic Green form along $Y \cap Z$, well-defined up to d- and d^c-boundaries of logarithmic Green forms along $Y \cap Z$. In the context of logarithmic Green forms and logarithmic boundaries, the results form Proposition 2.1.6 to Proposition 2.1.11 still hold. The formulations and proofs may be adjusted easily. In Example 2.1.12, the invertible meromorphic section s gives rise to a logarithmic Green form $\log \|s\|^{-2}$. But for a hermitian pseudo-divisor $(\hat{L}, Y, s)$, it is clear that $\log \|s\|^{-2}$ is not always a logarithmic form along Y.

2.2 Local Heights on Complex Spaces

In this section, we introduce local heights of compact cycles on a reduced complex analytic space X. They are analogues of intersection numbers using the $*$-product of Green forms from the previous section. Note that no smoothness assumptions are required on X. We measure the local height with respect to hermitian pseudo-divisors, which are more general than the divisors of meromorphic sections of hermitian line bundles. The assumption X reduced is harmless, otherwise we pass simply to X_{red} without changing local heights of cycles.

2.2.1 *Cycles* on X are locally finite formal sums of irreducible closed analytic subsets and Cartier divisors are defined similarly as in 1.2.4. The same one-to-one correspondence between invertible meromorphic sections of line bundles and Cartier divisors holds on complex spaces, in fact it is true for any ringed space ([EGA IV], Proposition 21.2.11). For a Cartier divisor, we can associate a Weil divisor by a similar construction as in 1.2.6. Hence we get a proper intersection product of Cartier divisors and cycles.

If $\varphi : X' \to X$ is a proper morphism of reduced complex analytic spaces, then we have a push-forward map of cycles denoted by φ_*. It is enough to define $\varphi_*(X')$ for X' irreducible and $\varphi(X') = X$, the general case is reduced to this special case by restriction. If $\dim(X') > \dim(X)$, then we set $\varphi_*(X') = 0$. In the equidimensional case, there is a lower dimensional closed analytic subset of X such that outside, the map φ is a d-sheeted finite étale covering of smooth complex manifolds and we define $\varphi_*(X') = d \cdot X$.

These constructions are worked out for rigid analytic varieties in 3.1, they are the non-archimedean analogues of complex spaces. The same results and proofs hold here, but we omit the details.

2.2.2 Let Z be an irreducible compact analytic subset of X of dimension t. Let $Y_0, \dots, Y_r$ be closed analytic subsets of X with $Y_0 \cap \cdots \cap Y_r \cap Z = \emptyset$ and let $p_0 + \cdots + p_r = 2t - 2r$ be a partition. For $g_j \in G_{Y_j}^{p_j}(X)$, the restriction of $g_0 * \cdots * g_r$ to Z is represented by a smooth $2t$-form. By Stokes, the integral over Z is independent of the choice of the reresentative, hence the following makes sense:

2.2.3 Definition.

$$\langle g_0 * \cdots * g_r \mid Z \rangle := \frac{1}{2} \int_Z g_0 * \cdots * g_r$$

2.2.4 Let $Z = \sum_i n_i Z_i$ be a compact cycle (i.e. the support $|Z|$ is compact) such that its components Z_i are disjoint from $Y_0 \cap \cdots \cap Y_r$. Then

$$\langle g_0 * \cdots * g_r \mid Z \rangle := \sum_i n_i \langle g_0 * \cdots * g_r \mid Z_i \rangle$$

is *well-defined* using the convention $\langle g_0 * \cdots * g_r \mid Z_i \rangle = 0$ if $p_0 + \cdots + p_r \neq 2 \dim Z_i - 2r$. Note that $\langle g_0 * \cdots * g_r \mid Z \rangle$ is multilinear in the variables $g_0, \dots, g_r$ if all terms are well-defined. By multilinearity, we extend the definition to all Green forms $g_j \in G_{Y_j}(X)$ if the terms in every degree are well-defined.

2.2.5 Proposition. *Let $\varphi : X' \to X$ be a morphism of reduced complex analytic spaces. Let Z' be a compact cycle on X' and let $Y_0, \dots, Y_r$ be closed analytic subsets of X with $|Z'| \cap \varphi^{-1}(Y_0) \cap \cdots \cap \varphi^{-1}(Y_r) = \emptyset$. If $g_j \in G_{Y_j}(X)$ for $j = 0, \dots, r$, then*

$$\langle \varphi^*(g_0) * \cdots * \varphi^*(g_r) \mid Z' \rangle = \langle g_0 * \cdots * g_r \mid \varphi_*(Z') \rangle.$$

In the last section, we weren't able to formulate the *projection formula*, since the push-forward of a smooth form is no longer a form. But here, the situation is better. Note that the restriction of φ to Z' is proper, hence the push-forward is well-defined.

Proof: To prove the claim, we may assume that Z' is a prime cycle and that $X' = Z', X = \varphi(X')$. Here, we have used (2.1.10) which also gives

$$\varphi^*(g_0) * \cdots * \varphi^*(g_r) \equiv \varphi^*(g_0 * \cdots * g_r)$$

as an identity of differential forms on X'. Then the claim follows from the formula

$$\int_{\varphi_*(Z')} \eta = \int_{Z'} \varphi^* \eta$$

valid for any differential form on Z'. Note that $\varphi_*(Z')$ is X' multiplied by the degree of the covering. □

2.2.6 Proposition. *Let Z be a compact cycle on X of pure dimension t. Let $g_0, \dots, g_r$ be Green forms on X with singularities along the closed analytic subsets $Y_0, \dots, Y_r$. We assume that Y_0 is disjoint from the support of Z, hence g_0 restricts to a smooth form on Z. Then*

$$\langle g_0 * \cdots * g_r \mid Z \rangle = \frac{1}{2} \int_Z g_0 \wedge \omega_1 \wedge \omega_1 \wedge \cdots \wedge \omega_r$$

where $\omega_j = dd^c g_j$ *on* $X \setminus Y_j$.

Proof: We may assume that Z is prime and $Z = X$, then the claim follows immediately from Proposition 2.1.11. $\square$

2.2.7 Definition. Let $(\hat{L}_0, Y_0, s_0), \ldots, (\hat{L}_t, Y_t, s_t)$ be hermitian pseudo-divisors on X (see (2.1.13)-(2.1.16)) and let Z be a compact cycle on X with

$$Y_0 \cap \cdots \cap Y_t \cap |Z| = \emptyset.$$

In (2.1.16), we have seen that $\log \|s_j\|^{-2} \in G^0_{Y_j}(X)$. Then we define the *local height* of Z with respect to the given hermitian pseudo-divisors by

$$\lambda(Z) := \lambda_{(\hat{L}_0,Y_0,s_0),\ldots,(\hat{L}_t,Y_t,s_t)}(Z) := \langle \log \|s_0\|^{-2} * \cdots * \log \|s_t\|^{-2} \mid Z \rangle.$$

Note that only the t dimensional components of Z matter for $\lambda(Z)$. From the above, we deduce immediately the following properties:

2.2.8 Corollary. *The local height is multilinear with respect to the hermitian pseudo-divisors, and linear in* Z *under the hypothesis that all terms are well-defined. The local height doesn't depend on the supports of the pseudo-divisors.*

2.2.9 Corollary. *If* $\varphi : X' \to X$ *is a morphism of reduced complex spaces and if* Z' *is a compact cycle on* X' *such that* $\varphi(|Z'|) \cap Y_0 \cap \cdots \cap Y_t = \emptyset$, *then we have*

$$\lambda_{\varphi^*(\hat{L}_0,Y_0,s_0),\ldots,\varphi^*(\hat{L}_t,Y_t,s_t)}(Z') = \lambda_{(\hat{L}_0,Y_0,s_0),\ldots,(\hat{L}_t,Y_t,s_t)}(\varphi_* Z')$$

2.2.10 Corollary. *If* $Y_0 \cap |Z| = \emptyset$, *then*

$$\lambda(Z) = -\int_Z \log \|s_0\| \wedge c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_t).$$

2.2.11 Corollary. *Under the assumptions of Definition 2.2.7, let us consider a second smooth hermitian metric* $\| \ \|'$ *on* L_0 *giving rise to the local height*

$$\lambda'(Z) := \lambda_{(\hat{L}'_0,Y_0,s_0),\ldots,(\hat{L}_t,Y_t,s_t)}(Z).$$

then $\rho := \log(\|s_0\|/\|s_0\|')$ *extends to a* C^∞*-function on* X *which doesn't depend on* s_0 *and we have*

$$\lambda'(Z) - \lambda(Z) = \int_Z \rho\, c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_t).$$

Proof: The claim for ρ follows from the fact that we may choose in the definition of ρ any local holomorphic section instead of s without changing its value. The formula is a consequence of linearity in the first argument and of Corollary 2.2.10 using the constant section 1 and the metric $\| \ \|'/\| \ \|$ on the trivial bundle O_X. $\square$

2.2.12 Corollary. *If the Chern form of one of the hermitian line bundles is* 0, *then the local height doesn't depend on the metrics of the other line bundles.*

2.3 Refined $*$-Product

Let X be a complex compact algebraic variety endowed with its analytic structure.

In this section, we introduce Green currents g_Z for cycles Z. They satisfy a certain partial differential equation. The basic example is the Green current $[\log \|s\|^{-2}]$ for $\mathrm{div}(s)$ where s is an invertible meromorphic section of a hermitian line bundle $\hat{L}$ and the PDE is just the Poincaré Lelong equation. For properly intersecting cycles on a smooth projective variety, the $*$-product of Green currents was introduced by Gillet-Soulé [GS2]. It is an archimedean analogue for the proper intersection product of cycles. Since we are only interested in products of the form $\log \|s\|^{-2} * g_Z$, no regularity assumption on X has to be required here.

First, we handle the proper intersection case. Then a refined $*$-product is defined with similar properties as in algebraic geometry (cf. section 1.2). By Hironaka's resolution of singularities, the proof of commutativity is reduced to the proper intersection case on a smooth variety. This is formally easy, but with additional effort, one could also give a direct argument without using resolution of singularites. At the end, we compare with the Burgos approach from section 2.1.

For simplicity, we assume that our spaces are compact algebraic varieties. By the GAGA principle [Se1], all analytic morphisms, line bundles, sections, subvarieties and cycles are algebraic. But the whole section is valid for any algebraic variety if we work only with algebraic objects. Instead of compactness, then we have to assume that every push-forward comes with a proper morphism.

2.3.1 Definition. A *Green current* for a cycle Z of pure dimension t is a current $g_Z \in D_{t+1,t+1}(X)$ with

$$dd^c g_Z = [\omega] - \delta_Z$$

for a smooth differential form ω_Z of bidegree (p,p) on X. Here [ ] denotes the associated current and δ_Z is the current of integration over Z (cf. 2.1.1). By linearity, we extend the definitions to cycles.

2.3.2 If $\hat{L}$ is a hermitian line bundle with invertible meromorphic section s, then the Poincaré Lelong equation gives

$$dd^c[\log \|s\|^{-2}] = [c_1(\hat{L})] - \delta_Z$$

with $c_1(\hat{L})$ the Chern form of $\hat{L}$ and $\delta_{\mathrm{div}(s)}$ the current of integration over the Weil divisor associated to $\mathrm{div}(s)$. For the case of manifolds, we refer to [GH], p. 388, and the singular case follows from Hironaka's resolution of singularities.

2.3.3 Proposition. *If $\varphi : X' \to X$ is a proper morphism of irreducible complex algebraic varieties and $g_{Z'}$ is a Green currrent for a cycle Z' on X' such that $\varphi_*[\omega_{Z'}]$ is the current associated to a smooth differential form on X (e.g. this holds if X and φ are smooth), then $\varphi_*(g_{Z'})$ is a Green current for $\varphi_*(Z')$.*

Proof: This follows from

$$dd^c \varphi_*(g_{Z'}) = \varphi_*(dd^c g_{Z'}) = \varphi_*[\omega_{Z'}] - \varphi_* \delta_{Z'}$$

and the fact $\varphi_* \delta_{Z'} = \delta_{\varphi_*(Z')}$. If X and φ is smooth, then integration along the fibres shows that $\varphi_*[\omega_{Z'}]$ is associated to a smooth differential form on X. □

2.3.4 Example. Let Y be an irreducible closed subvariety of X and let $f \in K(Y)^*$. We view $\mathrm{div}(f)$ as a cycle on X using the closed embedding $i : Y \to X$. By Example 2.3.2 and Proposition 2.3.3, we have

$$dd^c i_*[\log |f|^{-2}] = -\delta_{\mathrm{div}(f)},$$

i.e. $i_*[\log |f|^{-2}]$ is a Green current for $\mathrm{div}(f)$ on X. By abuse of notation, we denote this Green current also by $[\log |f|^{-2}]$.

2.3.5 Definition. Let Z be a prime cycle on X and let $\hat{L}$ be a hermitian line bundle on X with invertible meromorphic section s. If the Cartier divisor $\mathrm{div}(s)$ intersects Z properly, then we define the $*$-product

$$\log \|s\|^{-2} * g_Z := \log \|s\|^{-2} \wedge \delta_Z + c_1(\hat{L}) \wedge g_Z$$

where the terms on the right hand side are currents on X mapping a smooth differential form η of X to

$$\int_Z \log \|s\|^{-2}\eta$$

and

$$g_Z(c_1(\hat{L}) \wedge \eta),$$

respectively. If the support of $\mathrm{div}(s)$ contains Z, then we choose a non-zero meromorphic section s_Z of $L|_Z$. Then the $*$-product is defined by the same formula as above using

$$\log \|s\|^{-2} \wedge \delta_Z := i_*[\log \|s_Z\|^{-2}]$$

where $i : Z \to X$ is the closed embedding. We proceed by linearity to define $\log \|s\|^{-2} \wedge \delta_Z$ for any cycle Z on X. Then the above formula defines the $*$-product for any Green current g_Z of any cycle Z.

2.3.6 Proposition. *If* $\mathrm{div}(s)$ *and* Z *intersect properly, then* $\log \|s\|^{-2} * g_Z$ *is a Green current for* $\mathrm{div}(s).Z$.

Proof: Note first that

$$dd^c(c_1(\hat{L}) \wedge g_Z) = [c_1(\hat{L}) \wedge \omega_Z] - c_1(\hat{L}) \wedge \delta_Z.$$

Next, we claim

$$dd^c(\log \|s\|^{-2} \wedge \delta_Z) = c_1(\hat{L}) \wedge \delta_Z - \delta_{\mathrm{div}(s).Z}.$$

To prove it, we may assume Z prime. Let $i : Z \to X$ be the closed embedding. Then

$$\begin{aligned} dd^c(\log \|s\|^{-2} \wedge \delta_Z) &= dd^c i_*[\log \|s|_Z\|^{-2}] \\ &= i_*([c_1(\hat{L}|_Z)] - \delta_{\mathrm{div}(s|_Z)}) \\ &= c_1(\hat{L}) \wedge \delta_Z - \delta_{\mathrm{div}(s).Z}. \end{aligned}$$

We conclude that

$$dd^c\left(\log \|s\|^{-2} * g_Z\right) = [c_1(\hat{L}) \wedge \omega_Z] - \delta_{\mathrm{div}(s).Z}$$

proving the proposition. □

2.3.7 Remark. If $\mathrm{div}(s)$ doesn't intersect Z properly, then $\log \|s\|^{-2} * g_Z$ is only defined up to Green currents $\sum_W [\log |f_W|^{-2}]$ where $f_W \in K(W)^*$ and W ranges over the components of non-proper intersection of $\mathrm{div}(s).Z$, i.e. the components of Z contained in $|\mathrm{div}(s)|$. In the following proposition, we fix a representative for $\log \|s\|^{-2} * g_Z$.

2.3.8 Proposition. *Then there is a cycle* Z' *on* X *representing* $\mathrm{div}(s).Z \in CH(|\mathrm{div}(s)| \cap |Z|)$ *such that* $\log \|s\|^{-2} * g_Z$ *is a Green current for* Z' *with*

$$dd^c\left(\log \|s\|^{-2} * g_Z\right) = [c_1(\hat{L}) \wedge \omega_Z] - \delta_{Z'}.$$

Proof: As above, it is enough to show

$$dd^c\left(\log \|s\|^{-2} \wedge \delta_Z\right) = c_1(\hat{L}) \wedge \delta_Z - \delta_{Z'}$$

for a suitable cycle Z' representing the class $\mathrm{div}(s).Z \in CH(|\mathrm{div}(s)| \cap |Z|)$. Again, we may assume that Z is prime. If Z is contained in $|\mathrm{div}(s)|$, then let $Z' := \mathrm{div}(s_Z)$ where s_Z is the non-zero meromorphic section of $L|_Z$ used in the definition of $\log \|s\|^{-2} \wedge \delta_Z$. If Z is not contained in $|\mathrm{div}(s)|$, then $Z' := \mathrm{div}(s|_Z)$ as in Proposition 2.3.6. In any case, Z' represents the class $\mathrm{div}(s).Z$. □

2.3.9 Definition. A K_1-chain $\mathbf{f} = \sum f_W$ on X is a formal sum of $f_W \in K(W)^*$ where W ranges over all irreducible closed subvarieties of X and $f_W \neq 1$ only for finitely many f_W (cf. 1.2.8). By Example 2.3.4,

$$[\log |\mathbf{f}|^{-2}] := \sum_W [\log |f_W|^{-2}]$$

is a Green current for

$$\mathrm{div}(\mathbf{f}) := \sum_W \mathrm{div}(f_W).$$

2.3.10 Remark. Using this notion, we can say that $\log \|s\|^{-2} * g_Z$ is well-defined up to $[\log |\mathbf{f}|^{-2}]$ for K_1-chains $\mathbf{f}$ on $|\mathrm{div}(s)| \cap |Z|$. If Z has pure dimension t, then an obvious comparison of the degree of the currents show that only K_1-chains $\mathbf{f}$ matter which live on simultaneous components of $|\mathrm{div}(s)| \cap |Z|$ and of $|Z|$. So this carries all information from Proposition 2.3.8.

We generalize the above action of invertible meromorphic sections on Green currents to a relative situation. This allows us to consider pull-backs if the range is contained in the support of the Cartier divisor.

2.3.11 Definition. Let $\varphi : X' \to X$ be a morphism of complex compact algebraic varieties. Let s be an invertible meromorphic section of the hermitian line bundle $\hat{L}$ on X and let $g_{Z'}$ be a Green current for the cycle Z' on X'. Then we define

$$\log \|s\|^{-2} *_\varphi g_{Z'} := \log \|s\|^{-2} \wedge_\varphi \delta_{Z'} + c_1(\varphi^* \hat{L}) \wedge g_{Z'}$$

where the current $\log \|s\|^{-2} \wedge_\varphi \delta_{Z'}$ is defined as follows: We proceed by linearity, so we may assume that Z' is a prime cycle. If $\varphi(Z')$ is not contained in $|\mathrm{div}(s)|$, then $\varphi^*(s)|_{Z'}$ is well-defined as an invertible meromorphic section of $\varphi^*(L)|_{Z'}$ and we set

$$\log \|s\|^{-2} \wedge_\varphi \delta_{Z'} := \log \|\varphi^* s\|^{-2} \wedge \delta_{Z'} = i_*[\log \|\varphi^* s|_{Z'}\|^{-2}]$$

where $i : Z' \to X'$ is the closed immersion. If $\varphi(Z') \subset |\mathrm{div}(s)|$, then we choose any invertible meromorphic section $s_{Z'}$ of $\varphi^*(L')|_{Z'}$ and we define

$$\log \|s\|^{-2} \wedge_\varphi \delta_{Z'} := i_*[\log \|s_{Z'}\|^{-2}].$$

2.3.12 Remark. Again, the current $\log \|s\|^{-2} \wedge_\varphi \delta_{Z'}$ and hence $\log \|s\|^{-2} * g_{Z'}$ is well-defined up to $[\log |\mathbf{f}'|^{-2}]$ for K_1-chains $\mathbf{f}'$ on $\varphi^{-1}(|\mathrm{div}(s)|) \cap |Z'|$. Similarly as in Proposition 2.3.8, we deduce that $\log \|s\|^{-2} \wedge_\varphi \delta_{Z'}$ is a Green current for a cycle representing

$$\varphi^* \mathrm{div}(s).Z' \in CH\left(\varphi^{-1}|(|\mathrm{div}(s)|) \cap |Z'|\right).$$

If $\varphi^*(s)$ is a well-defined invertible meromorphic section of $\varphi^* L$ (i.e. no component of X' is mapped into $|\mathrm{div}(s)|$), then we have

$$\log \|s\|^{-2} *_\varphi g_{Z'} = \log \varphi^* \|s\|^{-2} * g_{Z'}.$$

If $\mathbf{f}'$ is a K_1-chain on X', then we have

$$\varphi_*[\log |\mathbf{f}'|] \equiv [\log |\varphi_* \mathbf{f}'|].$$

Here, $\varphi_* \mathbf{f}'$ is given by the norms (cf. 1.2.9) and the claim follows from the fact that we may resrict to the finite generically étale case where the norm is given by the sum over the fibre points (cf. [GS2], 3.6).

2.3.13 Proposition. *Let $\varphi : X' \to X$ be a morphism of complex compact algebraic varieties. Assume that $g_{Z'}$ is a Green current for a cycle Z' on X' such that $\varphi_*[\omega_{Z'}]$ is a smooth differential form on X (compare with Proposition 2.3.3). For an invertible meromorphic section s of a hermitian line bundle $\hat{L}$ on X, the projection formula*

$$\varphi_*(\log \|s\|^{-2} *_\varphi g_{Z'}) = \log \|s\|^{-2} * \varphi_* g_{Z'}$$

holds up to $[\log |\mathbf{f}|^{-2}]$ *for* K_1*-chains* $\mathbf{f}$ *on* $|\mathrm{div}(s)| \cap \varphi(|Z'|)$.

Proof: The obvious identity

$$\varphi_* \left(c_1(\varphi^* \hat{L}) \wedge g_{Z'} \right) = c_1(\hat{L}) \wedge \varphi_* g_{Z'}$$

of currents on X shows that it is enough to prove

$$\varphi_* \left(\log \varphi^* \|s\|^{-2} \wedge \delta_{Z'} \right) = \log \|s\|^{-2} \wedge \delta_{\varphi_* Z'} \tag{2.7}$$

up to $\log |\mathbf{f}|$ for a K_1-chain $\mathbf{f}$ on $|\mathrm{div}(s)| \cap \varphi(|Z'|)$. We may assume Z' prime.

First, we consider the case when $\varphi(Z')$ is not contained in $|\mathrm{div}(s)|$. We may assume that $X' = Z'$ and that φ is surjective. We have to prove

$$\varphi_*[\log \varphi^* \|s\|^{-2}] = \deg \varphi \cdot [\log \|s\|^{-2}].$$

The equidimensional case is clear from integration along the fibres since φ is generically smooth. If $\dim(X') < \dim(X)$, the formula is obvious since the degree is 0 by definition.

Finally, let $\varphi(Z') \subset |\mathrm{div}(s)|$. Let $s_{\varphi(Z')}$ be a non-trivial meromorphic section of $L|_{\varphi(Z')}$. If $\psi : Z' \to \varphi(Z')$ is induced by φ, then the first case shows

$$\psi_*[\log \psi^* \|s_{\varphi(Z')}\|^{-2}] = \deg \psi \cdot [\log \|s_{\varphi(Z')}\|^{-2}]$$

as an identity of currents on Z'. This implies (2.7) immediately. □

2.3.14 Remark. We introduce an equivalence $S \equiv T$ between currents on X if $S - T$ may be written as a sum of a d- and a d^c-boudary. Note that any current $T \equiv 0$ in $D_{t+1,t+1}(X)$ is a Green current for the zero-cycle. If $g_Z \equiv g'_Z$ are Green currents for Z, then

$$\log \|s\|^{-2} * g_Z \equiv \log \|s\|^{-2} * g'_Z$$

which is obvious from the definitions.

2.3.15 Proposition. *Let s, s' be invertible meromorphic sections of hermitian line bundles $\hat{L}, \hat{L}'$ on X. If g_Z is a Green current for a cycle Z on X, then*

$$\log \|s\|^{-2} * \left(\log \|s'\|^{-2} * g_Z \right) \equiv \log \|s'\|^{-2} * \left(\log \|s\|^{-2} * g_Z \right)$$

up to $\log |\mathbf{f}|^{-2}$ *for* K_1*-chains* $\mathbf{f}$ *on* $|\mathrm{div}(s)| \cap |\mathrm{div}(s')| \cap |Z|$.

Proof: It is enough to show

$$\begin{aligned} &\log \|s\|^{-2} \wedge \delta_{\mathrm{div}(s').Z} + c_1(\hat{L}) \wedge \left(\log \|s'\|^{-2} \wedge \delta_Z \right) \\ &\equiv \log \|s'\|^{-2} \wedge \delta_{\mathrm{div}(s).Z} + c_1(\hat{L}') \wedge \left(\log \|s\|^{-2} \wedge \delta_Z \right) \end{aligned} \tag{2.8}$$

up to K_1-chains on $|\mathrm{div}(s)| \cap |\mathrm{div}(s')| \cap |Z|$. Here, we choose the appropriate representatives $\mathrm{div}(s).Z$ and $\mathrm{div}(s').Z$ from Propostion 2.3.8. To prove (2.8), we may assume Z prime. Let s_Z, s'_Z be the meromorphic sections of $L|_Z$ used in the definitions of $\log \|s\|^{-2} \wedge \delta_Z$ and $\log \|s'\|^{-2} \wedge \delta_Z$. Then (2.8) reads as

$$\log \|s\|^{-2} \wedge \delta_{\mathrm{div}(s'_Z)} + c_1(\hat{L}) \wedge i_*[\log \|s'_Z\|^{-2}] \equiv \log \|s'\|^{-2} \wedge \delta_{\mathrm{div}(s_Z)} + c_1(\hat{L}') \wedge i_*[\log \|s_Z\|^{-2}]$$

where i is the closed embedding of Z into X. But this is the push-forward of the commutativity law

$$\log \|s_Z\|^{-2} * [\log \|s'_Z\|^{-2}] \equiv \log \|s'_Z\|^{-2} * [\log \|s_Z\|^{-2}]$$

on Z with respect to the embedding i. Assuming $Z = X$, we have reduced the claim to prove

$$\log \|s\|^{-2} * [\log \|s'\|^{-2}] \equiv \log \|s'\|^{-2} * [\log \|s\|^{-2}] \tag{2.9}$$

up to K_1-chains on $|\mathrm{div}(s)| \cap |\mathrm{div}(s')|$. Let $\pi : X' \to X$ be a birational morphism of irreducible complex compact algebraic varieties. Then we have

$$\pi_*[\log \pi^* \|s\|^{-2}] = \log \|s\|^{-2} \quad , \quad \pi_*[c_1(\pi^* \hat{L})] = [c_1(\hat{L})].$$

By the projection formula in Proposition 2.3.13, it is enough to prove the commutativity (2.9) on X'. Together with Hironaka's resolution of singularities and Chow's lemma, we conclude that we may assume that X is a smooth complex projective variety and that $|\mathrm{div}(s)| \cap |\mathrm{div}(s')|$ is a divisor. Using bilinearity, we easily reduce to the case where either the intersection of $\mathrm{div}(s)$ and $\mathrm{div}(s')$ is proper or $s = s'$. The first case follows from the well-known commutativity law of the $*$-product for logarithmic Green forms ([GS2], Corollary 2.2.9) and the latter case is trivial. □

2.3.16 Proposition.

a) If f is an invertible meromorphic function on X and g_Z is a Green current for a cycle Z on X, then

$$\log |f|^{-2} * g_Z = [\log |\mathbf{h}|^{-2}]$$

for a suitable K_1-chain $\mathbf{h}$ on $|Z|$.

b) If $\mathbf{f}$ is a K_1-chain on a closed subvariety Y of X and if s is an invertible meromorphic section of the hermitian line bundle $\hat{L}$ of X, then there is a K_1-chain $\mathbf{h}$ on $|\mathrm{div}(s)| \cap |Y|$ such that

$$\log \|s\|^{-2} * [\log |\mathbf{f}|^{-2}] \equiv [\log |\mathbf{h}|^{-2}].$$

Proof: To prove a), note that

$$\log |f|^{-2} * g_Z = \log |f|^{-2} \wedge \delta_Z$$

and hence we may assume that Z is prime. If Z is not contained in the support of the Cartier divisor $\mathrm{div}(f)$, then we choose $h = f|_Z$. If $Z \subset |\mathrm{div}(f)|$, then h can be any non-trivial rational function on Z. By definition of the refined $*$-product, this proves a).

For b), we may assume that $\mathbf{f}$ is just a rational function f on an irreducible closed subvariety Y of X. If Y is not contained in the support of $\mathrm{div}(s)$, then we restrict $\hat{L}$ and s to Y. If $Y \subset |\mathrm{div}(s)|$, then it is also enough to prove the claim on Y but with s replaced by any non-trivial meromorphic section of $L|_Y$. Hence we may assume $X = Y$ irreducible in both cases. Then b) follows from a) and commutativity (Proposition 2.3.15). □

2.3.17 Remark. There are also relative versions of the results above. The straightforward generalizations of formulations and the proofs above are left to the reader.

On a smooth projective variety X, it is shown in [Bu], Theorem 4.4, that the $*$-product of logarithmic Green forms for properly intersecting cycles is the same as the $*$-product of the corresponding Green currents in the sense of Gillet-Soulé [GS2]. We formulate this result for the divisor case, which is relevant for us.

Let Z be a cycle on X of codimension p. Then every Green form g_Z of degree $< 2p$ logarithmic along $|Z|$ is locally integrable, hence defines a current $[g_Z]$ on X.

2.3.18 Definition. We call $g_Z \in G^{p-1,p-1}_{|Z|}(X)$ a *logarithmic Green form* for Z if g_Z is logarithmic along $|Z|$ and $[g_Z]$ is a Green current for Z.

2.3.19 Remark. Let s be an invertible meromorphic section of the hermitian line bundle $\hat{L}$ on the smooth projective variety X. We assume that $\mathrm{div}(s)$ and Z intersect properly, where Z is a cycle of codimension p as above. Let g_Z be a logarithmic Green form for Z, then Burgos ([Bu], Theorem 4.4) shows

$$[\log \|s\|^{-2} * g_Z] \equiv \log \|s\|^{-2} * [g_Z]$$

where on the left hand side, the $*$-product is for logarithmic Green forms as in 2.1.18 and on the right hand side, we have the $*$-product from Definition 2.3.5 using Green currents. Moreover, Burgos proves that there is a logarithmic Green form for every cycle and that the map $g_Z \to [g_Z]$ gives a bijection from the equivalence classes of logarithmic Green forms for cycles to the equivalence classes of Green currents. The equivalence is up to d- and d^c-boundaries of logarithmic Green forms on the left hand side and up to d- and d^c-boundaries of currents on the right hand side.

2.4 Local Heights on Complex Algebraic Varieties

Let X be a complex compact algebraic variety endowed with the complex topology (for generalizations, see Remark 2.4.22).

We use refined intersection theory from the previous section to define local heights of cycles. If Z is a t-dimensional cycle and $s_0, \dots, s_t$ are invertible meromorphic sections of hermitian line bundles $\hat{L}_0, \dots, \hat{L}_t$, then the local height of Z is given by integrating the refined $*$-product of the Green currents $\log \|s_j\|^{-2}$ over Z. It is well-defined if the intersection of the supports is empty. The basic properties of local heights are immediate consequences of the results from section 2.3. We develop them independently of sections 2.1 and 2.2. At the end, we show that the approach in section 2.2 using Green forms leads to the same local heights on compact algebraic varieties as the one with Green currents used here.

A very useful tool for local heights is the induction formula 2.4.6. It expresses the local height of Z in terms of the local height of the $t-1$-dimensional cycle $\mathrm{div}(s_t).Z$ and of an integral over Z. Even if the reader is not familiar with the $*$-product, he can use the formula to understand heights. In this way, he may interpretate the local height in terms of finitely many integrals reducing to the zero-dimensional case. The induction formula was used by Faltings ([Fa3], Proposition 2.6) to prove non-negativity of heights of subvarieties. This will be picked up in chapter 5.

As an example, we give the local height of a cycle on a multiprojective space in terms of the Chow form. For global heights over number fields, this was done in [BoGS]. We use the local calculation from [Gu2].

2.4.1 In the last section, we have introduced the $*$-product as a left action of invertible meromorphic sections s of hermitian line bundles on Green currents g_Z for cycles Z. Now we consider several consecutive actions of such sections like

$$\log \|s_1\|^{-2} * \cdots * \log \|s_k\|^{-2} * g_Z$$

which we write without brackets. It is understood that $\log \|s_k\|^{-2}$ acts first on g_Z, then $\log \|s_{k-1}\|^{-2}$ acts on the Green current $\log \|s_k\|^{-2} * g_Z$, and so forth. If Z is prime, embedded by $i : Z \to X$, then

$$\log \|s\|^{-2} \wedge \delta_Z = i_*(\log \|s\|^{-2} *_i 1_Z)$$

up to K_1-chains on $|\mathrm{div}(s)| \cap Z$ (cf. Definition 2.3.11) where 1_Z is the unit Green current on Z, i.e. the constant function 1 viewed as a Green current on Z via

$$dd^c 1_Z = 0 = 1_Z - \delta_Z.$$

More generally, we define

$$\log \|s_1\|^{-2} * \cdots * \log \|s_k\|^{-2} \wedge \delta_Z := i_* \left(\log \|s_1\|^{-2} *_i \cdots *_i \log \|s_k\|^{-2} *_i 1_Z\right)$$

as a current on X, well-defined up to K_1-chains on $|\mathrm{div}(s_1) \cap \cdots \cap |\mathrm{div}(s_k)| \cap |Z|$. If Z is an arbitrary cycle, then we proceed by linearity.

However, for a left action, there is no associativity law. But the following is our substitute:

2.4.2 Proposition. *If s_j is an invertible meromorphic section of a hermitian line bundle $\hat{L}_j$ for $j = 1, \ldots, k$ and if Z is a cycle on X, then*

$$\log \|s_1\|^{-2} * \cdots * \log \|s_k\|^{-2} * g_Z = \log \|s_1\|^{-2} * \cdots * \log \|s_k\|^{-2} \wedge \delta_Z + c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_k) \wedge g_Z$$

up to K_1-chains on $|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_k)| \cap |Z|$.

Proof: We proceed by induction on k. The case $k = 1$ is by definition of the $*$-product. Now we prove the induction step for $k \geq 2$. By Proposition 2.3.8, we know that $\log \|s_k\|^{-2}$ is a Green current for a cycle Y representing $\mathrm{div}(s_k).Z \in CH(|\mathrm{div}(s_k)| \cap |Z|)$. The induction hypothesis shows

$$\begin{aligned} \log \|s_1\|^{-2} * \cdots * \log \|s_k\|^{-2} * g_Z &= \log \|s_1\|^{-2} * \cdots * \log \|s_{k-1}\|^{-2} \wedge \delta_Y \\ &\quad + c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_{k-1}) \wedge \left(\log \|s_k\|^{-2} * g_Z\right) \\ &= \log \|s_1\|^{-2} * \cdots * \log \|s_{k-1}\|^{-2} \wedge \delta_Y \\ &\quad + c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_{k-1}) \wedge \log \|s_k\|^{-2} \wedge \delta_Z \\ &\quad + c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_k) \wedge g_Z \end{aligned}$$

up to K_1-chains on $|\mathrm{div}(s_1)| \cap \cdots \cap \mathrm{div}(s_k)| \cap |Z|$. To handle the latter, use Proposition 2.3.16 and Remark 2.3.17. We may assume that Z is a prime cycle embedded by i into X. If Z is not contained in $|\mathrm{div}(s_j)|$, then let $s_{j,Z} := s_j|_Z$. If $Z \subset |\mathrm{div}(s_j)|$, then let $s_{j,Z}$ any non-trivial meromorphic section of $L_j|_Z$. It is clear from the definitions that the following identities hold

$$\begin{aligned} \log \|s_1\|^{-2} * \cdots * \log \|s_{k-1}\|^{-2} \wedge \delta_Y &= i_* \left(\log \|s_{1,Z}\|^{-2} * \cdots * \log \|s_{k-1,Z}\|^{-2} \wedge \delta_Y\right) \\ c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_{k-1}) \wedge \log \|s_k\|^{-2} \wedge \delta_Z &= i_* \left(c_1(\hat{L}_1|_Z) \wedge \cdots \wedge c_1(\hat{L}_{k-1}|_Z) \wedge [\log \|s_{k,Z}\|^{-2}]\right) \\ \log \|s_1\|^{-2} * \cdots * \log \|s_k\|^{-2} \wedge \delta_Z &= i_* \left(\log \|s_{1,Z}\|^{-2} * \cdots * [\log \|s_{k,Z}\|^{-2}]\right) \end{aligned}$$

up to K_1-chains on $|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_k)| \cap Z$. Note that in the brackets on the right hand sides, we perform operations of currents on Z. The induction hypothesis now applied on Z gives

$$\begin{aligned} &\log \|s_{1,Z}\|^{-2} * \cdots * \log \|s_{k-1,Z}\|^{-2} * [\log \|s_{k,Z}\|^{-2}] \\ &= \log \|s_{1,Z}\|^{-2} * \cdots * \log \|s_{k-1,Z}\|^{-2} \wedge \delta_Y + c_1(\hat{L}_1|_Z) \wedge \cdots \wedge c_1(\hat{L}_{k-1}|_Z) \wedge [\log \|s_{k,Z}\|^{-2}] \end{aligned}$$

up to K_1-chains on $|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_k)| \cap Z$. Note that we may assume $Y = \mathrm{div}(s_{k,Z})$. If we use the above identities, we deduce the claim. □

2.4.3 Definition. Let $s_0, \ldots, s_t$ be invertible meromorphic sections of hermitian line bundles $\hat{L}_0, \ldots, \hat{L}_t$ and let Z be a cycle on X of pure dimension t. We say that the *local height* $\lambda(Z)$ of Z with respect to $(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)$ is *well-defined* if

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset$$

and in this case, we set

$$\lambda(Z) := \lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z) = \left(\log\|s_0\|^{-2} * \cdots * \log\|s_t\|^{-2} \wedge \delta_Z\right)\left(\frac{1}{2}\right).$$

By linearity, we extend the definition to not necessarily pure dimensional cycles using the convention $\lambda(Z_i) = 0$ for all components with $\dim(Z_i) \neq t$.

2.4.4 Proposition. *The local height is multilinear and symmetric in the variables $(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)$, and linear in Z, under the hypothesis that all terms are well-defined.*

Proof: Only symmetry is not trivial. It follows from Proposition 2.3.15. □

2.4.5 Proposition. Let $s_0, \ldots, s_t$ be invertible meromorphic sections of hermitian line bundles $\hat{L}_0, \ldots, \hat{L}_t$ on X and let Z be a prime cycle of dimension t on X whose local height is well-defined. For $j = 0, \ldots, t$, let $s_{j,Z} := s_j|_Z$ whenever this is possible. If $Z \subset |\mathrm{div}(s_j)|$, then $s_{j,Z}$ denotes a non-trivial meromorphic section of $L_j|_Z$. Then

$$\lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z) = \lambda_{(\hat{L}_0|_Z,s_{0,Z}),\ldots,(\hat{L}_t|_Z,s_{t,Z})}(Z).$$

So we may compute the local height of Z really on Z which is useful in many proofs. The proof of the proposition is obvious from the definitions.

2.4.6 Proposition. *With the same hypothesis and notation as in Proposition 2.4.5, the induction formula*

$$\begin{aligned}\lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z) &= \lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_{t-1},s_{t-1})}\left(\mathrm{div}(s_{t,Z})\right) \\ &\quad - \int_Z \log\|s_{t,Z}\| c_1(\hat{L}_0) \wedge \cdots \wedge c_1(\hat{L}_{t-1})\end{aligned}$$

holds.

Proof: By Proposition 2.4.5, we may assume $Z = X$. Now Proposition 2.4.2 shows

$$\begin{aligned}\log\|s_0\|^{-2} * \cdots * \log\|s_{t-1}\|^{-2} * [\log\|s_t\|^{-2}] &= \log\|s_0\|^{-2} * \cdots * \log\|s_{t-1}\|^{-2} \wedge \delta_{\mathrm{div}(s_t)} \\ &\quad + c_1(\hat{L}_0) \wedge \cdots \wedge c_1(\hat{L}_{t-1}) \wedge [\log\|s_t\|^{-2}].\end{aligned}$$

Applying this to the constant $\frac{1}{2}$, we get the induction formula. □

2.4.7 Proposition. *Let $\varphi : X' \to X$ be a morphism of complex compact algebraic varieties and for $j = 1, \ldots, t$, let s_j be an invertible meromorphic section of a hermitian line bundle $\hat{L}_j$ on X such that $\varphi^*(s_j)$ is well-defined, i.e. no component of X' maps into $|\mathrm{div}(s_j)|$. Let Z' be a cycle on X' such that*

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap \varphi(|Z'|) = \emptyset.$$

Then

$$\lambda_{\varphi^*(\hat{L}_0,s_0),\ldots,\varphi^*(\hat{L}_t,s_t)}(Z') = \lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(\varphi_* Z').$$

Proof: Clearly, we may assume Z' prime. By Proposition 2.4.5, we reduce to the case $X' = Z'$ and φ surjective. We proceed by induction on t. The claim is obvious for $t = 0$. By induction formula, we have

$$\begin{aligned}\lambda_{\varphi^*(\hat{L}_0,s_0),\ldots,\varphi^*(\hat{L}_t,s_t)}(Z') &= \lambda_{\varphi^*(\hat{L}_0,s_0),\ldots,\varphi^*(\hat{L}_{t-1},s_{t-1})}\left(\mathrm{div}(\varphi^* s_t).Z'\right) \\ &\quad - \int_{Z'} \log\|\varphi^* s_t\| c_1(\varphi^*\hat{L}_0) \wedge \cdots \wedge c_1(\varphi^*\hat{L}_{t-1}). \end{aligned} \qquad (2.10)$$

By induction hypothesis and by

$$\varphi_* \left(\operatorname{div}(\varphi^* s_t).Z'\right) = \operatorname{div}(s_t).\varphi_* Z',$$

we get

$$\lambda_{\varphi^*(\hat{L}_0,s_0),\ldots,\varphi^*(\hat{L}_{t-1},s_{t-1})}\left(\operatorname{div}(\varphi^* s_t).Z'\right) = \lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_{t-1},s_{t-1})}\left(\operatorname{div}(s_t).\varphi_* Z'\right). \tag{2.11}$$

On the other hand, the transformation formula of integrals proves

$$\int_{Z'} \log \|\varphi^* s_t\| c_1(\varphi^* \hat{L}_0) \wedge \cdots \wedge c_1(\varphi^* \hat{L}_{t-1}) = \int_{\varphi_* Z'} \log \|s_t\| c_1(\hat{L}_0) \wedge \cdots \wedge c_1(\hat{L}_{t-1}). \tag{2.12}$$

Now (2.10)-(2.12) and the induction formula again prove the claim. □

2.4.8 Proposition. *Let $s_0, \ldots, s_t$ be invertible meromorphic sections of hermitian line bundles $\hat{L}_0, \ldots, \hat{L}_t$ on X such that the local height $\lambda(Z)$ of the t-dimensional cycle Z is well-defined. We assume that $c_1(\hat{L}_0) = 0$. Let Y be a cycle representing* $\operatorname{div}(s_1) \ldots \operatorname{div}(s_t).Z \in CH_0\left(|\operatorname{div}(s_1)| \cap \cdots \cap |\operatorname{div}(s_t)| \cap |Z|\right)$. *Then*

$$\lambda(Z) = -\log \|s_0(Y)\|$$

where the right hand side is defined by linearity in the components of the zero-dimensional cycle Y.

Proof: First, we check that $\log \|s_0(Y)\|$ is well-defined. If $|\operatorname{div}(s_1)| \cap \cdots \cap |\operatorname{div}(s_t)| \cap |Z|$ is empty, then Y is the zero cycle and hence $\log \|s_0(Y)\| = 0$. So we may assume that the intersection above is non-empty. Since $\lambda(Z)$ is well-defined, we conclude that $|\operatorname{div}(s_0)| \cap |Y| = \emptyset$, therefore $\log \|s_0(Y)\|$ makes sense. If Y' is another representative of $\operatorname{div}(s_1) \ldots \operatorname{div}(s_t).Z$, then there is a K_1-chain $\mathbf{f}$ on $|\operatorname{div}(s_1)| \cap \cdots \cap |\operatorname{div}(s_t)| \cap |Z|$ with $\operatorname{div}(\mathbf{f}) = Y - Y'$. We have

$$\begin{aligned} \log \|s_0(Y')\| - \log \|s_0(Y)\| &= \left(\log \|s_0\|^{-2} \wedge \delta_{\operatorname{div}(\mathbf{f})}\right)(\tfrac{1}{2}) \\ &= \left(\log \|s_0\|^{-2} * [\log |\mathbf{f}|^{-2}]\right)(\tfrac{1}{2}). \end{aligned}$$

By Proposition 2.3.16, the $*$-product corresponds to a K_1-chain on $|\operatorname{div}(s_0)| \cap \cdots \cap |\operatorname{div}(s_t)| \cap |Z| = \emptyset$, hence

$$\log \|s_0(Y)\| = \log \|s_0(Y')\|.$$

We conclude that $\log \|s_0(Y)\|$ is independent of the choice of the representative Y.

To prove the formula in the proposition, we may assume Z prime. By Proposition 2.4.5, we may compute the local height on Z, i.e.

$$\lambda(Z) = \left(\log \|s_{0,Z}\|^{-2} * \cdots * [\log \|s_{t,Z}\|^{-2}]\right)(\tfrac{1}{2}).$$

We may assume that Y has the Green current $g_Y := \log \|s_{1,Z}\|^{-2} * \cdots * [\log \|s_{t,Z}\|^{-2}]$ on Z (use Proposition 2.3.8). Then

$$\begin{aligned} \lambda(Z) &= \left(\log \|s_{0,Z}\|^{-2} * g_Y\right)(\tfrac{1}{2}) \\ &= \left(\log \|s_{0,Z}\|^{-2} \wedge \delta_Y\right)(\tfrac{1}{2}) \\ &= -\log \|s_{0,Z}(Y)\|. \end{aligned}$$

If Z is not contained in $|\mathrm{div}(s_0)|$, then $s_{0,Z} = s_0|_Z$ proves the claim. But if $Z \subset |\mathrm{div}(s_0)|$, then $\lambda(Z)$ well-defined implies that $|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset$. Hence Y is the zero cycle and

$$\log \|s_{0,Z}(Y)\| = 0 = \log \|s_0(Y)\|$$

proves the claim. □

2.4.9 Corollary. *If one hermitian line bundle has first Chern form zero, then the local height $\lambda(Z)$ does not depend on the metrics of the other line bundles.*

2.4.10 Corollary. *Let f be an invertible meromorphic function on X, let $s_1, \ldots, s_t$ be invertible meromorphic sections of the hermitian line bundles $\hat{L}_1, \ldots, \hat{L}_t$ on X and let Z be a t-dimensional cycle on X. We assume that*

$$|\mathrm{div}(f)| \cap |\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$

If we endow the trivial bundle with the trivial metric, then

$$\lambda_{(\hat{O}_X, f), (\hat{L}_1, s_1), \ldots, (\hat{L}_t, s_t)}(Z) = -\log |f(Y)|$$

for any representative Y of $\mathrm{div}(s_1) \ldots \mathrm{div}(s_t).Z \in CH_0\left(|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z|\right)$.

2.4.11 Proposition. *Let $s_0, \ldots, s_t$ be invertible meromorphic sections of hermitian line bundles $\hat{L}_0, \ldots, \hat{L}_t$ and let Z be a cycle. If $|\mathrm{div}(s_0)| \cap |Z| = \emptyset$, then*

$$\lambda(Z) = -\int_Z \log \|s_0\| c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_t).$$

Proof: This follows from symmetry and the induction formula. □

2.4.12 Proposition. *Let $s_0, \ldots, s_t$ be invertible meromorphic sections of hermitian line bundles $\hat{L}_0, \ldots, \hat{L}_t$ and let Z be a t-dimensional cycle such that the local height is well-defined. Let $\| \ \|'$ be a second smooth hermitian metric on L_0 and let ρ be the C^∞-function* $\log \|s_0\| / \log \|s_0\|'$ *on X as in Corollary 2.2.11. Then*

$$\begin{aligned} &\lambda_{(\hat{L}_0', s_0), (\hat{L}_1, s_1), \ldots, (\hat{L}_t, s_t)}(Z) - \lambda_{(\hat{L}_0, s_0), (\hat{L}_1, s_1), \ldots, (\hat{L}_t, s_t)}(Z) \\ &\qquad = \int_Z \rho \cdot c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_t). \end{aligned}$$

Proof: This follows from Proposition 2.4.11 with the same arguments as in Corollary 2.2.11. □

2.4.13 Definition. Let $\hat{L}$ be a hermitian line bundle on X. For compact manifolds, the notion of semipositive curvature form is well-known. For singular varieties, the metric of $\hat{L}$ is called *semipositive* if for every smooth variety X' and every morphism $\varphi : X' \to X$, the hermitian line bundle $\varphi^* \hat{L}$ has semipositive curvature form.

2.4.14 Corollary. *Under the hypothesis of Proposition 2.4.12, we additionaly assume that $\hat{L}_1, \ldots, \hat{L}_t$ have semipositive curvature forms and that Z is an effective cycle. Then*

$$\begin{aligned} \min_{z \in |Z|} \rho(z) \deg_{L_1, \ldots, L_t}(Z) &\leq \lambda_{(\hat{L}_0', s_0), (\hat{L}_1, s_1), \ldots, (\hat{L}_t, s_t)}(Z) - \lambda_{(\hat{L}_0, s_0), (\hat{L}_1, s_1), \ldots, (\hat{L}_t, s_t)}(Z) \\ &\leq \max_{z \in |Z|} \rho(z) \deg_{L_1, \ldots, L_t}(Z). \end{aligned}$$

Proof: By our semipositivity assumptions, we know that $c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_t)$ is a positive measure and the claim follows from

$$\int_Z c_1(\hat{L}_1) \wedge \cdots \wedge c_1(\hat{L}_t) = \deg_{L_1, \ldots, L_t}(Z).$$

□

2.4.15 Remark. Let $\mathbb{P}^n$ be the projective space over any algebraically closed field K. As it is usual in diophantine geometry, we always fix a set of coordinates $\mathbf{x} = [x_0 : \cdots : x_n]$ of $\mathbb{P}^n$. For a t-dimensional irreducible closed subvariety Y of $\mathbb{P}^n$, the *Chow form* $F_Y(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t)$ is a multihomogeneous polynomial of multidegree $(\deg(Y), \ldots, \deg(Y))$ in the dual coordinates $\boldsymbol{\xi}$ of $\boldsymbol{x}$. It is uniquely determined up to multiples by

$$\operatorname{div}(F_Y) = \{(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t) \in (\check{\mathbb{P}}^n)^{t+1} \mid Y \cap \boldsymbol{\xi}_0 \cap \cdots \cap \boldsymbol{\xi}_t = \emptyset\}$$

where we view $\boldsymbol{\xi}_j$ as a hyperplane in $\mathbb{P}^n$. Note that the line bundle $O_{\mathbb{P}^n \times \check{\mathbb{P}}^n}(1,1)$ has a canonical global section

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = x_0\xi_0 + \cdots + x_n\xi_n.$$

Let p (resp. $\check{p}_j$) be the projection of $\mathbb{P}^n \times (\check{\mathbb{P}}^n)^{t+1}$ to the factor $\mathbb{P}^n$ (resp. to the j-th factor $\check{\mathbb{P}}^n$ of $(\check{\mathbb{P}}^n)^{t+1}$). Then we get canonical global sections $G_j(\boldsymbol{x}, \boldsymbol{\xi})$ of $p^*O_{\mathbb{P}^n}(1) \otimes \check{p}_j^* O_{\check{\mathbb{P}}^n}(1)$ and we have

$$\operatorname{div}(F_Y) = \check{p}_* \left(p^*Y.\operatorname{div}(G_0) \ldots \operatorname{div}(G_t)\right) \tag{2.13}$$

where $\check{p}$ is the projection of $\mathbb{P}^n \times (\check{\mathbb{P}}^n)^{t+1}$ to $(\check{\mathbb{P}}^n)^{t+1}$ and the right hand side is a transversal intersection. By linearity, we may use (2.13) to define F_Z for all cycles Z.

2.4.16 Now let $K = \mathbb{C}$, then we have the Fubini-Study metric on $O_{\mathbb{P}^n}(1)$ given by

$$\|s(\boldsymbol{x})\| = \frac{|s(\boldsymbol{x})|}{\left(|x_0|^2 + \cdots + |x_n|^2\right)^{1/2}}$$

for any global section s. The corresponding metrized line bundle is denoted by $\bar{O}_{\mathbb{P}^n}(1)$. Let $s_0, \ldots, s_t$ be global sections of $O_{\mathbb{P}^n}(1)$ with dual coordinates $\boldsymbol{s}_0, \ldots, \boldsymbol{s}_t$. On the unit sphere $S^{2n+1} := \{\boldsymbol{x} \in \mathbb{C}^{n+1} \mid \|\boldsymbol{x}\| = 1\}$, we consider the Lebesque probability measure dP_j. On the product $S := (S^{2n+1})^{t+1}$, let dP be the product measure of the dP_j. Then the local height $\lambda(Z)$ of a t-dimensional cycle on $\mathbb{P}^n$ with respect to $(\bar{O}_{\mathbb{P}^n}(1), s_0), \ldots, (\bar{O}_{\mathbb{P}^n}(1), s_t)$ is given by

$$\lambda(Z) = \int_S \log |F_Z(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t)| \, dP(\boldsymbol{\xi}) - \log |F_Z(\boldsymbol{s}_0, \ldots, \boldsymbol{s}_t)| + \frac{t+1}{2} \deg(Z) \sum_{j=1}^{n} \frac{1}{j}. \tag{2.14}$$

This is a special case of the multiprojective situation described below.

2.4.17 Remark. Let $\mathbb{P} := \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_t}$ be a multiprojective space over an algebraically closed field K. Let $O_{\mathbb{P}}(e_j)$ be the pull-back of $O_{\mathbb{P}^{n_j}}(1)$, then

$$\mathbb{Z}^{t+1} \xrightarrow{\sim} \operatorname{Pic}(\mathbb{P}) \quad , \quad \boldsymbol{k} \mapsto O_{\mathbb{P}}(k_0, \ldots, k_t) := \otimes_{j=0}^{t} O_{\mathbb{P}}(e_j)^{\otimes k_j}.$$

For a line bundle $O_{\mathbb{P}}(\boldsymbol{k})$, Remark 2.4.15 provides us with a canonical global section

$$G(\boldsymbol{x}, \boldsymbol{\xi}) \in \Gamma(\mathbb{P} \times \check{\mathbb{P}}, O_{\mathbb{P} \times \check{\mathbb{P}}}(\boldsymbol{k}, \boldsymbol{k})).$$

Let $p, \check{p}$ be the projection of $\mathbb{P} \times \check{\mathbb{P}}$ to the factors. Similarly as above, we get canonical global sections G_j of $p^*\mathcal{O}_{\mathbb{P}}(e_j) \otimes \check{p}^*\mathcal{O}_{\check{\mathbb{P}}}(e_j)$ for $j = 0, \ldots, t$. For a t-dimensional cycle Z of $\mathbb{P}$, *the Chow form* F_Z is defined up to multiples by

$$\operatorname{div}(F_Z) = \check{p}_* \left(p^*Z.\operatorname{div}(G_0) \ldots \operatorname{div}(G_t)\right). \tag{2.15}$$

Again, on the right hand side we have a transversal intersection. The Chow form is a multihomogeneous polynomial $F_Z(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t)$ in the dual coordinates $\boldsymbol{\xi}_j$ of $\mathbb{P}^{n_j}$ and of multidegree $(\delta_0(Z), \ldots, \delta_t(Z))$ for

$$\delta_j(Z) := \deg_{O_{\mathbb{P}}(e_0), \ldots, O_{\mathbb{P}}(e_{j-1}), O_{\mathbb{P}}(e_{j+1}), \ldots, O_{\mathbb{P}}(e_t)}(Z).$$

2.4.18 Example. For $K = \mathbb{C}$, we denote by $\bar{O}_{\mathbb{P}}(e_j)$ the metrized line bundle on $\mathbb{P}$ endowed with the pull-back of the Fubini-Study metric. Let s_j be a global section of $O_{\mathbb{P}}(e_j)$ for $j = 0, \ldots, t$. Again, we consider the unit sphere S^{2n_j+1} in $\mathbb{C}^{n_j+1}$ with the Lebesque probability measure dP_j. Let $S := S^{2n_0+1} \times \cdots \times S^{2n_t+1}$ endowed with the product measure of the dP_j. By [Gu2], Proposition 1.7, the local height of Z with respect to $(\bar{O}_{\mathbb{P}}(e_0), s_0), \ldots, (\bar{O}_{\mathbb{P}}(e_t), s_t)$ is given by

$$\lambda(Z) = \int_S \log |F_Z(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t)| \, dP(\boldsymbol{\xi}) - \log |F_Z(\boldsymbol{s}_0, \ldots, \boldsymbol{s}_t)| + \frac{1}{2} \sum_{i=0}^{t} \delta_i(Z) \sum_{j=1}^{n_i} \frac{1}{j}. \tag{2.16}$$

2.4.19 Remark. By multilinearity and functoriality, this covers all local heights on any multiprojective space. We demonstrate it by deducing the identity (2.14) from (2.16). In the situation of Example 2.4.16, we choose $\mathbb{P} := (\mathbb{P}^n)^{t+1}$. We have a canonical morphism $\varphi : \mathbb{P}^n \to \mathbb{P}$ with $\varphi^* O_{\mathbb{P}}(e_j) \cong O_{\mathbb{P}^n}(1)$. Using

$$\lambda(\varphi_* Z) = \lambda(Z)$$

and

$$F_Z(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t) = F_{\varphi_* Z}(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t)$$

easily deduced from (2.13) and (2.15), we get formula (2.14).

2.4.20 Now we compare the local heights defined in section 2.2 with the local heights from section 2.4. We still assume that X is a complex compact algebraic variety. Let $s_0, \ldots, s_t$ be invertible meromorphic sections of hermitian line bundles $\hat{L}_0, \ldots, \hat{L}_t$. For $j = 0, \ldots, t$, we get a hermitian pseudo-divisor $(\hat{L}_j, Y_j, s_j)$ where Y_j is the support of the Cartier-divisor $\mathrm{div}(s_j)$. Let Z be a cycle on X with $Y_0 \cap \cdots \cap Y_t \cap |Z| = \emptyset$. From Definition 2.2.7, we get a local height $\lambda_{(\hat{L}_0, Y_0, s_0), \ldots, (\hat{L}_t, Y_t, s_t)}(Z)$ which we are going to compare with the local height $\lambda_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(Z)$ from Definition 2.4.3.

2.4.21 Proposition. *With the notation introduced above, we have*

$$\lambda_{(\hat{L}_0, Y_0, s_0), \ldots, (\hat{L}_t, Y_t, s_t)}(Z) = \lambda_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(Z).$$

Proof: By Proposition 2.4.5 and Corollary 2.2.9, we may assume that $X = Z$ prime. By Chow's lemma, X is birationally covered by an irreducible projective variety. By Hironaka, the singularities of the latter may be resolved by finitely many blow up's. Using Proposition 2.4.7 and Corollary 2.2.9, we conclude that we may assume that $X = Z$ is an irreducible smooth projective variety. The idea is now to reduce to the case of proper intersection where it is known, that the $*$-product of logarithmic Green forms in the sense of Burgos is the same as the $*$-product of Green currents from Gillet-Soulé. This was explained in detail in Remark 2.3.19.

Let g_Y be a logarithmic Green form for an cycle Y on X and let s be an invertible meromorphic section of a hermitian line bundle $\hat{L}$ on X. To prove the proposition, it is enough to show that $\log \|s\|^{-2} * g_Y$ may be represented in $\tilde{G}_{|\mathrm{div}(s)| \cap |Y|}(X)$ by a logarithmic Green form $g_{Y'}$ for a representative Y' of $\mathrm{div}(s').Y \in CH\,(|\mathrm{div}(s')| \cap |Y|)$ such that the associated Green current satisfies

$$[g_{Y'}] \equiv \log \|s\|^{-2} * [g_Y].$$

To prove it, we may assume that Y is prime. If $|\mathrm{div}(s)|$ and Y intersect properly, the claim is known. So we may assume $Y \subset |\mathrm{div}(s)|$. There is an invertible meromorphic section s' of L such that $|\mathrm{div}(s')|$ intersects Y properly. By definition, we may assume

$$\log \|s\|^{-2} * [g_Y] = \log \|s'\|^{-2} * [g_Y].$$

It is also clear that $\log \|s'\|^{-2} * g_Y$ is a logarithmic Green form for $Y' := \mathrm{div}(s').Z$ (Remark 2.1.18). Hence it is enough to show that $\log \|s'\|^{-2} * g_Y$ and $\log \|s\|^{-2} * g_Y$ induce the same classes in $\tilde{G}_Y(X)$. By commutativity, we may consider $g_Y * \log \|s'\|^{-2}$ and $g_Y * \log \|s\|^{-2}$. To compute this $*$-products in $\tilde{G}_Y(X)$, we may use the partitions $\sigma_{Y,|\mathrm{div}(s')|} = \sigma_{Y,|\mathrm{div}(s)|} = 0, \sigma_{|\mathrm{div}(s')|,Y} = \sigma_{|\mathrm{div}(s)|,Y} = 1$. We conclude that

$$g_Y * \log \|s'\|^{-2} \equiv dd^c \left(\sigma_{Y,|\mathrm{div}(s')|} g_Y\right) \wedge \log \|s'\|^{-2} + \sigma_{|\mathrm{div}(s')|,Y} g_Y \wedge c_1(\hat{L}) \equiv g_Y \wedge c_1(\hat{L})$$

and

$$g_Y * \log \|s\|^{-2} \equiv dd^c \left(\sigma_{Y,\mathrm{div}(s)|} g_Y\right) \wedge \log \|s\|^{-2} + \sigma_{|\mathrm{div}(s)|,Y} g_Y \wedge c_1(\hat{L}) \equiv g_Y \wedge c_1(\hat{L})$$

in $\tilde{G}_Y(X)$ proving the claim. □

2.4.22 Remark. We have worked on compact algebraic varieties to ensure that the restriction of a line bundle to a prime cycle has a non-trivial meromorphic section. To deal with local heights, we can always reduce to irreducible compact spaces (cf. Proposition 2.4.5).

There is a natural generalization to Moishezon spaces. An irreducible compact complex space X is called a Moishezon space, if the field of meromorphic functions on X has transcendence degree equal to $\dim(X)$. By [Moi], Theorem 4, the cruical property above also holds, i.e. the restriction of every holomorphic line bundle to an irreducible closed analytic subset Y has a non-trivial meromorphic section on Y. Note that for a Moishezon space, every irreducible closed analytic subset Y is also a Moishezon space ([Moi], Theorem 3). Hence all results and proofs from section 3 and 4 hold also for Moishezon spaces (without any algebraicity assumptions). To extend the proof of Proposition 2.4.21, one has to note that for every Moishezon space X, there is a birational proper morphism from an irreducible complex algebraic variety onto X ([Moi], Theorem 7).

Chapter 3

Operations of Divisors in the Non-Archimedean Case

3.1 Divisors on Rigid Analytic Varieties

Let K be a field with a non-trivial non-archimedean complete absolute value $| \ |$.

First, we recall the basic notions for rigid analytic varieties. Then we give a proper intersection product of Cartier divisors with cycles having similar properties as for algebraic varieties (cf. section 1.2). The basic reference for rigid analytic varieties is [BGR] and the proper intersection product is from [Gu3].

3.1.1 The *Gauss norm* of a polynomial is defined by

$$\left|\sum a_{\nu_1\dots\nu_n} x_1^{\nu_1}\dots x_n^{\nu_n}\right| = \max_{\nu_1\dots\nu_n} |a_{\nu_1\dots\nu_n}|.$$

The completion of the polynomial ring $K[x_1,\dots,x_n]$ with respect to the Gauss norm is called the *Tate algebra* $K\langle x_1,\dots,x_n\rangle$. A quotient of a Tate algebra is called a *K-affinoid algebra*. They are noetherian ([BGR], Proposition 6.1.1/3).

3.1.2 Definition. A *K-affinoid variety* $\mathrm{Sp}\mathcal{A}$ is a pair $(\mathrm{Max}\mathcal{A},\mathcal{A})$, where $\mathcal{A}$ is a K-affinoid algebra and $\mathrm{Max}\mathcal{A}$ is the spectrum of maximal ideals. A morphism of K-affinoid varieties is induced by a homomorphism of the corresponding K-affinoid algebras in the opposite direction.

3.1.3 Remark. The closed unit ball $\mathbb{B}^n$ is a K-affinoid variety with the Tate algebra. If $\mathcal{A}$ is the K-affinoid algebra given as the quotient of $K\langle x_1,\dots,x_n\rangle$ by the ideal I, then the corresponding K-affinoid variety may be viewed as a subset of the closed unit ball given by finitely many generators of I.

3.1.4 The supremum seminorm on a K-affinoid algebra $\mathcal{A}$ is defined by

$$|a|_{\sup} := \sup_{x\in\mathrm{Sp}\mathcal{A}} |a(x)|$$

which is a norm if and only if $\mathcal{A}$ is reduced ([BGR], Proposition 6.2.1/4). Let

$$\begin{aligned} \mathcal{A}^\circ &:= \{a\in\mathcal{A} : |a|_{\sup}\le 1\}, \\ \mathcal{A}^{\circ\circ} &:= \{a\in\mathcal{A} : |a|_{\sup}< 1\}, \\ \tilde{\mathcal{A}} &:= \mathcal{A}^\circ/\mathcal{A}^{\circ\circ}. \end{aligned}$$

The *reduction* of the K-affinoid variety $X=\mathrm{Sp}\mathcal{A}$ is the affine scheme $\tilde{X} := \mathrm{Spec}\tilde{\mathcal{A}}$ of finite type over the reduction $\tilde{K} = K^\circ/K^{\circ\circ}$ of K.

3.1.5 On a K-affinoid variety $X = \mathrm{Sp}\mathcal{A}$, there is a *G-topology* (Grothendieck-topology). A basis of this G-topology consists of rational domains

$$\{x \in X : |f_i(x)| \leq 1, |g_j(x)| \geq 1, i = 1, \dots, m, j = 1, \dots, n\}$$

where $f_1, \dots, f_m, g_1, \dots, g_n \in \mathcal{A}$. For details, we refer to [BGR]. We just note that only finite unions of rational domains are admissible open in the G-topology. This allows us to define a canonical sheaf of rings $\mathcal{O}_X$ on the G-topology of X with $\mathcal{O}_X(X) = \mathcal{A}$.

3.1.6 A locally G-ringed space is a G-topological space X with a sheaf $\mathcal{O}_X$ of K-algebras such that the stalks are local rings. A *rigid analytic variety* is a locally G-ringed space which has an admissible open covering by K-affinoid varieties.

3.1.7 Example. The *affine space* $\mathbb{A}^n$ is obtained by gluing the closed balls with center 0 and radius r with varying r in the value group. Such a closed ball is by given by the K-affinoid algebra $K\langle r^{-1}x_1, \dots, r^{-1}x_n\rangle$. For an affine scheme Y of finite type over K viewed as a closed subscheme of $\mathbb{A}^n$, the associated K-affinoid variety Y^{an} is the closed analytic subvariety of the rigid analytic version of $\mathbb{A}^n$ given by the same set of equations. If X is a scheme locally of finite type over K, then a gluing process along affine open subsets defines the rigid analytic variety X^{an} associated to X. Note that X and X^{an} have the same $\bar{K}$-rational points.

3.1.8 Definition. A *cycle* on an rigid analytic variety X is a locally finite formal sum of irreducible closed analytic subsets Y of X. We write $Z = \sum n_Y Y$ for a cycle and the irreducible closed analytic subsets Y with multiplicities $n_Y \neq 0$ are called the *components* of Z. The cycles are graded by dimension, a t-dimensional cycle has only components of dimension t.

3.1.9 Up to now, X denotes a rigid analytic variety over K. A *meromorphic function* f on X is locally given on an open K-affinoid variety $\mathrm{Sp}\mathcal{A}$ by $f = a/s$, where $a, s \in \mathcal{A}$ and s is not a zero-divisor. If a is also not a zero-divisor, then f is called an *invertible meromorphic function.*

3.1.10 Definition. A *Cartier divisor* D on X consists of the data $\{U_\alpha, \gamma_\alpha\}_{\alpha \in I}$, where $\{U_\alpha\}_{\alpha \in I}$ is a covering of X, admissible with respect to the G-topology, and γ_α is an invertible meromorphic function on X such that $\gamma_\alpha/\gamma_\beta$ is invertible in $\mathcal{O}_X(U_\alpha \cap U_\beta)$ for every $\alpha, \beta \in I$. Two Cartier divisors $\{U_\alpha, \gamma_\alpha\}_{\alpha \in I}$ and $\{U'_\beta, \gamma'_\beta\}_{\beta \in J}$ are considered to be equal if $\gamma_\alpha/\gamma'_\beta$ is invertible in $\mathcal{O}_X(U_\alpha \cap U'_\beta)$ for all α, β.

3.1.11 Remark. Let L be a line bundle on X and let s be an invertible meromorphic section of L, i.e. there is a covering $\{U_\alpha\}_{\alpha \in I}$ of X, admissible with respect to the G-topology, and trivializations of L over U_α such that s is given by the invertible meromorphic functions γ_α on U_α. We conclude easily that $\{U_\alpha, \gamma_\alpha\}_{\alpha \in I}$ is a Cartier divisor $\mathrm{div}(s)$ on X. Conversely, if $D = \{U_\alpha, \gamma_\alpha\}_{\alpha \in I}$ is a Cartier divisor on X, the subsheaf of the sheaf of meromorphic functions which is generated on U_α by γ_α^{-1} is the sheaf of sections of a line bundle L and 1 gives rise to an invertible meromorphic section s of L with corresponding Cartier divisor $D = \mathrm{div}(s)$.

3.1.12 Locally, X is equal to $\mathrm{Sp}\mathcal{A}$ for a K-affinoid algebra $\mathcal{A}$. For an invertible meromorphic function γ on the noetherian algebra $\mathcal{A}$, we can define the Weil divisor $\mathrm{cyc}(\mathrm{div}(\gamma))$ as usual: If $\gamma \in \mathcal{A}$, then the multiplicity in the prime divisor Y of $\mathrm{Sp}\mathcal{A}$ is the length of the artinian $\mathcal{A}_\wp$-module $\mathcal{A}_\wp/\langle\gamma\rangle$ where $\wp$ is the prime ideal of Y. In general, γ is the quotient of two such functions and we proceed by linearity.

If $\{U_\alpha, \gamma_\alpha\}_{\alpha \in I}$ is a Cartier divisor on X, then locally defined Weil divisors $\mathrm{cyc}(\mathrm{div}(\gamma_\alpha))$ patch together to a divisor $\mathrm{cyc}(D)$ on X called the *Weil divisor* of D. For details, see ([Gu3], 2.5).

3.1.13 Next, we are going to define the push-forward of a cycle with respect to a proper morphism $\varphi : X' \to X$ of rigid analytic varieties over K (cf. [BGR], 9.6.2). By linearity, it is enough to consider a prime cycle Y' on X'. By the proper mapping theorem ([Kie], Satz 4.1), the range $Y = \varphi(Y')$ is a closed analytic subset of X. If $\dim(Y) = \dim(Y')$, then outside a lower dimensional closed analytic subset W of X, the restriction ψ of φ to Y' is finite and we define

$$\varphi_*(Y') := \deg(\psi) Y$$

where $\deg(\psi)$ is the degree of this finite map. If $\dim(Y) = \dim(Y')$, then we set $\varphi_*(Y') = 0$. For details, see ([Gu3], 2.6.

3.1.14 Let D be a Cartier divisor on X. The *support* $|D|$ of D is the complement of the set of points $x \in X$ which have a neighbourhood U where D is given by an invertible function in $\mathcal{O}_X(U)$. Clearly, the support $|D|$ is a closed analytic subset of X. The support of a cycle Z is the union of its components.

We say that D *intersects* Z *properly* if no component of Z is contained in $|D|$. For generalization to more than one Cartier divisor, we refer to Definition 5.3.4.

3.1.15 Let $\varphi : X' \to X$ be a morphism of rigid analytic spaces over K and let D be a Cartier divisor on X. The *pull-back* $\varphi^*(D)$ is well-defined if D may be given by $\{U_\alpha, a_\alpha/b_\alpha\}_{\alpha \in I}$ where a_α, b_α are not zero-divisors in $\mathcal{O}_X(U_i)$ and where $a_\alpha \circ \varphi, b_\alpha \circ \varphi$ are not zero-divisors in $\mathcal{O}_{X'}(\varphi^{-1}U_\alpha)$. Then we set

$$\varphi^* D = \left\{ \varphi^{-1} U_\alpha, \frac{a_\alpha \circ \varphi}{b_\alpha \circ \varphi} \right\}_{\alpha \in I}.$$

If X' is reduced, then $\varphi^*(D)$ is well-defined if and only if no irreducible component of X' is mapped into $|D|$. Similarly, we define the pull-back of invertible meromorphic sections.

3.1.16 Definition. Let D be a Cartier divisor intersecting Z properly on X. To define the *proper intersection product* $D.Z$, we proceed by linearity, i.e. we may assume that Z is a prime cycle. Then D restricts to a well-defined Cartier divisor $D|_Z$ on Z (see pull back above) and we define

$$D.Z := \operatorname{cyc}(D|_Z)$$

as a cycle on X.

3.1.17 Proposition. *Let $\varphi : X' \to X$ be a morphism of rigid analytic varieties and let D be a Cartier divisor on X such that $\varphi^*(D)$ is well-defined. Let Z' be a prime cycle on X' such that $\varphi|_{Z'}$ is a proper morphism and such that D intersects $\varphi(Z')$ properly. Then the projection formula*

$$\varphi_*(\varphi^*(D).Z') = D.\varphi_*(Z')$$

holds.

Proof: This follows from ([Gu3], Proposition 2.10a)). □

3.1.18 Proposition. *Let D and D' be Cartier divisors on X and let Z be a cycle on X such that D, D' intersect Z properly. Then we have the commutativity law*

$$D.D'.Z = D'.D.Z.$$

Proof: This follows from ([Gu3], Proposition 2.11) applied to every component Z_j. □

3.1.19 For every closed analytic subvariety W of X, the *associated cycle* $\operatorname{cyc}(W)$ is defined as a linear combination of the irreducible components of W by the following local construction. If $U = \operatorname{Sp}\mathcal{A}$ is an admissible open K-affinoid subspace of X, then there is a one-to-one correspondence between closed analytic subvarieties of U and closed subschemes of $\operatorname{Spec}\mathcal{A}$ and we define $\operatorname{cyc}(W \cap U)$ by the corresponding cycle on the noetherian $\operatorname{Spec}\mathcal{A}$. The construction patches together to give a well-defined cycle $\operatorname{cyc}(W)$ on U (cf. [Gu3], 2.7).

3.1.20 Let $\varphi : X' \to X$ be a flat morphism of rigid analytic varieties, i.e. $\mathcal{O}_{X',x'}$ is a flat $\mathcal{O}_{X,\varphi(x')}$-module for all $x' \in X'$. Then the *pull-back* $\varphi^*(Y)$ of an irreducible closed analytic subset Y of X is defined as the cycle associated to the inverse image of Y in the sense of closed analytic subvarieties. By linearity, we extend the definition to all cycles on X.

3.1.21 Proposition. *Let us consider a Cartesian diagram*

$$\begin{tikzcd} X_2' \arrow[r, "\psi'"] \arrow[d, "\varphi'"] & X_2 \arrow[d, "\varphi"] \\ X_1' \arrow[r, "\psi"] & X_1 \end{tikzcd}$$

of rigid analytic varieties with φ proper and ψ flat. Then φ' is proper, ψ' is flat and $\psi^* \circ \varphi_* = (\varphi')_* \circ (\psi')^*$ holds on cycles of X_2.

Proof: We refer to [Gu3], Proposition 2.12. □

3.1.22 Proposition. *Let $\varphi : X' \to X$ be a flat morphism of rigid analytic varieties, let D be a Cartier divisor on X such that $\varphi^*(D)$ is well-defined and let Z be a cycle on X intersecting D properly. Then $\varphi^*(D)$ intersects $\varphi^*(Z)$ properly on X' and*

$$\varphi^*(D.Z) = \varphi^*(D).\varphi^*(Z).$$

Proof: This is shown in [Gu3], Proposition 2.10. □

3.1.23 Remark. Let X be a scheme locally of finite type over K. Obviously, Cartier divisors D and cycles Z give rise to corresponding Cartier divisors D^{an} and cycles Z^{an} on X^{an}. This process is compatible with forming Weil divisors, proper push-forward, flat pull-back, forming the associated cycles and proper intersection products. For details, we refer to ([Gu3], §6).

3.2 Divisors on Formal Analytic Varieties

In this section, K is an algebraically closed field with a non-trivial non-archimedean complete absolute value $|\ |$.

From arithmetic intersection theory, one knows that the analogue of complex analysis at non-archimedean places is intersection theory on models over the discrete valuation rings. Since we want to deal with arbitrary valuation rings K° of height 1 which are not noetherian if the valuation ring is not discrete, we have to use valuation theory instead.

The objects best suited for this purpose are reduced formal analytic varieties over K. They may be seen as rigid analytic varieties ("the generic fibre") with a coarser topology ("the integral structure") giving rise to a reduction ("the special fibre"). Cartier divisors and cycles are defined similarly as on K°-models in algebraic geometry, for vertical cycles however, we allow coefficients from the value group. The reason is that the order of a function in an irreducible component of the special fibre is given by the supremum norm. This allows to define a generically proper intersection product of a Cartier divisor with a cycle with all the properties, we expect from algebraic geometry (cf. section 1.2).

The assumption K algebraically closed is made for simplicity, for generalizations, we refer to section 3.3. There, we also remove the assumption that our underlying spaces have to be reduced which does not harm unless we want to consider flat pull-backs of cycles. The material is borrowed from [Gu3], sections 3-5, which the reader should consult for proofs.

3.2.1 For a K-affinoid algebra $\mathcal{A}$, one has a reduction map $\pi : \mathrm{Sp}\mathcal{A} \to \mathrm{Spec}\tilde{\mathcal{A}}$. A subset of the K-affinoid variety $X = \mathrm{Sp}\mathcal{A}$ is called *formal open* if it is the inverse image of an open subset of $\tilde{X}$. This defines a topology on X called the *formal topology.* Since the G-topology is finer than the formal topology, the sheaf $\mathcal{O}_X$ restricts to define a ringed space Spf$\mathcal{A}$ on the formal topology of X. We call Spf$\mathcal{A}$ a *formal K-affinoid variety.* Morphisms of formal K-affinoid varieties are induced by homomorphisms of K-affinoid algebras. Note that a formal K-affinoid variety has not to be a locally ringed space.

3.2.2 A *formal analytic variety* over K is a pair $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, where $\mathfrak{X}$ is a topological space and $\mathcal{O}_{\mathfrak{X}}$ is a sheaf of K-algebras which is locally isomorphic to formal K-affinoid varieties. The latter are called *formal open affinoid subspaces* of $\mathfrak{X}$. Their reductions paste together to the *reduction* $\tilde{\mathfrak{X}}$ of $\mathfrak{X}$ which is a reduced scheme locally of finite type over $\tilde{K}$.

3.2.3 A *morphism* of formal analytic varieties over K is a morphism of K-ringed spaces inducing locally morphisms of formal K-affinoid varieties. We can glue the formal open affinoid subspaces in the sense of rigid geometry to define a rigid analytic variety $\mathfrak{X}^{an}$ associated to the given formal analytic variety $\mathfrak{X}$. We call $\mathfrak{X}^{an}$ the *generic fibre* of $\mathfrak{X}$. Often, we denote it by X. Note that $\mathfrak{X}$ and $\mathfrak{X}^{an}$ have the same underlying set but $\mathfrak{X}^{an}$ has a G-topology which is finer than the topology of $\mathfrak{X}$. Every morphism $\varphi : \mathfrak{X} \to \mathfrak{Y}$ of formal analytic varieties induces a morphism $\varphi^{an} : \mathfrak{X}^{an} \to \mathfrak{Y}^{an}$ of rigid analytic varieties.

3.2.4 Example. Let $\mathfrak{X}$ be an affine scheme of finite type over the valuation ring K°. Then $\mathfrak{X} = \mathrm{Spec}K^\circ[x_1, \ldots, x_n]/I$ for an ideal I and we have the associated formal K-affinoid variety $\mathfrak{X}^{f-an} = \mathrm{Spf}K\langle x_1, \ldots, x_n\rangle/\langle I\rangle$. Note that the K-rational points of $\mathfrak{X}^{f-an}$ are the K-rational points of $\mathfrak{X}$ contained in the closed unit ball $\mathbb{B}^n$. If $\mathfrak{X}$ is any scheme locally of finite type over K, then we use these local construction to glue a formal analytic variety $\mathfrak{X}^{f-an}$ over K. This construction is functorial. The points of the generic fibre of $\mathfrak{X}^{f-an}$ are the K°-integral points of X.

3.2.5 Definition. Let $\mathfrak{X}$ be a reduced formal analytic variety over K. A *horizontal cycle* on $\mathfrak{X}$ is just a cycle on the generic fibre $\mathfrak{X}^{an}$. A *vertical cycle* is a locally finite formal sum $\sum \lambda_W W$ where W ranges over all irreducible closed subsets of the reduction $\tilde{\mathfrak{X}}$ and the multiplicities λ_W are from the value group $\log|K^\times|$. Note that the latter are not in $\mathbb{Z}$ if the absolute value is not discrete. A *cycle* on $\mathfrak{X}$ is a formal sum of a horizontal cycle and a vertical cycle.

3.2.6 Remark. We grade the cycles by an analogue of the *relative dimension*, i.e. the horizontal cycles have their usual dimension and the vertical cycles have relative dimension equal to their dimension in the special fibre minus 1. In particular, a K-rational point of the special fibre has relative dimension -1. This will be convenient for working with models.

3.2.7 The notions of Cartier divisors and line bundles on formal analytic varieties come from the corresponding notions on formal schemes (cf. [Gu3], Definition 2.2). A *Cartier divisor* D on a formal analytic variety $\mathfrak{X}$ over K is given by the data $\{\mathcal{U}_\alpha, \gamma_\alpha\}_{\alpha\in I}$ where $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ is a formal open covering of $\mathfrak{X}$ and γ_α is an invertible meromorphic function on $\mathcal{U}_\alpha$ such that

$$|(\gamma_\alpha/\gamma_\beta)\,(x)| = 1$$

for every $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Two Cartier divisors D, D' are considered to be equal if

$$\left|\left(\gamma_\alpha/\gamma'_\beta\right)(x)\right| = 1$$

for every $x \in \mathcal{U}_\alpha \cap \mathcal{U}'_\beta$. The *support* $|D|$ has a horizontal part $|D^{an}|$, where D^{an} is the corresponding Cartier divisor on $\mathfrak{X}^{an}$, and a vertical part: Let $\mathcal{U}$ be the set of points $x \in \mathfrak{X}$ which have a formal K-affinoid neighbourhood Spf$\mathcal{A}$ where D is given by an invertible element of $\mathcal{A}^\circ$. Then $\mathcal{U}$ is formal open and $\tilde{\mathfrak{X}} \setminus \tilde{\mathcal{U}}$ is called the vertical part of $|D|$. Note that D restricts to a Cartier divisor on $\tilde{\mathcal{U}}$.

3.2.8 A *line bundle* $\mathcal{L}$ on the formal analytic variety $\mathfrak{X}$ over K is given by a formal open covering $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ and transition functions $g_{\alpha\beta} \in \mathcal{O}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ such that $|g_{\alpha\beta}(x)| = 1$ for all

$x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. A *meromorphic section* s of $\mathcal{L}$ is given by meromorphic function γ_α on $\mathcal{U}_\alpha$ such that $\gamma_\alpha = g_{\alpha\beta}\gamma_\beta$. We call s *invertible* if all γ_α are invertible meromorphic functions. Then $\{\mathcal{U}_\alpha, \gamma_\alpha\}_{\alpha\in I}$ is called the Cartier divisor associated to the invertible meromorphic section s.

Conversely, if $D = \{\mathcal{U}_\alpha, \gamma_\alpha\}_{\alpha\in I}$ is a Cartier divisor on $\mathfrak{X}$, then the transition functions $g_{\alpha\beta} := \gamma_\alpha/\gamma_\beta$ define a line bundle $O(D)$ on $\mathfrak{X}$ with canonical invertible meromorphic section s_D given by γ_α on the trivialization over $\mathcal{U}_\alpha$. Clearly, the Cartier divisor associated to s_D is D. Note that every line bundle on $\mathfrak{X}$ gives canonically rise to a line bundle $\tilde{\mathcal{L}}$ on the reduction $\tilde{\mathfrak{X}}$.

3.2.9 Let $\mathfrak{X} = \mathrm{Spf}\mathcal{A}$ be a reduced formal K-affinoid variety over K. For an irreducible component W of the reduction $\tilde{\mathfrak{X}}$ and $a \in \mathcal{A}$, let

$$|a(W)| := \sup |a(x)|$$

where x ranges over all points of $\mathfrak{X}$ with reduction not contained in another irreducible component of $\tilde{\mathfrak{X}}$ than W. This induces a multiplicative norm on $\mathcal{A}$. If a is not a zero-divisor, then $|a(W)| \neq 0$ and we define the *order* of a in W by

$$\mathrm{ord}(a, W) := -\log |a(W)|.$$

3.2.10 If D is a Cartier divisor on a reduced formal analytic variety over K and W is an irreducible component of $\tilde{\mathfrak{X}}$, then we use the local construction in 3.2.9 to define the order of D in W. More precisely, we choose formal open affinoid subspaces $\mathcal{U} = \mathrm{Spf}\mathcal{A}$ of $\mathfrak{X}$ such that D is given on $\mathcal{U}$ by a/b for $a, b \in \mathcal{A}$ which are not zero-divisors and such that $\tilde{\mathcal{U}} \cap W \neq \emptyset$. It is shown in [Gu3], Lemma 3.9, that

$$\mathrm{ord}(D, W) := \mathrm{ord}(a, W \cap \tilde{\mathcal{U}}) - \mathrm{ord}(b, W \cap \tilde{\mathcal{U}})$$

is a well-defined real function on Cartier-divisors.

3.2.11 Definition. The *Weil divisor* $\mathrm{cyc}(D)$ associated to a Cartier divisor on a reduced formal analytic variety $\mathfrak{X}$ over K is defined by

$$\mathrm{cyc}(D) = \mathrm{cyc}(D^{an}) + \sum_W \mathrm{ord}(D, W)W$$

where D^{an} is the Cartier divisor on the generic fibre $\mathfrak{X}^{an}$ induced by D and W ranges over all irreducible components of $\tilde{\mathfrak{X}}$.

3.2.12 A morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ of reduced formal analytic varieties is called *proper* if and only if φ^{an} is a proper morphism of rigid analytic varieties and $\tilde{\varphi}$ is a proper morphism of schemes. Note that if the formal analytic varieties are quasicompact and if φ^{an} is proper, then $\tilde{\varphi}$ and hence φ is proper. This is a result of Lütkebohmert, see [Gu3], Remark 3.14. Let $\mathfrak{Z}'$ be a cycle on $\mathfrak{X}'$ with horizontal part Z' and vertical part $\mathfrak{Z}'_v$. Then the *push-forward* of $\mathfrak{Z}'$ with respect to the proper morphism φ is defined by

$$\varphi_*(\mathfrak{Z}') = \varphi_*^{an}(Z') + \tilde{\varphi}_*(\mathfrak{Z}'_v).$$

3.2.13 Let $\mathfrak{X}$ be a reduced formal analytic variety over K with generic fibre $X = \mathfrak{X}^{an}$ and let Y be a closed analytic subvariety of X. Then Y gives rise to a closed formal analytic subvariety $\bar{Y}$ of $\mathfrak{X}$. We just use the same ideal of vanishing on formal open affinoid subspaces. It is clear that the generic fibre of $\bar{Y}$ is Y. If we use this operation on the components, we can associate to every cycle Z on X a canonical horizontal cycle $\bar{Z}$ determined by the property $Z = \bar{Z}^{an}$.

3.2.14 Definition. A K_1-*chain* on $\tilde{\mathfrak{X}}$ is a formal linear combination $\sum_W \lambda_W f_W$ where W ranges over all irreducible closed subsets of $\tilde{\mathfrak{X}}$, $\lambda_W \in \log|K^\times|$ is non-zero only for finitely many

W and f_W is a non-zero rational function on W. By linearity, we define $\mathrm{div}(\mathbf{f})$ for a K_1-chain $\mathbf{f}$ on $\tilde{\mathfrak{X}}$ viewed as a vertical cycle on $\mathfrak{X}$.

3.2.15 Let D be a Cartier divisor on the reduced formal analytic variety $\mathfrak{X}$ and let Y be a horizontal prime cycle on $\mathfrak{X}$. We say that D *intersects* Y *properly in the generic fibre* if D^{an} intersects Y properly in $\mathfrak{X}^{an}$. Then $D|_{\bar{Y}}$ is a well-defined Cartier divisor on $\bar{Y}$ and

$$D.Y := \iota_* \mathrm{cyc}\left(D|_{\bar{Y}}\right)$$

is called the *generically proper intersection product* of D and Y where $\iota : \bar{Y} \to \mathfrak{X}$ is the inclusion. If $\mathfrak{Z}$ is any cycle on $\mathfrak{X}$, then we say that D intersects $\mathfrak{Z}$ properly in the generic fibre if D intersects the horizontal part Z of $\mathfrak{Z}$ properly in X^{an}. By linearity, we extend the generically proper intersection product $D.Z$. Let $\mathfrak{Z}_v$ be the vertical part of $\mathfrak{Z}$, then

$$D.\mathfrak{Z}_v := c_1\left(\widetilde{\mathcal{O}(D)}\right) \cap \mathfrak{Z}_v$$

is well-defined up to $\mathrm{div}(\mathbf{f})$ for K_1-chains $\mathbf{f}$ on $|D| \cap \mathfrak{Z}_v$. If D intersects $\mathfrak{Z}$ properly in the generic fibre, the *intersection product* is defined by

$$D.\mathfrak{Z} := D.Z + D.\mathfrak{Z}_v.$$

Then the intersection product is bilinear under the assumption that all intersections are proper in the generic fibre ([Gu3], Proposition 4.4).

3.2.16 Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism of reduced formal analytic varieties over K. Let D be a Cartier divisor on $\mathfrak{X}$ such that no irreducible component of the generic fibre of $\mathfrak{X}'$ is mapped into $|D^{an}|$. Then the *pull-back* $\varphi^*(D)$ is a well-defined Cartier divisor on $\mathfrak{X}'$. If $D = \{\mathcal{U}_\alpha, \gamma_\alpha\}_{\alpha \in I}$, then $\varphi^*(D)$ is given by $\{\varphi^{-1}(\mathcal{U}_\alpha), \gamma_\alpha \circ \varphi\}_{\alpha \in I}$.

3.2.17 Proposition. Under the hypothesis of 3.2.16, let $\mathfrak{Z}'$ be a prime cycle on $\mathfrak{X}'$ such that D intersects $\varphi(\mathfrak{Z}')$ properly in the generic fibre. If φ is proper, then the *projection formula*

$$\varphi_*\left(\varphi^*(D).\mathfrak{Z}'\right) = D.\varphi_*(\mathfrak{Z}')$$

holds up to $\mathrm{div}(\mathbf{f})$ for K_1-chains $\mathbf{f}$ on $|D| \cap \tilde{\varphi}(|\mathfrak{Z}'_v|)$. This follows immediately from [Gu3], Proposition 4.5.

3.2.18 Theorem. Let D and D' be Cartier divisors on the reduced formal analytic variety $\mathfrak{X}$ over K. Let $\mathfrak{Z}$ be a cycle on $\mathfrak{X}$ such that D, D' and $\mathfrak{Z}$ intersect properly in the generic fibre. Then

$$D.D'.\mathfrak{Z} = D'.D.\mathfrak{Z}$$

holds up to $\mathrm{div}(\mathbf{f})$ for a K_1-chain $\mathbf{f}$ on $|D| \cap |D'| \cap |\tilde{\mathfrak{Z}}|$ where $|\tilde{\mathfrak{Z}}|$ is the union of $|\mathfrak{Z}_v|$ and the reductions of $\bar{Y}$ for the horizontal components Y of $\mathfrak{Z}$.

Proof: We may assume $\mathfrak{Z}$ prime. The proof for horizontal $\mathfrak{Z}$ is rather involved, we refer to [Gu3], Theorem 5.9, applied to $\mathfrak{X} = \bar{\mathfrak{Z}}$. For vertical $\mathfrak{Z}$, the claim follows from [Fu], Corollary 2.4.2. □

3.3 Divisors on Admissible Formal Schemes

Let K be an algebraically closed field with a fixed non-trivial complete non-archimedean absolute value $|\ |$. The valuation ring and the residue field of K are denoted by K° and $\tilde{K}$, respectively.

In this section, we generalize the results of the preceding section to admissible formal schemes. Formal analytic varieties are good enough for models of reduced rigid analytic varieties. But if we consider flat morphisms, we have to consider arbitrary rigid analytic varieties and their models over the valuation ring will be admissible formal schemes. They are analogous to flat models in algebraic geometry. For every admissible formal scheme, there is an associated formal analytic variety and we can use the latter and section 3.2 to extend the divisoral operations. In fact, there is an equivalence between admissible formal schemes with reduced special fibre and reduced formal analytic varieties. Most important is that the special fibres of the two are isomorphic, we will use it frequently to switch between categories. This equivalence holds under the assumption that K is algebraically closed. For generalizations to arbitrary complete fields, see Remark 3.3.19.

The basic reference for admissible formal schemes is [BL3] and [BL4]. The generically proper intersection product with Cartier divisors is from [Gu3], §3-5. New is the flat pull-back rule at the end. Its proof uses noetherian normalization to reduce to the zero-dimensional case and then techniques from valuation theory.

3.3.1 Let $K^\circ\langle x_1, \ldots, x_n\rangle$ be the algebra of formal power series $\sum a_{\nu_1\ldots\nu_n} x_1^{\nu_1}\ldots x_n^{\nu_n}$ with coefficients $a_{\nu_1\ldots\nu_n} \in K^\circ$ and $\lim_{|\nu|\to\infty} a_\nu = 0$. A K°-algebra A is called *admissible* if it is flat over K° and has the form $A \cong K^\circ\langle x_1, \ldots, x_n\rangle/I$ for an ideal I. For such a quotient, flatness is equivalent with no K°-torsion ([BL3],Proposition 1.1).

3.3.2 Definition. A formal scheme $\mathfrak{X}$ over K° is called *admissible* if it is locally isomorphic to the affine formal scheme Spf$A$ associated to an admissible K°-algebra A. A morphism of admissible formal schemes over K° is just a morphism of formal schemes.

3.3.3 We associate to an admissible formal scheme $\mathfrak{X}$ over K° a formal analytic variety $\mathfrak{X}^{f-an}$. Locally, $\mathfrak{X}$ is isomorphic to Spf$A$ for an admissible K°-algebra A. Then $\mathcal{A} := A \otimes_{K^\circ} K$ is a K-affinoid algebra. Applying the same gluing process as for $\mathfrak{X}$, we get a formal analytic variety $\mathfrak{X}^{f-an}$ locally isomorphic to the formal affinoid variety Spf$\mathcal{A}$. Here, we use that a morphism of affine formal schemes is given by an opposite homomorphism of the corresponding algebras and this induces a canonical homomorphism between the K-affinoid algebras. The generic fibre of $\mathfrak{X}^{f-an}$ is a rigid analytic variety over K denoted by $\mathfrak{X}^{an}$. In this way, we get a functor from the category of admissible formal schemes to the category of formal analytic varieties and hence to the category of rigid analytic varieties.

3.3.4 Similarly as in 3.3.3, we can define a functor from the category of admissible formal schemes over K° to the category of schemes of finite type over $\tilde{K}$ associating to $\mathfrak{X}$ its *special fibre* $\tilde{\mathfrak{X}}$. Locally, $\tilde{\mathfrak{X}}$ is given by Spec$A \otimes_{K^\circ} \tilde{K}$. Note that $\mathfrak{X}$ and $\tilde{\mathfrak{X}}$ have the same underlying topological space. We have a natural finite morphism

$$i : \widetilde{\mathfrak{X}^{f-an}} \to \tilde{\mathfrak{X}}$$

locally induced by $A \otimes_{K^\circ} \tilde{K} \to \widetilde{A \otimes_{K^\circ} K}$ ([BL2], §1).

3.3.5 There is some sort of converse procedure to 3.3.3. For a formal analytic variety $\mathfrak{X}$, there is an associated formal scheme $\mathfrak{X}^{f-sch}$. If $\mathfrak{X}$ is locally given by Spf$\mathcal{A}$ for a K-affinoid algebra $\mathcal{A}$, then $\mathfrak{X}^{f-sch}$ is locally given by Spf$\mathcal{A}^\circ$. If we assume $\mathcal{A}$ reduced, then K algebraically closed implies that $\mathcal{A}^\circ$ is an admissible K°-algebra and that $\mathcal{A}^\circ \otimes_{K^\circ} \tilde{K} \cong \tilde{\mathcal{A}}$ is reduced ([BGR], Theorem 6.4.3/1). Hence we get an equivalence between the category of reduced formal analytic varieties and the category of admissible formal schemes over K° with reduced special fibre. Moreover, the natural $i : \tilde{\mathfrak{X}} \to \widetilde{\mathfrak{X}^{f-sch}}$ is an isomorphism ([BL2], Lemma 1.1, which was stated for discrete valuation rings but the proof holds in general).

3.3.6 Example. Let $\mathfrak{X}$ be a scheme flat and locally of finite type over K° and let $\pi \in K^\circ, |\pi| < 1$. We claim that the formal completion $\mathfrak{X}^{f-sch}$ along the special fibre (more precisely with respect to the ideal $\langle \pi \rangle$) is an admissible formal scheme over K°. Locally, the scheme $\mathfrak{X}$ is isomorphic to Spec$A$ for a flat K°-algebra of finite type. Then the completion $\hat{A}$ of A with respect to the $\langle \pi \rangle$-adic topology is an admissible K°-algebra (use [BL3], Proposition 1.1, Lemma 1.6). The formal completion $\mathfrak{X}^{f-sch}$ is locally given by Spec$\hat{A}$ proving that $\mathfrak{X}^{f-sch}$ is admissible. Note that the points of the generic fibre of $(\mathfrak{X}^{f-sch})$ are in one-to-one correspondence to the K°-integral points of the generic fibre $\mathfrak{X} \otimes_{K^\circ} K$ of $\mathfrak{X}$. Moreover, $(\mathfrak{X}^{f-sch})^{an}$ is an analytic subdomain of $(\mathfrak{X} \otimes_{K^\circ} K)^{an}$. Best, this is seen for $\mathfrak{X} = \mathbb{A}^n_{K^\circ}$. Then $\mathfrak{X}^{f-sch} = \mathrm{Spf} K^\circ \langle x_1, \ldots, x_n \rangle$ with generic fibre the closed unit ball $\mathbb{B}^n$ which is an analytic subdomain of $(\mathbb{A}^n_K)^{an}$ consisting of the K°-integral points of $\mathfrak{X}$. The general case may be easily reduced to this situation.

3.3.7 Remark. Let $\mathfrak{X}$ be an admissible formal scheme over K° with generic fibre $X = \mathfrak{X}^{an}$ and let Y be a closed analytic subvariety of X. Then there is a unique closed subscheme $\mathfrak{Y}$ of $\mathfrak{X}$ such that $\mathfrak{Y}$ is an admissible formal scheme over K° with generic fibre Y ([Gu3], Proposition 3.3). We denote $\mathfrak{Y}$ by $\bar{Y}$.

3.3.8 The *cycles* on an admissible formal scheme $\mathfrak{X}$ over K° are similarly defined as for formal analytic varieties. They are formal sums $\mathfrak{Z} = Z + \mathfrak{Z}_v$, where $Z = \mathfrak{Z}^{an}$ is a cycle on $\mathfrak{X}^{an}$ called the *horizontal part* of $\mathfrak{Z}$ and the *vertical part* $\mathfrak{Z}_v$ is a cycle on the special fibre $\tilde{\mathfrak{X}}$ with coefficients in the value group $\log |K^\times|$. Let us denote the group of cycles by $Z(\mathfrak{X}, v)$ and the group of vertical cycles by $Z(\tilde{\mathfrak{X}}, v)$. Again, we grade it by the *relative dimension.*

The concept of *Cartier divisors, line bundles* and *meromorphic sections* is defined on any ringed space (cf. [EGA IV], 21.1), hence it makes sense on $\mathfrak{X}$. We could also repeat the definitions from section 6. Moreover, we can easily translate the definitions of *proper intersection in the generic fibre* and *K_1-chains* on the special fibre to an admissible formal scheme over K°. We leave the details to the reader.

3.3.9 Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism of admissible formal schemes over K°. It is called *proper* if the morphisms $\varphi^{an} : (\mathfrak{X}')^{an} \to \mathfrak{X}^{an}$ and $\tilde{\varphi} : \tilde{\mathfrak{X}}' \to \tilde{\mathfrak{X}}$ are both proper. Note that if the formal schemes are quasi-compact, then φ^{an} proper implies that φ is proper ([Gu3], Remark 3.14).

If φ is proper and $\mathfrak{Z}' = Z' + \mathfrak{Z}'_v$ is a cycle on $\mathfrak{X}'$, then the *push-forward* of $\mathfrak{Z}'$ is defined by

$$\varphi_*(\mathfrak{Z}') := \varphi_*^{an}(Z') + \tilde{\varphi}_*(\mathfrak{Z}'_v) \in Z(\mathfrak{X}, v).$$

3.3.10 The morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ of admissible formal schemes over K° is called *flat* if $\mathcal{O}_{\mathfrak{X}',x'}$ is a flat $\mathcal{O}_{\mathfrak{X},\varphi(x')}$-module for every $x' \in \mathfrak{X}'$. If φ is locally given by a homomorphism $A \to A'$ of admissible K°-algebras, then this is equivalent to flatness of A' as an A-algebra. It follows that φ^{an} and $\tilde{\varphi}$ are both flat. If φ is flat and $\mathfrak{Z} = Z + \mathfrak{Z}_v$ is a cycle on $\mathfrak{X}$, then the *pull-back* of $\mathfrak{Z}$ is defined by

$$\varphi^*(\mathfrak{Z}) := \varphi^{an*}(Z) + \tilde{\varphi}^*(\mathfrak{Z}_v) \in Z(\mathfrak{X}', v).$$

3.3.11 Proposition. *Let us consider the Cartesian diagram*

$$\begin{tikzcd} \mathfrak{X}'_2 \arrow[r, "\psi'"] \arrow[d, "\varphi'"] & \mathfrak{X}_2 \arrow[d, "\varphi"] \\ \mathfrak{X}'_1 \arrow[r, "\psi"] & \mathfrak{X}_1 \end{tikzcd}$$

of admissible formal schemes over K° with φ proper and ψ flat. Then φ' is proper, ψ' is flat and the fibre-square rule

$$\psi^* \circ \varphi_* = \varphi'_* \circ (\psi')^*$$

holds on $Z(\mathfrak{X}_2, v)$.

Proof: Note first that due to flatness of ψ, $\mathfrak{X}_2'$ is the product of $\mathfrak{X}_1'$ and $\mathfrak{X}_2$ in the category of formal schemes and ψ' is flat (see [BL2], section 1). We conclude that we have corresponding Cartesian diagrams of generic and special fibres. Then the claim follows from the corresponding results for rigid analytic varieties (Proposition 3.1.21) and for algebraic schemes (1.2.11). $\square$

3.3.12 Let D be a Cartier divisor on the admissible formal scheme $\mathfrak{X}$ and let $\mathfrak{Z}$ be a cycle on $\mathfrak{X}$ intersecting D properly in the generic fibre. We want to define the *generically proper intersection product* $D.\mathfrak{Z}$.

First, we assume that $\mathfrak{Z}$ is horizontal and prime, i.e. equal to an irreducible and reduced analytic subset Y on $\mathfrak{X}^{an}$. Let $\mathfrak{Y}^{f-an}$ be the formal analytic structure on Y induced by $\mathfrak{X}^{f-an}$ and let $\mathfrak{Y}$ be the formal scheme associated to $\mathfrak{Y}^{f-an}$. By 3.3.4 and 3.3.5, $\mathfrak{Y}$ is an admissible formal scheme over K° and we have a canonical morphism $\iota : \mathfrak{Y} \to \mathfrak{X}$ which is finite on special fibres. Then $\iota^* D$ may be seen as a Cartier divisor on $\mathfrak{Y}^{f-an}$ giving rise to an associated Weil divisor $\mathrm{cyc}(\iota^* D)$ on $\mathfrak{Y}^{f-an}$ and we define

$$D.Y := \iota_* \left(\mathrm{cyc}(\iota^* D)\right)$$

using that the special fibres of $\mathfrak{Y}^{f-an}$ and $\mathfrak{Y}$ are canonically isomorphic. By linearity, we extend the definition to all horizontal cycles on $\mathfrak{X}$ intersecting D properly in the generic fibre.

If $\mathfrak{Z}$ is a vertical cycle $\mathfrak{Z}_v$, then as in section 3.2, we set

$$D.\mathfrak{Z}_v := c_1\left(\widetilde{\mathcal{O}(D)}\right) \cap \mathfrak{Z}_v$$

which is well-defined up to $\mathrm{div}(\mathbf{f})$ for K_1-chains on $|D| \cap |\mathfrak{Z}_v|$. The *Weil divisor* associated to D is defined by

$$\mathrm{cyc}(D) := D.\mathrm{cyc}(X).$$

Note that the pull-back of a Cartier-divisor with respect to a morphism of admissible formal schemes over K° is well-defined if the pull-back on generic fibres is well-defined in the sense of 3.1.15. The next two results are easily deduced from the corresponding statements in section 3.2 (see [Gu3]).

3.3.13 Proposition. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a proper morphism of admissible formal schemes over K°, let D be a Cartier divisor on $\mathfrak{X}$ such that $\varphi^*(D)$ is a well-defined Cartier divisor on $\mathfrak{X}'$ and let $\mathfrak{Z}'$ be a prime cycle on $\mathfrak{X}'$ such that D intersects $\varphi(\mathfrak{Z}')$ properly in the generic fibre. Then the projection formula*

$$\varphi_* \left(\varphi^*(D).\mathfrak{Z}'\right) = D.\varphi_*(\mathfrak{Z}')$$

holds up to $\mathrm{div}(\mathbf{f})$ *for K_1-chains* $\mathbf{f}$ *on $|D| \cap \tilde{\varphi}(|\mathfrak{Z}_v'|)$ (which plays only a role if $\mathfrak{Z}$ is vertical).*

3.3.14 Theorem. *Let D and D' be Cartier divisors on the admissible formal scheme $\mathfrak{X}$ over K°. Let $\mathfrak{Z}$ be a cycle on $\mathfrak{X}$ such that D, D' intersect $\mathfrak{Z}$ properly in the generic fibre. Then*

$$D.D'.\mathfrak{Z} = D'.D.\mathfrak{Z}$$

holds up to K_1-chains on $|D| \cap |D'| \cap |\tilde{\mathfrak{Z}}|$ where $|\tilde{\mathfrak{Z}}|$ is the union of $|\mathfrak{Z}_v|$ and the reductions of $\bar{Y}$ for the horizontal components Y of $\mathfrak{Z}$.

Now we state a new result proving compatibility of flat pull-back and the proper intersection with divisors.

3.3.15 Proposition. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a flat morphism of admissible formal schemes over K°, let D be a Cartier divisor on $\mathfrak{X}$ such that $\varphi^*(D)$ is a well-defined Cartier-divisor on $\mathfrak{X}'$ and let $\mathfrak{Z}$ be a cycle on $\mathfrak{X}$ intersecting D properly in the generic fibre $\mathfrak{X}^{an}$. Then $\varphi^*(D)$ intersects $\varphi^*(\mathfrak{Z})$ properly in the generic fibre and we have*

$$\varphi^*(D).\varphi^*(\mathfrak{Z}) = \varphi^*(D.\mathfrak{Z})$$

up to K_1-chains on $\varphi^{-1}(|D|) \cap \tilde{\varphi}^{-1}(|\mathfrak{Z}_v|)$.

Proof: By 1.2.17, the result holds for $\mathfrak{Z}$ vertical. Hence it is enough to prove the claim for $Z = \mathfrak{Z}$ horizontal and prime. For the horizontal parts, the claim follows from Proposition 3.1.22. Using the fibre square rule, we may even assume that $Z = \mathfrak{X}^{an}$ and that $\mathfrak{X}$ is the formal scheme associated to a formal analytic variety, i.e. $\mathfrak{X}$ is an admissible formal scheme over K° with reduced special fibre (cf. 3.3.5). So we have to prove

$$\varphi^* \mathrm{cyc}(D) = \mathrm{cyc}(\varphi^* D) \tag{3.1}$$

by checking the multiplicities in the irreducible components of $\tilde{\mathfrak{X}}'$. By passing to formal open subspaces, we may assume that $\mathfrak{X}$ and $\mathfrak{X}'$ are formal affine, that D is given by an equation $a \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ and that $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{X}}'$ are both irreducible. The vertical part of the left hand side of (3.1) is

$$-\log |a(\tilde{\mathfrak{X}})| \, \mathrm{cyc}(\widetilde{\mathfrak{X}'}).$$

The vertical part of the right hand side of (3.1) is

$$-\sum_{V'} \log |a(V')| \, [\tilde{K}(V') : \tilde{K}(\widetilde{\mathfrak{X}'_{red}})] \, \mathrm{cyc}\left(\widetilde{\mathfrak{X}'}\right)$$

where V' is ranging over all irreducible components of the special fibre of $(\mathfrak{X}')^{f-an}$. By flatness of $\tilde{\varphi}$ and finiteness of the canonical map between the special fibres of $(\mathfrak{X}')^{f-an}$ and $\mathfrak{X}'$, we conclude that every V' is mapped onto $\tilde{\mathfrak{X}}$ and hence $|a(V')| = |a(\tilde{\mathfrak{X}})|$. The proposition follows now from the following result:

3.3.16 Lemma. *Let $\mathfrak{X}$ be an admissible formal scheme over K°, let*

$$\mathrm{cyc}(\mathfrak{X}^{an}) = \sum_j m_j X_j$$

be the decomposition of the generic fibre into irreducible components X_j and let $\mathfrak{X}_j^{f-an}$ be the formal analytic structure on X_j induced by $\mathfrak{X}^{f-an}$. Then we have

$$\mathrm{cyc}(\tilde{\mathfrak{X}}) = \sum_j m_j \, (\tilde{\imath}_j)_* (\mathrm{cyc}(\tilde{\mathfrak{X}}_j))$$

where $\tilde{\imath}_j$ is the canonical finite morphism mapping the special fibre $\tilde{\mathfrak{X}}_j$ of $\mathfrak{X}_j^{f-an}$ to $\tilde{\mathfrak{X}}$.

Proof: Again, we may assume that $\mathfrak{X} = \mathrm{Spf}\mathcal{A}$ is formal affine and that $\tilde{\mathfrak{X}}$ is irreducible. Using noetherian normalization with respect to $\tilde{\mathfrak{X}}_{red}$, we may reduce the problem to the zero-dimensional case with base field the completion $\hat{Q}$ of the field of fractions of a Tate-algebra (cf. [Gu3], Lemma 5.6). Note that $\hat{Q}$ is not algebraically closed but stable ([BGR], Theorem 5.3.2/1, Proposition 3.6.2/3) and the value group $|\hat{Q}^\times| = |K^\times|$ is still divisible. The definition of a stable complete field is recalled in Remark 3.3.21. By [Fu], Lemma A1.3, the multiplicity of $\mathrm{cyc}(\tilde{\mathfrak{X}})$ in $\tilde{\mathfrak{X}}_{red}$ is

$$\ell(\tilde{A}) = \frac{\dim_{\tilde{Q}} \tilde{A}}{[\tilde{Q}(\tilde{\mathfrak{X}}_{red}) : \tilde{Q}]}$$

where ℓ denotes the length of the local artinian ring $\tilde{A}$ (using $\tilde{\mathfrak{X}}$ irreducible and zero-dimensional). Note that $\mathcal{A} = A \otimes_{\hat{Q}^\circ} \hat{Q}$ is a finite dimensional $\hat{Q}$-algebra and hence

$$\mathcal{A} \cong \prod_{\wp \in \mathrm{Spec}\mathcal{A}} \mathcal{A}_\wp$$

where the localization $\mathcal{A}_\wp$ of $\mathcal{A}$ in $\wp$ is an artinian $\hat{Q}$-algebra. By [Fu], Lemma A1.3, we have

$$\mathrm{cyc}(\mathfrak{X}^{an}) = \sum_\wp \ell(\mathcal{A}_\wp)\wp = \sum_\wp \frac{\dim_{\hat{Q}}(\mathcal{A}_\wp)}{[\hat{Q}(\wp) : \hat{Q}]}\,\wp.$$

Note that $\hat{Q}(\wp) = (\mathcal{A}_\wp)_{red}$ is a finite dimensional field extension of $\hat{Q}$. By stability and divisibility, we have $[\hat{Q}(\wp) : \hat{Q}] = [\tilde{Q}(\wp) : \tilde{Q}]$. We conclude that the right hand side of the claim in the proposition has multiplicity

$$\sum_\wp \ell(\mathcal{A}_\wp)[\tilde{Q}(\wp) : \tilde{Q}(\tilde{\mathfrak{X}}_{red})] = \sum_\wp \frac{\dim_{\hat{Q}}(\mathcal{A}_\wp)}{[\tilde{Q}(\tilde{\mathfrak{X}}_{red}) : \tilde{Q}]}$$

in $\tilde{\mathfrak{X}}_{red}$. Hence the claim follows from the following result:

3.3.17 Lemma. *Let A be an admissible algebra over the valuation ring Q° of a non-trivial non-archimedean complete absolute value on a field Q. If $\tilde{A}$ is a finite dimensional $\tilde{Q}$ algebra, then A is a free Q°-algebra of rank $\dim_{\tilde{Q}}(\tilde{A})$.*

Proof: Let us choose $b_1, \ldots, b_r \in A$ such that $\tilde{b}_1, \ldots, \tilde{b}_r$ form a $\tilde{Q}$-basis of $\tilde{A}$. As a quotient of a Tate algebra, A is topologically generated by $\xi_1, \ldots, \xi_n \in A$ as a Q°-algebra. There is an element $\pi \in Q$, $|\pi| < 1$, such that $\xi_1, \ldots, \xi_n$ and every $b_i b_j$ may be written in the form $\sum_j \lambda_j b_j$, $\lambda_j \in Q^\circ$, up to an error in $A\pi$. Now any $a \in A$ may be written as a finite sum

$$a \equiv \sum_\nu \lambda_\nu \xi_1^{\nu_1} \cdots \xi_n^{\nu_n} \pmod{\pi A}.$$

Using the above, this is a linear combination of $b_1, \ldots, b_r$ up to an error in πA. This proves

$$A = Q^\circ b_1 + \cdots + Q^\circ b_r + \pi A.$$

By induction, we get

$$A = Q^\circ b_1 + \cdots + Q^\circ b_r + \pi^n A$$

for all $n \in \mathbb{N}$. Since A is complete and separated with respect to the π-adic topology ([BL2], Proposition 1.1), we conclude that $b_1, \ldots, b_r$ generate A as a Q°-module. If a non-trivial Q°-linear combination of $b_1, \ldots, b_r$ equals 0, then we may assume that all coefficients are in Q° and at least one has absolute value 1. By reduction to $\tilde{A}$, a non-trivial linear combination of $\tilde{b}_1, \ldots, \tilde{b}_r$ is $\tilde{0}$ which is a contradiction. Hence $b_1, \ldots, b_r$ is a Q°-basis of A. □

3.3.18 Remark. This ends also the proofs of Lemma 3.3.16 and hence of Proposition 3.3.15. Note that the argument for Lemma 3.3.17 works for any Q°-algebra of topological finite presentation.

3.3.19 Remark. For simplicity, we have assumed K algebraically closed. This can be easily removed, for details of the following construction, we refer to [Gu3]. Let K be a field with a non-trivial non-archimedean complete absolute value $|\ |$. The latter extends uniquely to an absolute value of the completion $\mathbb{K}$ of the algebraic closure of K also denoted by $|\ |$. Note that $\mathbb{K}$ is also algebraically closed ([BGR], Proposition 3.4.1/3).

Then we have base change for rigid analytic varieties and admissible formal schemes. For an admissible formal scheme $\mathfrak{X}$ over K°, it is clear that the base change $X\hat{\otimes}_K\mathbb{K}$ of the generic fibre X is the generic fibre of the base change $\mathfrak{X}\hat{\otimes}_{K^\circ}\mathbb{K}^\circ$. As above, we may define cycles, push-forward and flat pull-back on $\mathfrak{X}$. To define the generically proper intersection product $D.\mathfrak{Z}$ of a Cartier divisor D with a cycle $\mathfrak{Z}$ on $\mathfrak{X}$, we perform base change to $\mathbb{K}^\circ$. Then $D.\mathfrak{Z}$ is defined over K°, well-defined up to K_1-chains on $|D| \cap |\mathfrak{Z}_v|$ which are defined over $\tilde{\mathbb{K}}$. Then Proposition 3.3.11, Proposition 3.3.13, Theorem 3.3.14 and Proposition 3.3.15 hold in this generality allowing the K_1-chains in question to be defined over $\tilde{\mathbb{K}}$.

3.3.20 Remark. Now let $K \subset K'$ be a field extension with a non-trivial complete absolute value $|\ |$ on K extending to a complete absolute value on K'. As above, we have base change to K'° for admissible formal schemes over K°. Then all the constructions above are compatible with base change ([Gu3], Lemma 3.12).

3.3.21 Remark. A field with a non-trivial non-archimedean complete absolute value $|\ |$ is called *stable* if for every finite dimensional field extension L over K, we have

$$[L : K] = ef$$

where e and f are the ramification index and the residue degree of the unique extension to an absolute value on L. Then Lemma 3.3.16 can be generalized in the following way. Let A be an admissible K°-algebra for a complete stable field K. Then the irreducible components of $\mathrm{Sp}A \otimes_{K^\circ} K$ have the form $\mathrm{Sp}(A \otimes_{K^\circ} K)/\wp$, where $\wp$ ranges over the minimal prime ideals of $A \otimes_{K^\circ} K$. Every irreducible component $V_\wp$ of the special fibre of $\mathrm{Sp}(A \otimes_{K^\circ} K)/\wp$ gives rise to a unique absolute value $|\ |_{V_\wp}$ on the quotient field of $(A \otimes_{K^\circ} K)/\wp$ which extends the multiplicative norm from 6.8, i.e.

$$|a(V_\wp)| = |a|_{V_\wp}$$

for every $a \in (A \otimes_{K^\circ} K)/\wp$. Let $e(V_\wp)$ be the ramification index of $|\ |_{V_\wp}$ over $|\ |$. Then $e(V_\wp)$ is finite and for a minimal prime ideal $\mathfrak{P}$ of $\tilde{A}$, we have

$$\ell(\tilde{A}_{\mathfrak{P}}) = \sum_{\wp} \ell((A \otimes_{K^\circ} K)_\wp) \sum_{V_\wp} [\tilde{K}(V_\wp) : \tilde{K}(\mathfrak{P})] e(V_\wp)$$

where $\wp$ ranges over all minimal prime ideals of $A \otimes_{K^\circ} K$ and $V_\wp$ ranges over all irreducible components of the special fibre of $\mathrm{Sp}(A \otimes_{K^\circ} K)/\wp$ lying over $\mathfrak{P}$. The proof is the same as for Lemma 3.3.16.

3.4 Arakelov-Cycle Groups

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $|\ |$. In this section, we consider a line bundle L on a quasi-compact and quasi-separated rigid analytic variety X over K.

An admissible formal scheme over the valuation ring K° of K is called a K°-model of X if its generic fibre is X. The idea of non-archimedean Arakelov-theory of Bloch-Gillet-Soulé [BlGS] is to consider all K°-models simultaneously. They used the projective limit of the Chow groups to get objects analogous to currents and the intersection product was interpretated as an analogue of the $*$-product at finite places. This approach was developped for complete discrete valuation rings and various regularity assumptions were imposed on the algebraic K°-models.

Here, in the context of formal geometry and divisors, we consider the set M_X of isomorphism classes of all K°-models of X which is a directed set with respect to the partial order given by dominance of models. The results of Raynaud and Bosch-Lütkebohmert imply that many properties of quasi-compact and quasi-separated rigid analytic varieties extend to sufficiently large K°-models. They are collected in Proposition 3.4.2 used frequently in the sequel.

The projective limit over M_X of the cycle groups modulo vertical rational equivalence is called the Arakelov-cycle group of X. Vertical rational equivalence plays the same role as the relation $\equiv$ used in section 2.1 where we have considered Green forms modulo d- and d^c-boundaries. Easily, we can extend the divisoral operations from section 3.3 to a proper intersection product on Arakelov-cycle groups. The role of hermitian line bundles is played here by formal K°-models of line bundles. This will be explored in section 4.3. Commutativity and projection formula still holds. However, flat pull-back is not well-defined on Arakelov-cycle groups similarly to the fact that flat pull-back of currents is not possible without additional assumptions.

Using Remark 3.3.19 , the assumption K algebraically closed may be removed using base change. For simplicity, we keep it here and in the following sections as we are interested in local heights which do not depend on the choice of the base field.

3.4.1 Definition . A K°-*model* of X is an admissible formal scheme $\mathfrak{X}$ over K° with generic fibre $\mathfrak{X}^{an} = X$. We denote the isomorphism classes of K°-models of X by M_X. A line bundle $\mathcal{L}$ on $\mathfrak{X}$ is said to be a K°-model of L if $\mathcal{L}^{an} = L$.

3.4.2 Proposition. *Let L be a line bundle on X.*

a) Every quasi-compact and quasi-separated rigid analytic variety over K has a K°-model.

b) If $\varphi : X' \to X$ is a morphism of quasi-compact and quasi-separated rigid analytic varieties over K and $\mathfrak{X}$ is a K°-model of X, then there is a K°-model $\mathfrak{X}'$ of X' and a morphism $\mathfrak{X}' \to \mathfrak{X}$ extending φ. The extension of φ is always unique.

c) The set M_X is partially ordered by defining $\mathfrak{X}' \geq \mathfrak{X}$ if and only if the identity on X extends to a morphism $\mathfrak{X}' \to \mathfrak{X}$. Then M_X is a directed set.

d) If $\varphi : X' \to X$ is a flat morphism of quasi-compact and quasi-separated rigid analytic varieties over K with extension to a morphism $\mathfrak{X}'_0 \to \mathfrak{X}_0$ of K°-models, then there is $\mathfrak{X}_1 \in M_X$ with $\mathfrak{X}_1 \geq \mathfrak{X}_0$ such that the projection $\mathfrak{X}'_1 := \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}'_0 \to \mathfrak{X}_1$ is a flat extension of φ. Here, the fibre product is understood in the category of admissible formal schemes which is the closed subscheme of the fibre product in the category of formal schemes given by the ideal of K°-torsion.

e) If Y is a closed analytic subvariety of X extending to a morphism $\mathfrak{Y} \to \mathfrak{X}$ of K°-models, then there is $\mathfrak{X}' \in M_X$ with a closed subscheme $\mathfrak{Y}' \in M_Y$ such that $\mathfrak{X}' \geq \mathfrak{X}$ and $\mathfrak{Y}' \geq \mathfrak{Y}$.

f) Every line bundle on a quasi-compact and quasi-separated rigid analytic variety has a K°-model.

Proof: The claims a)-c) follow from a theorem of Raynaud which is proved by Bosch and Lütkebohmert in [BL3], Theorem 4.1. Claim d) is proved in [BL2], Theorem 5.2, and e) is part of [BL4], Corollary 5.3. For f), we refer to [Gu3], Lemma 7.6. □

3.4.3 For $\mathfrak{X} \in M_X$, let $Z(\mathfrak{X}, v)$ be the group of cycles. Recall that we have

$$Z(\mathfrak{X}, v) = Z(X) \oplus Z(\tilde{\mathfrak{X}}, v)$$

where $Z(X)$ is the group of cycles on X and $Z(\tilde{\mathfrak{X}}, v)$ is the group of cycles on the special fibre $\tilde{\mathfrak{X}}$ with coefficients in the value group $\log|K^\times|$. Let $R(\tilde{\mathfrak{X}}, v)$ be the subgroup of $Z(\tilde{\mathfrak{X}}, v)$ generated by all divisors of K_1-chains on $\tilde{\mathfrak{X}}$. Let us consider the factor group

$$\tilde{Z}(\mathfrak{X}, v) := Z(\mathfrak{X})/R(\tilde{\mathfrak{X}}, v)$$

of cycles on $\mathfrak{X}$ up to vertical rational equivalence. By Proposition 3.4.2, we know that M_X is a non-empty directed set. So it makes sense to define the Arakelov-cycle group

$$\hat{Z}(X, v) := \varprojlim_{\mathfrak{X}} \tilde{Z}(\mathfrak{X}, v)$$

of X where the inverse limit is with respect to push-forward of the morphisms extending the identity. An element $\alpha \in \hat{Z}(X,v)$ will be described by a family $(\alpha_{\mathfrak{X}})_{\mathfrak{X}\in M_X}$, where $\alpha_{\mathfrak{X}} \in \tilde{Z}(\mathfrak{X},v)$. All the groups will be graded by relative dimension.

3.4.4 Proposition. *Let $\varphi : X' \to X$ be a proper morphism of quasi-compact and quasi-separated rigid analytic varieties over K. Then there is a unique map $\varphi_* : \hat{Z}(X',v) \to \hat{Z}(X,v)$ such that for any extension of φ to a morphism $\bar{\varphi} : \mathfrak{X}' \to \mathfrak{X}$ of K°-models and any $\alpha' \in \hat{Z}(X',v)$, we have*

$$\varphi_*(\alpha')_{\mathfrak{X}} = \bar{\varphi}_*(\alpha'_{\mathfrak{X}'}).$$

Proof: Note that $\bar{\varphi}$ is proper (see 3.2.12), hence the push-forward $\bar{\varphi}_*$ makes sense. For $\mathfrak{X} \in M_X$, we define $\varphi_*(\alpha')_{\mathfrak{X}} \in \tilde{Z}(\mathfrak{X},v)$ by choosing $\mathfrak{X}' \in M_{X'}$ with an extension $\bar{\varphi} : \mathfrak{X}' \to \mathfrak{X}$ of φ (Proposition 3.4.2b)) and then using the formula in the claim for definition. Using Proposition 3.4.2b) and c), it is obvious that the definition is independent of the choice of $\bar{\varphi}$ and that $\varphi_*(\alpha') \in \hat{Z}(X,v)$. □

3.4.5 Corollary. Let Y be a closed analytic subset of X and let $i : Y \to X$ be the corresponding embedding. Then $i_* : \hat{Z}(Y,v) \to \hat{Z}(X,v)$ is one-to-one which allows us to view $\hat{Z}(Y,v)$ as a subgroup of $\hat{Z}(X,v)$.

Proof: Let $\alpha \in \hat{Z}(Y,v) \setminus \{0\}$. There is $\mathfrak{Y} \in M_Y$ with $\alpha_{\mathfrak{Y}} \neq 0$. By Proposition 3.4.2e), we may assume that $\mathfrak{Y}$ is a closed subscheme of $\mathfrak{X} \in M_X$. The corresponding embedding gives $\tilde{Z}(\mathfrak{Y},v) \subset \tilde{Z}(\mathfrak{X},v)$ proving $i_*(\alpha)_{\mathfrak{Y}} \neq 0$. □

3.4.6 Proposition. Let s be an invertible meromorphic section of L and let $\mathcal{L}$ be a K°-model of L living on $\mathfrak{X} \in M_X$. For $\alpha \in \hat{Z}(X,v)$ with horizontal part intersecting $\mathrm{div}(s)$ properly, there is $\widehat{\mathrm{div}}_{\mathcal{L}}(s).\alpha \in \hat{Z}(X,v)$ uniquely determined by the following property: For every K°-model $\mathfrak{X}' \geq \mathfrak{X}$ with corresponding morphism $\pi : \mathfrak{X}' \to \mathfrak{X}$ extending the identity, there is a unique meromorphic section $\overline{s'}$ of $\pi^*(\mathcal{L})$ extending $\pi^*(s)$ and we require

$$\left(\widehat{\mathrm{div}}_{\mathcal{L}}(s).\alpha\right)_{\mathfrak{X}'} = \mathrm{div}(\overline{s'}).\alpha_{\mathfrak{X}'} \in \tilde{Z}(\mathfrak{X}',v).$$

Proof: Let $\mathfrak{X}_0 \in M_X$. By Proposition 3.4.2c), there is a formal K°-model $\mathfrak{X}'$ of X with $\mathfrak{X}' \geq \mathfrak{X}$ and $\mathfrak{X}' \geq \mathfrak{X}_0$. Let $\pi : \mathfrak{X}' \to \mathfrak{X}$ and $\pi_0 : \mathfrak{X}' \to \mathfrak{X}_0$ be the corresponding morphisms extending the identity and let $\overline{s'}$ be the unique meromorphic section of $\pi^*(\mathcal{L})$ extending s. Then $\overline{s'}$ is an invertible meromorphic section and we define

$$\left(\widehat{\mathrm{div}}_{\mathcal{L}}(s).\alpha\right)_{\mathfrak{X}_0} := \pi_{0*}\left(\mathrm{div}(\overline{s'}).\alpha_{\mathfrak{X}'}\right) \in \tilde{Z}(\mathfrak{X}_0,v).$$

By Proposition 3.4.2c) and projection formula (Proposition 3.3.13), this gives a well-defined element of $\hat{Z}(X,v)$. □

3.4.7 Remark. Note that in [Gu3], §8, the local cycle group was defined by using the projective limit over all K°-models of X_{red}. By Corollary 3.4.5, this gives a subgroup of our group here.

The basic results form section 3.3 carry immediately over to the local Chow groups by a componentwise verification:

3.4.8 Let $\varphi : X' \to X$ be a proper morphism of quasi-compact and quasi-separated rigid analytic varieties over K, let s be an invertible meromorphic section of the line bundle L on X such that $\varphi^*\mathrm{div}(s)$ is a well-defined Cartier divisor on X', let $\mathcal{L}$ be a formal K°-model of L living on $\mathfrak{X} \in M_X$ and let $\alpha' \in \hat{Z}(X',v)$ such that no component of the horizontal part of α' is mapped into $|\mathrm{div}(s)|$. By Proposition 3.4.2b), φ extends to a morphism $\bar{\varphi} : \mathfrak{X}' \to \mathfrak{X}$ for a suitable $\mathfrak{X}' \in M_{X'}$. Then we define

$$\varphi^*\widehat{\mathrm{div}}_{\mathcal{L}}(s).\alpha' := \widehat{\mathrm{div}}_{\bar{\varphi}^*\mathcal{L}}(s \circ \varphi).\alpha' \in \hat{Z}(X',v).$$

Clearly, this is independent of the choice of $\bar{\varphi}$.

3.4.9 Corollary. *Under the hypothesis above, the projection formula*

$$\varphi_* \left(\varphi^* \widehat{\mathrm{div}}_{\mathcal{L}}(s).\alpha' \right) = \widehat{\mathrm{div}}_{\mathcal{L}}(s).\varphi_*(\alpha') \in \hat{Z}(X, v)$$

holds.

3.4.10 Corollary. *Let s, s' be invertible meromorphic sections of line bundles L and L' on X and let $\alpha \in \hat{Z}(X, v)$. We assume that* $\mathrm{div}(s)$, $\mathrm{div}(s')$ *intersect the horizontal part of α properly in X. For K°-models $\mathcal{L}$, $\mathcal{L}'$ of L and L', we get*

$$\widehat{\mathrm{div}}_{\mathcal{L}}(s).\widehat{\mathrm{div}}_{\mathcal{L}'}(s').\alpha = \widehat{\mathrm{div}}_{\mathcal{L}'}(s').\widehat{\mathrm{div}}_{\mathcal{L}}(s).\alpha \in \hat{Z}(X, v).$$

3.5 Refined Intersection on Arakelov-Chow Groups

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $| \ |$. Let X be a proper scheme over K endowed with its rigid analytic structure (cf. Example 3.1.7).

In this section, we introduce rational equivalence on Arakelov-cycle groups giving rise to Arakelov-Chow groups. For defining a refined intersection product with divisors similarly as in section 1.2, we have to take care of supports. This makes it convenient to consider local Arakelov-Chow groups. Local means that all Arakelov-cycles have support in a given closed analytic subset Y of X and similarly for rational equivalence. In contrast to algebraic geometry, it is not isomorphic to the Arakelov-Chow group of Y because the support condition does not force vertical cycles to ly in the closure of Y. From section 3.4, we deduce projection formula and commutativity for the refined operations of divisors.

3.5.1 Remark. Note that the GAGA-principle holds on X, i.e. every line bundle L on X is algebraic, every meromorphic section of L is algebraic, every analytic subvariety is induced by a closed algebraic subscheme of X and every analytic morphism of proper schemes over K is algebraic ([Ko]; one can use also [Ber], Proposition 3.4.11). Hence the algebraic and the analytic cycle groups on X are the same. It is clear that the intersection product with Cartier divisors on X may be refined. This follows from the GAGA-principle and the refined intersection theory from section 1.2. We can also argue directly in the category of rigid analytic varieties. The algebraicity assumption is used, as in the complex case, to ensure that the restriction of a line bundle to an irreducible analytic subset has a non-trivial meromorphic section. It follows from [Gu3], Proposition 6.2, that the use of the algebraic approach in [Fu] gives the same refined intersection product as working directly on the rigid analytic variety.

3.5.2 Let $\mathbf{f} = (f_W)$ be a K_1-chain on X. For every $\mathfrak{X} \in M_X$, the rational function f_W on the irreducible analytic subset W extends to a rational function $\bar{f}_W^{\mathfrak{X}}$ on the closure of W in $\mathfrak{X}$. By projection formula 3.3.13, it is clear that the family $\mathrm{div}(\bar{f}_W^{\mathfrak{X}})$, $\mathfrak{X} \in M_X$, induces a well-defined element $\widehat{\mathrm{div}}(f_W) \in \hat{Z}(X, v)$. In general, let

$$\widehat{\mathrm{div}}(\mathbf{f}) := \sum_W \widehat{\mathrm{div}}(f_W) \in \hat{Z}(X, v).$$

3.5.3 Proposition. *Let $\varphi : X' \to X$ be a proper morphism over K. For a K_1-chain $\mathbf{f}'$ on X', we have*

$$\varphi_* \widehat{\mathrm{div}}(\mathbf{f}') = \widehat{\mathrm{div}}(\varphi_* \mathbf{f}') \in \hat{Z}(X, v).$$

Proof: We may assume that X' is irreducible and reduced, φ is surjective and $\mathbf{f}'$ is given by a single rational function g on X'. Let $\mathfrak{X} \in M_X$, $\mathfrak{X}' \in M_{X'}$ such that φ extends to a morphism $\bar{\varphi} : \mathfrak{X}' \to \mathfrak{X}$ (cf. Proposition 3.4.2b)). It is enough to show that

$$\bar{\varphi}_* \mathrm{cyc}\,(\mathrm{div}(\bar{g})) = \mathrm{cyc}\,(\mathrm{div}(\overline{\varphi_* g})) \tag{3.2}$$

holds as an identity of cycles on $\mathfrak{X}$. To prove (3.2), we have only to check the vertical parts, the equality of the horizontal parts is just 1.2.9. Note that the irreducible components of $\tilde{\mathfrak{X}}$ have the same dimension as X. So we may assume that $\dim X = \dim X'$, otherwise both sides of (3.2) are zero. Using 3.3.3-3.3.5, it is also clear that it is enough to prove (3.2) for formal analytic varieties $\mathfrak{X}$, $\mathfrak{X}'$. The Stein factorization gives a formal analytic variety $\mathfrak{X}''$ over K and proper morphisms $\varphi' : \mathfrak{X}' \to \mathfrak{X}''$, $\varphi'' : \mathfrak{X}'' \to \mathfrak{X}$ such that $\varphi = \varphi'' \circ \varphi'$ with the properties:

a) φ' is an isomorphism outside of $\pi^{-1}(S)$, where $\pi : \mathfrak{X} \to \tilde{\mathfrak{X}}$ is the reduction and S is a closed lower dimensional subset of $\tilde{\mathfrak{X}}$.

b) φ'' is a finite map.

For details, we refer to the proof of [Gu3], Proposition 4.5. For φ', the vertical parts of (3.2) also agree since φ' is an isomorphism outside $\pi^{-1}(S)$ and the irreducible components of $\widetilde{\mathfrak{X}'}$ lying over S do not contribute to the left hand side of ((3.2)) by dimensionality reasons. So we may assume that $\varphi = \varphi''$ is finite. By passing to formal open affinoid subspaces, we may assume that $\mathfrak{X} = \mathrm{Spf}\mathcal{A}$ and hence $\mathfrak{X}' = \mathrm{Spf}\mathcal{A}'$ for K-affinoid algebras $\mathcal{A}$, $\mathcal{A}'$. To see this, we have to check that the norm doesn't change: This follows from

$$K(X') \otimes_{K(X)} Q(\mathcal{A}) \cong Q(\mathcal{A}')$$

using finiteness of X' over X and that $\mathcal{A}$, $\mathcal{A}'$ are integral domains. We check the multiplicities in an irreducible component W of $\tilde{\mathfrak{X}}$ and, by passing again to formal open affinoid subspaces, we may assume that $W = \tilde{\mathfrak{X}}$. We may also assume that $f \in \mathcal{A}$, and so we have to check

$$\sum_V [\tilde{K}(V) : \tilde{K}(W)] \log|f(V)| = \log|N_{Q(\mathcal{A}')/Q(\mathcal{A})}(f)(W)| \tag{3.3}$$

where V ranges over all irreducible components of $\widetilde{\mathfrak{X}'} = \mathrm{Spec}\tilde{\mathcal{A}}'$ and where the norm is with respect to the quotient fields $Q(\mathcal{A}')/Q(\mathcal{A})$. Note that $|a(W)|$ is just the supremum norm on $\mathcal{A}$, which extends uniquely to an absolute value $|\ |_W$ on $Q(\mathcal{A})$. Similarly, we have absolute values $|\ |_V$ on $Q(\mathcal{A}')$ for every irreducible component V of $\widetilde{\mathfrak{X}'}$. By [Gu3], Lemma 3.17, they are just the extensions of $|\ |_W$ to absolute values on $Q(\mathcal{A}')$. Moreover, it is shown that $Q(\mathcal{A})$ is stable with respect to $|\ |_W$ and hence the classical formula

$$\sum_V e_{V/W} f_{V/W} = [Q(\mathcal{A}') : Q(\mathcal{A})] \tag{3.4}$$

holds, where $e_{V/W}$ is the ramification index and $f_{V/W}$ is the residue degree of $|\ |_V$ over $|\ |_W$. Let $\hat{Q}(\mathcal{A})$ be the completion of $Q(\mathcal{A})$ with respect to $|\ |_W$. We conclude that

$$Q(\mathcal{A}') \otimes_{Q(\mathcal{A})} \hat{Q}(\mathcal{A}) \cong \prod_V \hat{K}_V \tag{3.5}$$

where $\hat{K}_V$ is the completion of $Q(\mathcal{A}')$ with respect to $|\ |_V$. Here, we have used that the left hand side is a finite dimensional $\hat{Q}(\mathcal{A})$-algebra, hence it has a decomposition into finite dimensional local $\hat{Q}(\mathcal{A})$-algebras R_V with residue fields isomorphic to $\hat{K}_V$ (see e.g. [Se2], §2.1) and formula (3.4) implies $R_V \cong \hat{K}_V$ by using the well known inequality

$$e_{V/W} f_{V/W} \leq [\hat{K}_V : \hat{Q}(\mathcal{A})]. \tag{3.6}$$

for the local degree. By [Ja], Theorem 9.8, we have

$$|f|_V = |N_{\hat{K}_V/\hat{Q}(\mathcal{A})}(f)|_W^{\frac{1}{[\hat{K}_V:\hat{Q}(\mathcal{A})]}}.$$

Using (3.5), we get

$$|N_{Q(\mathcal{A}')/Q(\mathcal{A})}(f)|_W = \prod_V |N_{\hat{K}_V/\hat{Q}(\mathcal{A})}(f)|_W = \prod_V |f|_V^{[\hat{K}_V:\hat{Q}(\mathcal{A})]}. \tag{3.7}$$

We have already seen above that (3.4) implies equality in (3.6). All value groups involved are equal to $|K^\times|$, hence $e_{V/W} = 1$. By [Gu3], Lemma 3.18, the residue fields of $| \ |_W, | \ |_V$ are isomorphic to $\tilde{K}(W)$ and $\tilde{K}(V)$, respectively, and therefore

$$[\tilde{K}(V) : \tilde{K}(W)] = f_{V/W} = [\hat{K}_V : \hat{Q}(\mathcal{A})].$$

We conclude that (3.3) follows from (3.7). □

3.5.4 Definition. Let Y be a closed analytic subset of X. Then the *local Arakelov-Chow group with support in* Y is defined by

$$\widehat{CH}^Y(X,v) := \{\alpha \in \hat{Z}(X,v) \mid |\alpha^{an}| \subset Y\}/\{\widehat{\mathrm{div}}(\mathbf{f}) \mid \mathbf{f} \ K_1\text{-chain on } Y\}.$$

3.5.5 We always grade the local Chow group by relative dimension and write $\widehat{CH}_*^Y(X,v)$ for the corresponding graded module. Note that the dimension of vertical cycles is the dimension of the cycle on the special fibre shifted by -1. The local Chow groups play the role of a homology theory. If $Y = \emptyset$, then we write $\widehat{CH}_*^{fin}(X,v)$.

3.5.6 Remark. Let $\varphi : X' \to X$ be a proper morphism over K and let Y, Y' be closed analytic subsets of X and X' with $\varphi(Y') \subset Y$. Then Proposition 3.5.3 proves that φ_* descends to a well-defined graded homomorphism

$$\varphi_* : \widehat{CH}_*^{Y'}(X',v) \to \widehat{CH}_*^Y(X,v).$$

3.5.7 Definition. Let $\varphi : X' \to X$ be a morphism of proper schemes over K, let s be an invertible meromorphic section of a line bundle L on X, let $\mathcal{L}$ be a formal K°-model of L and let $\alpha' \in \hat{Z}_k(X',v)$. The goal is to define a *refined intersection product*

$$\widehat{\mathrm{div}}_{\mathcal{L}}(s).{}_\varphi\alpha' \in \widehat{CH}_{k-1}^{|(\alpha')^{an}|\cap\varphi^{-1}|\mathrm{div}(s)|}(X',v).$$

First, we assume that α' is a horizontal prime cycle Y' on X'. If $\mathrm{div}(s)$ intersects $\varphi(Y')$ properly in X, then $s_{Y'} := \varphi|_{Y'}^*(s)$ is a well-defined invertible meromorphic section of $\varphi|_{Y'}^*(L)$. If $\mathcal{L}$ lives on the K°-model $\mathfrak{X}$, then Proposition 3.4.2b) gives an extension $\bar{\varphi} : \mathfrak{X}' \to \mathfrak{X}$ of φ to a suitable $\mathfrak{X}' \in M_{X'}$. Let $\bar{Y}'$ be the closure of Y' in $\mathfrak{X}'$ and let $\bar{s}_{Y'}$ be the extension of $s_{Y'}$ to a meromorphic section of $\bar{\varphi}^*\mathcal{L}|_{\bar{Y}'}$. Then we define

$$\left(\widehat{\mathrm{div}}_{\mathcal{L}}(s).{}_\varphi Y'\right)_{\mathfrak{X}'} := \mathrm{div}(\bar{s}_{Y'}).Y' \in \tilde{Z}(\mathfrak{X}',v). \tag{3.8}$$

Clearly, this induces a well-defined element in $\hat{Z}(X',v)$ called $\widehat{\operatorname{div}}_{\mathcal{L}}(s)._{\varphi}Y'$ independent of the choice of $\bar{\varphi}$.

If $\varphi(Y') \subset |\operatorname{div}(s)|$, then let $s_{Y'}$ be any non-trivial invertible meromorphic section of $\varphi|_{Y'}^{*}(L)$. As above, we have (3.8) which is well-defined up to K_1-chains on Y'. It induces an element in $\widehat{CH}_{*}^{Y'}(X',v)$ denoted by $\widehat{\operatorname{div}}_{\mathcal{L}}(s)._{\varphi}Y'$. This finishes the definition for horizontal prime cycles and by linearity, we extend the definition to all horizontal cycles.

If α' is vertical, then we define

$$\widehat{\operatorname{div}}_{\mathcal{L}}(s)._{\varphi}\alpha' := c_1(\bar{\varphi}^{*}(\mathcal{L})) \cap \alpha' \in \hat{Z}(X',v)$$

where $\bar{\varphi}$ is now an extension of φ to K°-models and where the right hand side is defined componentwise. Using Proposition 3.4.2c), we easily deduce that the definition is independent of the choice of the extension.

By linearity, we extend the definition of $\widehat{\operatorname{div}}_{\mathcal{L}}(s)._{\varphi}\alpha'$ to all $\alpha' \in \hat{Z}(X',v)$. If $X' = X$ and the refined intersection product is with respect to the identity, we just write

$$\widehat{\operatorname{div}}_{\mathcal{L}}(s).\alpha \in \widehat{CH}_{*}^{|(\alpha)^{an}| \cap |\operatorname{div}(s)|}(X,v).$$

for $\alpha \in \hat{Z}(X,v)$. If $|\operatorname{div}(s)|$ and α^{an} intersect properly in X, then this refined intersection product is the same as in Proposition 3.4.6.

The proof of the following two results is trivial.

3.5.8 Proposition. *Let s be an invertible meromorphic section of a line bundle L on X, let $\mathcal{L}$ be a K°-model of L and let $\alpha_1, \alpha_2 \in \hat{Z}(X,v)$, then*

$$\widehat{\operatorname{div}}_{\mathcal{L}}(s).(\alpha_1 + \alpha_2) = \widehat{\operatorname{div}}_{\mathcal{L}}(s).\alpha_1 + \widehat{\operatorname{div}}_{\mathcal{L}}(s).\alpha_2 \in \widehat{CH}_{*}^{|\operatorname{div}(s)| \cap (|\alpha_1^{an}| \cup |\alpha_2^{an}|)}(X,v).$$

3.5.9 Proposition. *Let s_1, s_2 be invertible meromorphic sections of line bundles L_1, L_2 on X with K°-models $\mathcal{L}_1$, $\mathcal{L}_2$ and let $\alpha \in \hat{Z}(X,v)$. Then*

$$\left(\widehat{\operatorname{div}}_{\mathcal{L}_1}(s_1) + \widehat{\operatorname{div}}_{\mathcal{L}_2}(s_2)\right).\alpha = \widehat{\operatorname{div}}_{\mathcal{L}_1}(s_1).\alpha + \widehat{\operatorname{div}}_{\mathcal{L}_2}(s_2).\alpha \in \widehat{CH}_{*}^{(|\operatorname{div}(s_1)| \cup |\operatorname{div}(s_2)|) \cap |\alpha^{an}|}(X,v).$$

3.5.10 Proposition. *Let $\varphi : X' \to X$ be a morphism of proper schemes over K, let s be an invertible meromorphic section of a line bundle L on X, let $\mathcal{L}$ be a K°-model of L and let $\alpha' \in \hat{Z}(X',v)$. Then we have the projection formula*

$$\varphi_{*}\left(\widehat{\operatorname{div}}_{\mathcal{L}}(s)._{\varphi}\alpha'\right) = \widehat{\operatorname{div}}_{\mathcal{L}}(s).\varphi_{*}(\alpha') \in \widehat{CH}_{*}^{|\operatorname{div}(s)| \cap \varphi(|(\alpha')^{an}|)}(X,v).$$

Proof: Note that the morphism φ is proper. We may assume that $(\alpha')^{an}$ is a prime cycle Z'. The claim is an immediate consequence of Corollary 3.4.9 if $\operatorname{div}(s)$ intersects Z' properly. It remains to consider the case $\alpha' = Z'$ mapping into $|\operatorname{div}(s)|$. Then we may replace X' by Z', X by $\varphi(Z')$ and s by any non-trivial meromorphic section of $L|_{\varphi(Z')}$ to reduce to the old situation. □

3.5.11 Definition. Let s be an invertible meromorphic section of a line bundle L on X with K°-model $\mathcal{L}$ on X. Then we set

$$\operatorname{cyc}\left(\widehat{\operatorname{div}}_{\mathcal{L}}(s)\right) := \widehat{\operatorname{div}}_{\mathcal{L}}(s).\operatorname{cyc}(X)$$

which is a well-defined cycle, i.e. an element of $Z(X) \oplus \varprojlim_{\mathfrak{X} \in M_X} Z(\mathfrak{X},v)$. But usually, we only need its class in $\hat{Z}(X,v)$.

3.5.12 Theorem. *Let s, s' be invertible meromorphic section of line bundles L, L' on X with K°-models $\mathcal{L}$, $\mathcal{L}'$. Then the commutativity law*

$$\widehat{\operatorname{div}}_{\mathcal{L}}(s).\mathrm{cyc}\left(\widehat{\operatorname{div}}_{\mathcal{L}'}(s')\right) = \widehat{\operatorname{div}}_{\mathcal{L}'}(s').\mathrm{cyc}\left(\widehat{\operatorname{div}}_{\mathcal{L}}(s)\right)$$

holds in $\widehat{CH}_*^{|\operatorname{div}(s)|\cap|\operatorname{div}(s')|}(X,v)$.

Proof: If $|\operatorname{div}(s)|$ intersects $|\operatorname{div}(s')|$ properly in X, then this follows from Corollary 3.4.10. The reduction of the general case to the case of proper intersection is similar as in the proof of [Fu], Theorem 2.4. We may view the Cartier divisors $D := \operatorname{div}(s)$ and $D' := \operatorname{div}(s')$ in the algebraic category to make the analogy complete.

First, we may assume that both D and D' are effective Cartier divisors. The excess of intersection is defined by

$$\epsilon(D, D') := \max_Y \operatorname{ord}_Y(D)\operatorname{ord}_Y(D')$$

where Y ranges over all prime divisors of X. We proceed by induction on the excess. Note that $\epsilon(D, D') = 0$ is equivalent to proper intersection. Let $\pi : X' \to X$ be the blow up of X along the intersection scheme $D \cap D'$ with exceptional divisor E which is always an effective Cartier divisor. There are effective Cartier divisors C, C' on X' satisfying the properties

a) $\pi^*D = E + C$, $\pi^*D' = E + C'$;

b) $|C| \cap |C'| = \emptyset$;

c) $\max\{\epsilon(C, E), \epsilon(C', E)\} \leq \max\{\epsilon(D, D') - 1, 0\}$.

By Proposition 3.4.2, there exists a K°-model $\mathcal{E}$ of $O(E)$. The line bundle $\mathcal{L}$ lives on a K°-model $\mathfrak{X}$. By Proposition 3.4.2, there is an extension $\bar{\pi} : \mathfrak{X}' \to \mathfrak{X}$ of π to a K°-model $\mathfrak{X}'$ of X' and we may assume that $\mathcal{E}$ lives on $\mathfrak{X}'$. We get the decompositions

$$\bar{\pi}^*\mathcal{L} = \mathcal{E} \otimes \mathcal{C}$$

and

$$\bar{\pi}^*\mathcal{L}' = \mathcal{E} \otimes \mathcal{C}'$$

for line bundles $\mathcal{C}$ and $\mathcal{C}'$ on $\mathfrak{X}'$. Let s_E be the canonical global section of $O(E)$ and let $s_C := \pi^*s/s_E$, $s_{C'} := \pi^*s'/s_E$. By projection formula, we have

$$\begin{aligned}\widehat{\operatorname{div}}_{\mathcal{L}}(s).\mathrm{cyc}\left(\widehat{\operatorname{div}}_{\mathcal{L}'}(s')\right) &= \pi_*\left(\widehat{\operatorname{div}}_{\mathcal{E}}(s_E).\mathrm{cyc}\left(\widehat{\operatorname{div}}_{\mathcal{E}}(s_E)\right) + \widehat{\operatorname{div}}_{\mathcal{E}}(s_E).\mathrm{cyc}\left(\widehat{\operatorname{div}}_{\mathcal{C}'}(s_{C'})\right)\right) \\ &\quad + \pi_*\left(\widehat{\operatorname{div}}_{\mathcal{C}}(s_C).\mathrm{cyc}\left(\widehat{\operatorname{div}}_{\mathcal{E}}(s_E)\right) + \widehat{\operatorname{div}}_{\mathcal{C}}(s_C).\mathrm{cyc}\left(\widehat{\operatorname{div}}_{\mathcal{C}'}(s_{C'})\right)\right)\end{aligned}$$

up to K_1-chains on $|D| \cap |D'|$. We claim that the commutativity law holds for each of the summands. The first is trivial, the second and third are by induction on the excess and the last follows from the proper intersection case. Going back again by projection formula, we get the commutativity law for $\widehat{\operatorname{div}}_{\mathcal{L}}(s)$ and $\widehat{\operatorname{div}}_{\mathcal{L}'}(s')$ in the effective case.

Next, we assume that only D' is effective. Let $\pi : X' \to X$ be the blow up of X with respect to the ideal sheaf of denominators of D with exceptional divisor E. As above, we fix a K°-model $\mathcal{E}$ of $O(E)$. There is an effective Cartier divisor C on X' with

$$\pi^*D = C - E.$$

As above, we may extend π to a morphism $\bar{\pi} : \mathfrak{X}' \to \mathfrak{X}$ and we may assume that $\mathcal{E}$ lives on $\mathfrak{X}'$. Let $\mathcal{C}$ be the K°-model of $O(C)$ given by

$$\bar{\pi}^*\mathcal{L} = \mathcal{C} \otimes \mathcal{E}^{-1}.$$

For the pairs $\left(\widehat{\operatorname{div}}_{\mathcal{C}}(s_C), \pi^*\widehat{\operatorname{div}}_{\mathcal{L}'}(s')\right)$ and $\left(\widehat{\operatorname{div}}_{\mathcal{E}}(s_E), \pi^*\widehat{\operatorname{div}}_{\mathcal{L}'}(s')\right)$, we know that commutativity holds . By the same argument as above, we deduce from the projection formula that commutativity holds for $\widehat{\operatorname{div}}_{\mathcal{L}}(s)$ and $\widehat{\operatorname{div}}_{\mathcal{L}'}(s')$.

Finally, if neither D nor D' is effective, then let $\pi : X' \to X$ be the blow up of X with respect to the ideal sheaf of denominators of D as above. Using the same notation, we know that commutativity holds for the pairs $\left(\widehat{\operatorname{div}}_{\mathcal{C}}(s_C), \pi^*\widehat{\operatorname{div}}_{\mathcal{L}'}(s')\right)$ and $\left(\widehat{\operatorname{div}}_{\mathcal{E}}(s_E), \pi^*\widehat{\operatorname{div}}_{\mathcal{L}'}(s')\right)$ now using just effectivity in the first component. Again by projection formula, we conclude that $\widehat{\operatorname{div}}_{\mathcal{L}}(s)$ and $\widehat{\operatorname{div}}_{\mathcal{L}'}(s')$ commute. □

3.5.13 Remark. It is immediate from the definitions that for an invertible meromorphic function f on X and for $\alpha \in \hat{Z}(X, v)$, there is a K_1-chain $\mathbf{g}$ on $|\alpha^{an}|$ such that

$$\widehat{\operatorname{div}}(f).\alpha = \widehat{\operatorname{div}}(\mathbf{g}) \in \widehat{CH}_*^{|\alpha^{an}|}(X, v).$$

On the other hand, we have

3.5.14 Corollary. *Let s be an invertible meromorphic section of a line bundle L on X with K°-model $\mathcal{L}$ and let $\mathbf{f}$ be a K_1-chain on a closed analytic subset Y of X, then there is a K_1-chain $\mathbf{g}$ on $|\operatorname{div}(s)| \cap Y$ with*

$$\widehat{\operatorname{div}}_{\mathcal{L}}(s).\widehat{\operatorname{div}}(\mathbf{f}) = \widehat{\operatorname{div}}(\mathbf{g}).$$

Proof: We may assume that $\mathbf{f}$ is given by a single rational function f_Y on the irreducible closed subset Y of X. Moreover, projection formula reduces to the case $Y = X$. Note that if $Y \subset |\operatorname{div}(s)|$, then we use any invertible meromorphic section s_Y of $L|_Y$ instead of $s|_Y$. Then we apply commutativity (Theorem 3.5.12) and Remark 3.5.13 to get the claim. □

3.5.15 Remark. This proves that the intersection product with $\widehat{\operatorname{div}}_{\mathcal{L}}(s)$ descends to a homomorphism

$$\widehat{CH}_k^Y(X, v) \to \widehat{CH}_{k-1}^{Y \cap |\operatorname{div}(s)|}(X, v).$$

Moreover, it makes sense to consider multiple refined intersection products

$$\widehat{\operatorname{div}}_{\mathcal{L}_1}(s_1).\widehat{\operatorname{div}}_{\mathcal{L}_2}(s_2) \ldots \widehat{\operatorname{div}}_{\mathcal{L}_n}(s_n).\alpha.$$

It is always unterstood that we proceed from the right.

3.5.16 Corollary. For $i = 1, 2$, let $\varphi_i : X \to X_i$ be a morphism of proper schemes over K and let s_i be an invertible meromorphic section of a line bundle L_i on X_i with K°-model $\mathcal{L}_i$. For a closed analytic subset Y of X and $\alpha \in \widehat{CH}_*^Y(X, v)$, the identity

$$\widehat{\operatorname{div}}_{\mathcal{L}_1}(s_1)._{\varphi_1}\widehat{\operatorname{div}}_{\mathcal{L}_2}(s_2)._{\varphi_2}.\alpha = \widehat{\operatorname{div}}_{\mathcal{L}_2}(s_2)._{\varphi_2}\widehat{\operatorname{div}}_{\mathcal{L}_1}(s_1)._{\varphi_1}.\alpha$$

holds in $\widehat{CH}_*^{\varphi_1^{-1}|\operatorname{div}(s_1)| \cap \varphi_2^{-1}|\operatorname{div}(s_2)| \cap Y}(X, v)$.

Proof: First, we extend φ_i to a morphism of K°-models $\bar{\varphi}_i : \mathfrak{X}' \to \mathfrak{X}_i$ where $\mathcal{L}_i$ lives on $\mathfrak{X}_i$ (Proposition 3.4.2). If α is vertical, then the claim may be checked componentwise on sufficiently large K°-models $\mathfrak{X}$ where it is well-known from algebraic intersection theory ([Fu], Corollary 2.4.2). So we may assume that α is horizontal and prime, moreover even $\alpha = \operatorname{cyc}(X)$ as in the proof of Corollary 3.5.14.

If $\varphi_i(X)$ is not contained in $|\operatorname{div}(s_i)|$ for $i = 1, 2$, then $s_i' = s_i \circ \varphi$ is a well-defined invertible meromorphic section of $\varphi_i^*(L_i)$. For any $\beta \in \hat{Z}(X, v)$, we have

$$\widehat{\operatorname{div}}_{\mathcal{L}_i}(s_i)._{\varphi_i}\beta = \widehat{\operatorname{div}}_{\bar{\varphi}^*\mathcal{L}_i}(s_i').\beta \in \widehat{CH}_*^{|\beta^{an}| \cap \varphi^{-1}|\operatorname{div}(s_i)|}(X, v)$$

and the claim follows from Theorem 3.5.12. If $\varphi_i(X) \subset |\mathrm{div}(s_i)|$ for some $i \in \{1,2\}$, then let s_i' be any invertible meromorphic section of $\varphi^*(L_i)$. Then for $\beta \in \hat{Z}(X,v)$, we have

$$\widehat{\mathrm{div}}_{\mathcal{L}_i}(s_i).{}_{\varphi_i}\beta = \widehat{\mathrm{div}}_{\bar{\varphi}^*\mathcal{L}_i}(s_i').\beta \in \widehat{CH}_*^{|\beta^{an}|}(X,v).$$

Again Theorem 3.5.12 proves the claim. □

3.6 Refined Intersection Theory on Models

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $|\ |$. In this section, we consider only admissible formal schemes over the valuation ring K° with algebraic generic fibre. The formal schemes are denoted by $\mathfrak{X}, \mathfrak{X}', \ldots$ and the corresponding generic fibres by $X, X', \ldots$. Algebraic means that X is the rigid analytic variety associated to a proper scheme over K. By the GAGA principle and 3.3.9, all admissible formal schemes considered will be proper.

In the previous section, we have defined a refined intersection product with divisors on the projective limit over all models of a given algebraic generic fibre. Now we deduce a similar theory for individual models which takes care also of the supports in the special fibre. It has the advantage that flat pull-backs are well-defined.

One might ask here, why we can not identify the local Chow group with support with the Chow group on the support? Since the support may have vertical components, the corresponding closed subscheme is no longer admissible and formal schemes locally of finite type over arbitrary valuations rings of height one do not behave well with respect to localization.

3.6.1 A K_1-chain $\mathbf{f}$ on $\mathfrak{X}$ is a formal sum of a K_1-chain $\mathbf{f}^{hor}$ on X and a K_1-chain $\mathbf{f}^{ver}$ on $\tilde{\mathfrak{X}}$. We define

$$\mathrm{div}_{\mathfrak{X}}(\mathbf{f}) := \mathrm{div}_{\mathfrak{X}}(\mathbf{f}^{hor}) + \mathrm{div}_{\mathfrak{X}}(\mathbf{f}^{ver})$$

where we get $\mathrm{div}_{\mathfrak{X}}(\mathbf{f}^{hor})$ from 3.5.2 applied to the component $\mathfrak{X}$ and $\mathrm{div}_{\mathfrak{X}}(\mathbf{f}^{ver}) = \mathrm{div}_{\tilde{\mathfrak{X}}}(\mathbf{f}^{ver})$ is from algebraic geometry on the special fibre.

3.6.2 Recall that the support $|D|$ of a Cartier divisor D on $\mathfrak{X}$ has a horizontal part $|D^{an}|$ on X and a vertical part on $\tilde{\mathfrak{X}}$. It is clear that the reduction of the closure of the horizontal part in $\mathfrak{X}$ is contained in the vertical part. If Z is a horizontal prime cycle on $\mathfrak{X}$, then we define the support $|Z|$ to have horizontal part Z and vertical part equal to the reduction of $\bar{Z}$. If $\mathfrak{Z}$ is a vertical prime cycle, then the support is just its support as a cycle on $\tilde{\mathfrak{X}}$. In general, the support $|\mathfrak{Z}|$ of any cycle $\mathfrak{Z}$ on $\mathfrak{X}$ is the union of all supports of its components, performed for horizontal and vertical parts separately.

To deal with this situation, a pair $S := (Y, V)$ is called a *support* on $\mathfrak{X}$ if it is the support of a cycle. Here Y is a closed subset of X and V is a closed subset of $\tilde{\mathfrak{X}}$ containing the reduction of $\bar{Y}$. Componentwise, it makes sense to consider range and inverse images of supports, as well as intersections.

3.6.3 Definition. Let $S = (Y, V)$ be a support on $\mathfrak{X}$. We say that a K_1-chain $\mathbf{f}$ of $\mathfrak{X}$ is defined on S if $\mathbf{f}^{hor}$ is a K_1-chain on Y and $\mathbf{f}^{ver}$ is a K_1-chain on V. The *local Chow group of $\mathfrak{X}$ with support in S* is defined by

$$CH_*^S(\mathfrak{X}, v) := \{\mathfrak{Z} \in Z(\mathfrak{X}, v) \mid |\mathfrak{Z}| \subset S\}/\{\mathrm{div}_{\mathfrak{X}}(\mathbf{f}) \mid \mathbf{f}\ K_1\text{-chain on } S\}$$

where the grading is again by relative dimension. If $S = (\emptyset, \tilde{\mathfrak{X}})$, then we write $CH_*^{fin}(\mathfrak{X}, v)$.

3.6.4 Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism and let $\mathbf{g}$ be a K_1-chain on $\mathfrak{X}'$. Then we define the push-forward of $\mathbf{g}$ as the K_1-chain on $\mathfrak{X}$ given by

$$\varphi_*(\mathbf{g}) := \varphi_*^{an}(\mathbf{g}^{hor}) + \tilde{\varphi}_*(\mathbf{g}^{ver})$$

using the corresponding constructions on generic and special fibres. Then it follows immediately from Proposition 3.5.3 and 1.2.9 applied to the special fibre that

$$\varphi_* \left(\mathrm{div}_{\mathfrak{X}'}(\mathbf{g})\right) = \mathrm{div}_{\mathfrak{X}} \left(\varphi_*(\mathbf{g})\right).$$

If S' is a support on $\mathfrak{X}'$ with $\varphi(S')$ contained in the support S of $\mathfrak{X}$, then we get a well-defined map

$$CH_*^{S'}(\mathfrak{X}', v) \longrightarrow CH_*^{S}(\mathfrak{X}, v)$$

induced by push-forward.

3.6.5 Definition. Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism over K, let s be an invertible meromorphic section of a line bundle $\mathcal{L}$ on $\mathfrak{X}$ and let $\mathfrak{Z}' \in Z(\mathfrak{X}', v)$. By using the construction of Definition 3.5.7 componentwise, we get the *refined intersection product*

$$\mathrm{div}(s).{}_{\varphi}\mathfrak{Z}' \in CH_*^{|(\mathfrak{Z}')| \cap \varphi^{-1}|\mathrm{div}(s)|}(\mathfrak{X}', v).$$

Easily, we can prove the following results from the corresponding statements in section 3.3.

3.6.6 Proposition. *Let s be an invertible meromorphic section of a line bundle $\mathcal{L}$ on $\mathfrak{X}$ and let $\mathfrak{Z}'_1, \mathfrak{Z}'_2$ be cycles on $\mathfrak{X}'$, then*

$$\mathrm{div}(s).{}_{\varphi}(\mathfrak{Z}'_1 + \mathfrak{Z}'_2) = \mathrm{div}(s).{}_{\varphi}\mathfrak{Z}'_1 + \mathrm{div}(s).{}_{\varphi}\mathfrak{Z}'_2 \in CH_*^{(|\mathfrak{Z}'_1| \cup |\mathfrak{Z}'_2|) \cap \varphi^{-1}|\mathrm{div}(s)|}(\mathfrak{X}', v).$$

3.6.7 Proposition. *Let s_1, s_2 be invertible meromorphic sections of line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on $\mathfrak{X}$ and let $\mathfrak{Z}'$ be a cycle on $\mathfrak{X}'$, then*

$$\left(\mathrm{div}(s_1) + \mathrm{div}(s_2)\right).{}_{\varphi}\mathfrak{Z}' = \mathrm{div}(s_1).{}_{\varphi}\mathfrak{Z}' + \mathrm{div}(s_2).{}_{\varphi}\mathfrak{Z}' \in CH_*^{|\mathfrak{Z}'| \cap (\varphi^{-1}|\mathrm{div}(s_1)| \cup \varphi^{-1}|\mathrm{div}(s_2)|)}(\mathfrak{X}', v).$$

3.6.8 Proposition. *Let s be an invertible meromorphic section of a line bundle $\mathcal{L}$ on $\mathfrak{X}$ and let $\mathfrak{Z}'$ be a cycle on $\mathfrak{X}'$. Then we have the projection formula*

$$\varphi_* \left(\mathrm{div}(s).{}_{\varphi}\mathfrak{Z}'\right) = \mathrm{div}(s).\varphi_*(\mathfrak{Z}') \in CH_*^{|\mathrm{div}(s)| \cap \varphi|\mathfrak{Z}'|}(\mathfrak{X}', v).$$

3.6.9 Definition. For an invertible meromorphic section s of a line bundle $\mathcal{L}$ on $\mathfrak{X}$, the *associated Weil-divisor* is the cycle

$$\mathrm{cyc}(\mathrm{div}(s)) := \mathrm{div}(s).\mathrm{cyc}(X)$$

on $\mathfrak{X}$.

3.6.10 Theorem. *Let s, s' be invertible meromorphic sections of line bundles $\mathcal{L}$, $\mathcal{L}'$ on $\mathfrak{X}$. Then the commutativity*

$$\mathrm{div}(s).\mathrm{cyc}\left(\mathrm{div}(s')\right) = \mathrm{div}(s').\mathrm{cyc}\left(\mathrm{div}(s)\right)$$

holds in $CH_*^{|\mathrm{div}(s)| \cap |\mathrm{div}(s')|}(\mathfrak{X}, v)$.

Proof: The argument is completely analogous to the proof of Theorem 3.5.12. First, one notes that the claim follows from Theorem 3.3.14 if D and D' intersect properly in the generic fibre. Then the same blowing up process as in the proof of Theorem 3.5.12 is done to reduce to this case. We have only to take care that the meromorphic sections $s_C, s_{C'}$ and s_E of $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{E}$ have support over $|\mathrm{div}(s)| \cap |\mathrm{div}(s')|$. To achieve this, we use the following remark: Let $\pi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism extending a blowing up of the generic fibre with exceptional divisor E in X'. We assume

$$E \subset \pi^{-1}\left(|\mathrm{div}(s^{an})| \cap |\mathrm{div}(s'^{an})|\right).$$

Then the closure $\bar{E}$ of E in $\mathfrak{X}'$ is given by a coherent ideal sheaf $\mathcal{J}$ on $\mathfrak{X}'$ ([Gu3], Proposition 3.3). Replacing $\mathfrak{X}'$ by an admissible formal blowing up as in the proof of [Gu3], Lemma 7.6, we may assume that $\mathcal{J}$ is an invertible ideal sheaf. Note that we may choose the center of the admissible formal blowing up in the reduction of $\bar{E}$, hence if we set $\mathcal{E} := \mathcal{J}$ and if we consider s_E as a section of $\mathcal{E}$, then the support of $\mathrm{div}_{\mathcal{E}}(s_E)$ is contained in $\pi^{-1}(|\mathrm{div}(s)| \cap |\mathrm{div}(s')|)$. Choosing the models $\mathcal{E}$ in this way, we can use the proof of Theorem 3.5.12 to deduce the claim. □

3.6.11 Remark. As in section 9, we can prove that intersection product with $\mathrm{div}(s)$ factors through Chow groups, i.e. we have a well-defined homomorphism

$$CH_*^S(\mathfrak{X}, v) \to CH_*^{S \cap |\mathrm{div}(s)|}(\mathfrak{X}, v)$$

for any support S on $\mathfrak{X}$. Similarly as Corollary 3.5.16, we deduce

3.6.12 Corollary. For $i = 1, 2$, let $\varphi_i : \mathfrak{X} \to \mathfrak{X}_i$ be a morphism over K° and let s_i be an invertible meromorphic section of a line bundle $\mathcal{L}_i$ on $\mathfrak{X}_i$ with K°-model $\mathcal{L}_i$. For a support S on $\mathfrak{X}$ and for $\alpha \in CH_*^S(\mathfrak{X}, v)$, the identity

$$\mathrm{div}(s_1).{}_{\varphi_1}\mathrm{div}(s_2).{}_{\varphi_2}.\alpha = \mathrm{div}(s_2).{}_{\varphi_2}\mathrm{div}(s_1).{}_{\varphi_1}.\alpha$$

holds in $CH_*^{S \cap \varphi_1^{-1}|\mathrm{div}(s_1)| \cap \varphi_2^{-1}|\mathrm{div}(s_2)|}(\mathfrak{X}, v)$.

3.6.13 Lemma. Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a flat morphism and let $\mathbf{f}$ be a K_1-chain on the support S on $\mathfrak{X}$. Then there is a K_1-chain $\mathbf{f}'$ on $\varphi^{-1}(S)$ such that

$$\varphi^* \mathrm{div}_{\mathfrak{X}}(\mathbf{f}) = \mathrm{div}_{\mathfrak{X}'}(\mathbf{f}').$$

Proof: We may assume that $\mathbf{f}$ is given by a single component f_W on an irreducible closed analytic subset W of X. We may assume $W = X$, hence the K_1-chain is given by a rational function f on X. Then the claim follows immediately from Proposition 3.3.15. □

3.6.14 Remark. We conclude that the flat pull-back induces a well-defined homomorphism

$$\varphi^* : CH_*^S(\mathfrak{X}, v) \to CH_*^{\varphi^{-1}S}(\mathfrak{X}', v).$$

If φ has relative dimension d, then φ^* shifts the dimension of cycles by d.

3.6.15 Proposition. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a flat morphism, let s be an invertible meromorphic section of a line bundle $\mathcal{L}$ on $\mathfrak{X}$ and let $\alpha \in CH_*^S(\mathfrak{X}, v)$. Then we have*

$$\varphi^*(\mathrm{div}(s).\alpha) = \mathrm{div}(s).{}_\varphi\varphi^*(\alpha) \in CH_*^{\varphi^{-1}(|\mathrm{div}(s)| \cap S)}(\mathfrak{X}', v).$$

Proof: For vertical α, this follows from 1.2.17.. So we may assume that $\alpha = S$ is horizontal and prime. By Lemma 3.6.13, we easily reduce to $\alpha = X$. Then the claim follows immediately from Proposition 3.3.15. □

Chapter 4

Non-Archimedean Arakelov-Chow Cohomology

4.1 Chow Cohomology

In the next section, we formalize the properties of divisoral operations considered in 3.5 and 3.6 defining local Chow cohomology classes. In this section, we recall the definition and properties of Chow cohomology classes from [Fu], §17. The Chow cohomology ring has similar properties as we expect from cohomology and which can't be obtained for singular spaces simply by grading the classical Chow ring by codimension. The classical Chow ring plays here the role of homology.

All spaces are assumed to be schemes proper over the given ground field K. This is the case of interest for our applications, but the constructions and results are valid for all algebraic schemes over K. One just has to assume that every push-forward comes with a proper morphism.

4.1.1 Definition. Let Y be a closed subset of X and let $p \in \mathbb{Z}$. A *Chow cohomology class* $c \in CH^p_Y(X)$ is a family of homomorphisms

$$CH_k(X') \to CH_{k-p}(\psi^{-1}Y), \quad \alpha' \mapsto c \cap_\psi \alpha',$$

for all $k \in \mathbb{N}$ and for all morphisms $\psi : X' \to X$, satisfying the following axioms:

(C1) If $\varphi : X'' \to X'$ is a morphism and $\alpha'' \in CH_k(X'')$, then

$$\varphi_*(c \cap_{\psi\circ\varphi} \alpha'') = c \cap_\psi \varphi_*(\alpha'') \in CH_{k-p}(\psi^{-1}Y).$$

(C2) If $\varphi : X'' \to X'$ is a flat morphism of relative dimension d and $\alpha' \in CH_k(X')$, then

$$\varphi^*(c \cap_\psi \alpha') = c \cap_{\psi\circ\varphi} \varphi^*(\alpha') \in CH_{k+d-p}(\varphi^{-1}\psi^{-1}Y).$$

(C3) If s' is an invertible meromorphic section of a line bundle L' on X' and $\alpha' \in CH_k(X')$, then

$$\operatorname{div}(s').(c \cap_\psi \alpha') = c \cap_\psi (\operatorname{div}(s').\alpha') \in CH_{k-p-1}(|\operatorname{div}(s')| \cap \psi^{-1}Y).$$

4.1.2 Example. Let s be an invertible meromorphic section of a line bundle L on X with support in the closed subset Y of X. Then s induces a Chow cohomology class $c_1(s) \in CH^1_Y(X)$ by

$$c_1(s) \cap_\psi \alpha' := \operatorname{div}(s).{}_\psi\alpha'.$$

The axioms (C1)-(C3) follow from 1.2.16, 1.2.17 and 1.2.20. More generally, this holds for a pseudo-divisor with support in Y (cf. [Fu], chapter 2). Note that $c_1(s)$ plays the role of a refined Chern class.

4.1.3 Example. If E is a vector bundle on X, the Chern class $c_p(E)$ induces an element of $CH^p(X)$ by

$$c_p(E) \cap_\psi \alpha' := c_p(\psi^* E).\alpha'.$$

This follows from Example 4.1.2 and the splitting principle ([Fu], 3.2).

4.1.4 Remark. The groups $CH^p_Y(X)$ are completely analogous to the corresponding groups of Fulton ([Fu], Example 17.3.1). Note that there are two differences in the axioms. First, Fulton uses arbitrary algebraic schemes. If we restrict his approach to the category of separated algebraic schemes over K, which may be done without any problems, then the local Chow groups are the same. This follows from the fact that every algebraic scheme is a dense open subscheme of a proper scheme over K (cf. [Na]). The second difference is in axiom (C3), where Fulton requires that Chow cohomology classes commute with refined Gysin homomorphisms relative to regular embeddings. The equivalence follows from [Fu], Theorem 17.1. It implies that Chow cohomology classes commute with arbitrary intersections.

4.1.5 Let Y and Z be closed subsets of X. For $c \in CH^p_Y(X)$ and $c' \in CH^q_Z(X)$, the *cup product* $c \cup c' \in CH^{p+q}_{Y\cap Z}(X)$ is defined by

$$(c \cup c') \cap_\psi \alpha' := c \cap_\psi (c' \cap_\psi \alpha') \in CH_{k-p-q}\left(\psi^{-1}(Y \cap Z)\right).$$

The cup product has an obvious associativity property.

4.1.6 Let $\varphi : Z \to X$ be a morphism and let Y be a closed subset of X. For $c \in CH^p_Y(X)$, the *pull back* $\varphi^*(c) \in CH^p_{\varphi^{-1}Y}(Z)$ is defined by

$$\varphi^*(c) \cap_\psi \alpha' := c \cap_{\varphi\circ\psi} \alpha' \in CH_{k-p}(\psi^{-1}\varphi^{-1}Y)$$

for all $k \in \mathbb{Z}$, all morphisms $\psi : Z' \to Z$ and all $\alpha' \in CH_k(Z')$. The pull back is functorial (i.e. $\varphi_1^* \circ \varphi_2^* = (\varphi_2 \circ \varphi_1)^*$) and compatible with cup product.

4.1.7 Definition. Let $\varphi : X \to Z$ be a flat morphism of relative dimension d and let Y be a closed subset of X. For $c \in CH^p_Y(X)$, we define the *push-forward* $\varphi_*(c) \in CH^{p-d}_{\varphi Y}(Z)$ by

$$\varphi_*(c) \cap_\psi \alpha' := \varphi'_*\left(c \cap_{\psi'} \varphi'^*(\alpha')\right) \in CH_{k+d-p}(\psi^{-1}\varphi Y)$$

for all $k \in \mathbb{Z}$, all morphisms $\psi : Z' \to Z$, all $\alpha' \in CH_k(Z')$ and using the Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi'"] \arrow[d, "\psi'"] & Z' \arrow[d, "\psi"] \\ X \arrow[r, "\varphi"] & Z \end{tikzcd} .$$

For $\varphi_*(c)$, the axioms (C1) and (C2) follow easily from the fibre square rule 1.2.11 while axiom (C3) is a consequence of projection formula 1.2.16 and 1.2.17.

4.1.8 Proposition.

a) If $X \xrightarrow{\varphi} Z \xrightarrow{\psi} W$ are flat morphisms of relative dimensions d, e and if $c \in CH^p_Y(X)$, then

$$\psi_*(\varphi_*(c)) = (\psi \circ \varphi)_*(c) \in CH^{p-d-e}_{\psi\varphi Y}(X).$$

b) Let $c \in CH^p_Y(X)$ and let us consider the Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi'"] \arrow[d, "\psi'"] & Z' \arrow[d, "\psi"] \\ X \arrow[r, "\varphi"] & Z \end{tikzcd}$$

with φ flat of relative dimension d. Then we have the fibre square rule

$$\psi^*\varphi_*(c) = \varphi'_*\psi'^*(c) \in CH^{p-d}_{\psi^{-1}\varphi Y}(Z').$$

c) Let $\varphi : X \to Z$ be a flat morphism of relative dimension d and let $c \in CH^p_Y(X), c' \in CH^q_W(Z)$, then the projection formulas

$$\begin{aligned} \varphi_*(c) \cup c' &= \varphi_*\left(c \cup \varphi^*(c')\right) \\ c' \cup \varphi_*(c) &= \varphi_*\left(\varphi^*(c') \cup c\right) \end{aligned}$$

hold in $CH^{p+q-d}_{W\cap\varphi Y}(Z)$.

Proof: Claims a) and b) are immediate from the definitions. To prove c), the fibre square rule b) implies that it is enough to check the projection formula by evaluating at $\alpha \in CH_k(Z)$. Then

$$\begin{aligned} (\varphi_*(c) \cup c') \cap \alpha &= \varphi_*(c) \cap (c' \cap \alpha) \\ &= \varphi_*\left(c \cap \varphi^*(c' \cap \alpha)\right) \\ &= \varphi_*\left(c \cap \varphi^*(c') \cap \varphi^*(\alpha)\right) \\ &= \varphi_*\left(c \cup \varphi^*(c')\right) \cap \alpha \end{aligned}$$

and similarly, we prove the other formula. □

4.1.9 Remark. By [Fu], Example 17.3.3, we have $CH^p_Y(X) = 0$ for $p < 0$ or $p > \dim(X)$. We define the graded module

$$CH^*_Y(X) := \bigoplus_{p\in\mathbb{Z}} CH^p_Y(X).$$

If $Y = X$, then $CH^*(X) := CH^*_X(X)$ is an associative graded ring with $1 = \mathrm{cyc}(X)$ called the *Chow cohomology ring*. If X is smooth, then *Poincaré duality* below proves that it is commutative. If we assume resolution of singularities (as it holds in characteristic 0), then this implies commutativity of the Chow cohomology ring in general, but we don't need it.

4.1.10 Proposition. *Let X be a smooth scheme of pure dimension d. Then the canonical homomorphisms*

$$CH^p_Y(X) \to CH_{d-p}(Y), \quad c \mapsto c \cap \mathrm{cyc}(X),$$

are isomorphisms. For $Y = X$, they induce a ring isomorphisms between the Chow cohomology ring $CH^*(X)$ and the Chow ring $CH_*(X)$, i.e. the cup product is transferred into the intersection product.

Proof: Let us denote the canonical homomorphism by Φ. We claim that its inverse is defined by mapping $\beta \in CH_{d-p}(Y)$ to the class $\Omega(\beta) \in CH^p_Y(X)$ given by intersection with β on Y, i.e.

$$\Omega(\beta) \cap_\psi \alpha' := \beta._\psi\alpha' \in CH_{k-p}(\psi^{-1}Y)$$

for all $k \in \mathbb{Z}$, all morphisms $\psi : X' \to X$ and all $\alpha' \in CH_k(X')$. The refined intersection product on the right is defined in [Fu], 8.1, and its properties immediately imply that $\Omega(\beta)$ satiesfies axioms (C1)-(C3). To check that the maps are inverse, note first that

$$\Phi\left(\Omega(\beta)\right) = \beta.\mathrm{cyc}(X) = \beta.$$

For $c \in CH^p_Y(X)$ and α' as above, we have

$$\begin{aligned} \Omega\left(\Phi(c)\right)(\alpha') &= \Phi(c)._\psi\alpha' \\ &= \left(c \cap \mathrm{cyc}(X)\right)._\psi\alpha' \\ &= c \cap_\psi \left(\mathrm{cyc}(X)._\psi\alpha'\right) \end{aligned}$$

where the last step follows from the fact that arbitrary intersections commute with Chow cohomology classes (cf. Remark 4.1.4). We conclude that

$$\Omega\left(\Phi(c)\right)(\alpha') = c \cap_\psi \alpha'$$

proving that Ω is the inverse of Φ. For the last claim, we refer to [Fu], Corollary 17.4. □

4.2 Local Chow Cohomology on Models

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $|\ |_v$. All spaces denoted by fractur letters $\mathfrak{X}, \mathfrak{X}', \dots$ are assumed to be admissible formal schemes over the valuation ring K°. Their generic fibres are assumed to be proper schemes over K and they are denoted by greek letters $X, X', \dots$. By the GAGA principle and 3.3.9, our formal schemes will be proper.

The formalism of Chow cohomology, summarized in the last section, is very well adapted to Chern class operations and refined intersection theory. In this section, it will be developped in the situation of models. The local Chow cohomology groups $CH^p_S(\mathfrak{X}, v)$ with support in S act on the local Chow groups $CH_k^{S'}(\mathfrak{X}', v)$ for every $\mathfrak{X}'$ lying over $\mathfrak{X}$. Note that in contrast to the algebraic situation in section 4.1, here we have to keep track of the support S' as well since the local Chow groups $CH_k^{S'}(\mathfrak{X}', v)$ may not be identified with a Chow group of S'. The local Chow groups fulfill the same axioms as in the algebraic situation and we require additionaly that the action on vertical cycles is induced by a local Chow cohomology class of the special fibre. At the end of the section, we get rid of the K°-models by defining the local Arakelov-Chow cohomology groups of X with support in the closed subset Y to be the operations on the local Arakelov-Chow (homology) groups induced by a local Chow cohomology class from any K°-model of X.

4.2.1 Definition. Let S be a support on $\mathfrak{X}$ (cf. 3.6.2) and let $p \in \mathbb{Z}$. A *Chow cohomology class* $c \in CH^p_S(\mathfrak{X}, v)$ is a family of homomorphisms

$$CH_k^{S'}(\mathfrak{X}', v) \to CH_{k-p}^{S' \cap \psi^{-1} S}(\mathfrak{X}', v), \quad \alpha' \mapsto c \cap_\psi \alpha',$$

for all $k \in \mathbb{N}$, all morphisms $\psi : \mathfrak{X}' \to \mathfrak{X}$ and for all supports S' on $\mathfrak{X}'$. They are required to satisfy the following axioms:

(C1) If $\varphi : \mathfrak{X}'' \to \mathfrak{X}'$ is a morphism and $\alpha'' \in CH_k^{S''}(\mathfrak{X}'', v)$ for a support S'' on $\mathfrak{X}''$ with $\varphi(S'') \subset S'$, then

$$\varphi_*(c \cap_{\psi \circ \varphi} \alpha'') = c \cap_\psi \varphi_*(\alpha'') \in CH_{k-p}^{S' \cap \psi^{-1} S}(\mathfrak{X}', v).$$

(C2) If $\varphi : \mathfrak{X}'' \to \mathfrak{X}'$ is a flat morphism of relative dimension d and $\alpha' \in CH_k^{S'}(\mathfrak{X}', v)$, then

$$\varphi^*(c \cap_\psi \alpha') = c \cap_{\psi \circ \varphi} \varphi^*(\alpha') \in CH_{k+d-p}^{\varphi^{-1} S' \cap \varphi^{-1} \psi^{-1} S}(\mathfrak{X}'', v).$$

(C3) If s' is an invertible meromorphic section of a line bundle $\mathcal{L}'$ on $\mathfrak{X}'$ and $\alpha' \in CH_k^{S'}(\mathfrak{X}', v)$, then

$$\operatorname{div}(s').(c \cap_\psi \alpha') = c \cap_\psi (\operatorname{div}(s').\alpha') \in CH_{k-p-1}^{S' \cap |\operatorname{div}(s')| \cap \psi^{-1} S}(\mathfrak{X}', v).$$

(C4) The operation c on vertical cycles is induced by a cohomology class $\tilde{c} \in CH^p(\tilde{\mathfrak{X}}, v)$.

4.2.2 Remark. The group $CH^p(\tilde{\mathfrak{X}}, v)$ is the p-th Chow cohomology group on $\tilde{\mathfrak{X}}$ with coefficients in the value group $\log|K^\times|_v$ defined as operations on Chow groups with coefficients in the value group. The whole section 4.1 is valid for Chow cohomology groups with coefficients.

4.2.3 Example. Let s be an invertible meromorphic section of a line bundle $\mathcal{L}$ on $\mathfrak{X}$. Then s induces a Chow cohomology class $c_1^{\mathcal{L}}(s) \in CH^1_{|\mathrm{div}(s)|}(\mathfrak{X}, v)$ by

$$c_1^{\mathcal{L}}(s) \cap_\psi \alpha' := \mathrm{div}(s).{}_\psi\alpha'.$$

The axioms (C1)-(C4) follow easily from the properties of refined intersection theory deduced in section 3.6.

4.2.4 Remark. Since every morphism of generic fibres extends to suitable large K°-models, it is clear that if we restrict $c \in CH^p_S(\mathfrak{X}, v)$ to the generic fibre, we get $c^{an} \in CH^p_Y(X, v)$ for the horizontal part Y of the support S.

4.2.5 Let S and T be supports on $\mathfrak{X}$. For $c \in CH^p_S(\mathfrak{X}, v)$ and $c' \in CH^q_T(\mathfrak{X}, v)$, we define the *cup product* $c \cup c' \in CH^{p+q}_{S\cap T}(\mathfrak{X}, v)$ by

$$(c \cup c') \cap_\psi \alpha' := c \cap_\psi (c' \cap_\psi \alpha') \in CH_{k-p-q}^{S'\cap\psi^{-1}(S\cap T)}(\mathfrak{X}, v)$$

for all $\alpha' \in CH_k^{S'}(\mathfrak{X}', v)$. The axioms (C1)-(C4) follow from the corresponding axioms for each factor. If $c'' \in CH^r_U(\mathfrak{X}, v)$, then we have associativity

$$c \cup (c' \cup c'') = (c \cup c') \cup c'' \in CH^{p+q+r}_{S\cap T\cap U}(\mathfrak{X}, v).$$

4.2.6 Let $\varphi : \mathfrak{Y} \to \mathfrak{X}$ be a morphism and let S be a support on $\mathfrak{X}$. The *pull-back* of $c \in CH^p_S(\mathfrak{X})$ is defined by

$$\varphi^*(c) \cap_\psi \alpha' := c \cap_{\varphi\circ\psi} \alpha' \in CH_{k-p}^{S'\cap\psi^{-1}\varphi^{-1}S}(\mathfrak{Y}', v)$$

for all $k \in \mathbb{Z}$, all morphisms $\psi : \mathfrak{Y}' \to \mathfrak{Y}$, all supports S' on $\mathfrak{Y}'$ and all $\alpha' \in CH_k^{S'}(\mathfrak{Y}', v)$. Similarly as in the geometric case, one checks that $\varphi^*(c) \in CH^p_{\varphi^{-1}S}(\mathfrak{Y}, v)$. Again the pull-back is functorial and compatible with cup product.

4.2.7 Definition. Let $\varphi : \mathfrak{X} \to \mathfrak{Y}$ be a flat morphism of relative dimension d and let S be a support on $\mathfrak{X}$. For $c \in CH^p_S(\mathfrak{X}, v)$, the *push-forward* $\varphi_*(c) \in CH^{p-d}_{\varphi S}(\mathfrak{Y}, v)$ is defined by

$$\varphi_*(c) \cap_\psi \alpha' := \varphi'_* (c \cap_{\psi'} \varphi'^*(\alpha')) \in CH_{k+d-p}^{\varphi' S'\cap\psi^{-1}\varphi S}(\mathfrak{Y}', v)$$

for all $k \in \mathbb{Z}$, for all morphisms $\psi : \mathfrak{Y}' \to \mathfrak{Y}$, for all supports S' on $\mathfrak{X}'$ and for all $\alpha' \in CH_k^{S'}(\mathfrak{Y}', v)$ where we use the Cartesian diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "\varphi'"] \arrow[d, "\psi'"] & \mathfrak{Y}' \arrow[d, "\psi"] \\ \mathfrak{X} \arrow[r, "\varphi"] & \mathfrak{Y} \end{tikzcd} .$$

As in section 4.1, the axioms (C1) and (C2) follow easily from the fibre square rule, and axiom (C3) is a consequence of projection formula 3.6.8 and Proposition 3.6.15. Finally, axiom (C4) follows immediately from the corresponding axiom for $\tilde{\varphi}_*(\tilde{c})$.

The properties of push-forward are summarized in the following proposition. The proofs of axioms (C1)-(C3) are formally the same as in Proposition 4.1.8 and axiom (C4) is always a consequence of the corresponding property for $\tilde{c}$. The details are left to the reader.

4.2.8 Proposition.

a) *Let* $\mathfrak{X} \xrightarrow{\varphi} \mathfrak{Y} \xrightarrow{\psi} \mathfrak{Z}$ *be flat morphisms of relative dimensions* d *and* e*. For* $c \in CH^p_S(\mathfrak{X}, v)$*, we have*

$$\psi_*(\varphi_*(c)) = (\psi \circ \varphi)_*(c) \in CH^{p-d-e}_{\psi\varphi S}(\mathfrak{X}, v).$$

b) *Let* $c \in CH^p_S(\mathfrak{X})$ *and let* φ *be a flat morphism of relative dimension* d *in the Cartesian diagram*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "\varphi'"] \arrow[d, "\psi'"] & \mathfrak{Y}' \arrow[d, "\psi"] \\ \mathfrak{X} \arrow[r, "\varphi"] & \mathfrak{Y} \end{tikzcd} .$$

Then the fibre square rule

$$\psi^*\varphi_*(c) = \varphi'_*\psi'^*(c) \in CH^{p-d}_{\psi^{-1}\varphi S}(\mathfrak{Y}', v)$$

holds.

c) *Let* $\varphi : \mathfrak{X} \to \mathfrak{Y}$ *be a flat morphism of relative dimension* d *and let* $c \in CH^p_S(\mathfrak{X}, v), c' \in CH^q_T(\mathfrak{Y}, v)$*, then the projection formulas*

$$\begin{aligned} \varphi_*(c) \cup c' &= \varphi_*(c \cup \varphi^*(c')) \\ c' \cup \varphi_*(c) &= \varphi_*(\varphi^*(c') \cup c) \end{aligned}$$

hold in $CH^{p+q-d}_{T\cap\varphi S}(\mathfrak{Y}, v)$.

4.2.9 Proposition. *We have* $CH^p_S(\mathfrak{X}, v) = 0$ *for* $p < 0$ *or* $p > \dim(X) + 1$.

Proof: Let $c \in CH^p_S(\mathfrak{X}, v)$ with $p < 0$ or $p > \dim(X) + 1$. Let $\psi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism, let S' be a support on $\mathfrak{X}'$ and let $\alpha' \in CH^{S'}_k(\mathfrak{X}', v)$. If α' is vertical, then axiom (C4) and Remark 4.1.9 imply that $c \cap_\psi \alpha' = 0$. So we may assume that α' is horizontal. By axiom (C1), we easily reduce to the case α' prime and $\alpha' = Y' = X'$. Hence we have to prove

$$c \cap_\psi \mathrm{cyc}(X') = 0 \in CH^{\psi^{-1}S}_{\dim(X')-p}(\mathfrak{X}', v)$$

for an integral scheme X' with morphism $\psi : \mathfrak{X}' \to \mathfrak{X}$ of K°-models. Replacing $\mathfrak{X}'$ by the closure of $\psi^{an}X$, we may assume that ψ^{an} is dominant. As in the proof of the corresponding result for Chow cohomology groups (cf. [Fu], Example 17.3.3), we apply now the flattening theorem of Raynaud and Gruson ([RG], 5.5.2). We get a Cartesian diagram

$$\begin{tikzcd} X'_1 \arrow[r, "\psi'^{an}"] \arrow[d, "\varphi'^{an}"] & X_1 \arrow[d, "\varphi^{an}"] \\ X' \arrow[r, "\psi^{an}"] & X \end{tikzcd}$$

with φ^{an} a birational morphism and a closed subscheme Y'_1 of X'_1 such that the restriction of φ'^{an} to Y'_1 induces a birational morphism onto X' and such that the restriction of ψ'^{an} to Y'_1 induces a flat morphism to X_1. We extend φ^{an} to a morphism $\varphi : \mathfrak{X}_1 \to \mathfrak{X}$ of K°-models and let $\mathfrak{X}'_1 := \mathfrak{X}_1 \times_{\mathfrak{X}} \mathfrak{X}'$ be the product in the category of admissible formal schemes over K°. By axiom (C1), it is enough to prove

$$c \cap_{\psi\circ\varphi'} Y'_1 = 0 \in CH^{\varphi'^{-1}\psi^{-1}S}_{\dim(Y')-p}(\mathfrak{X}'_1, v).$$

By Proposition 3.4.2, there is a K°-model $\mathfrak{Y}_1'$ of Y_1' such that $\psi'^{an}|_{Y_1'}$ extends to a flat morphism $\rho : \mathfrak{Y}_1' \to \mathfrak{X}_1$ replacing $\mathfrak{X}_1$ by a larger model if necessary. It is enough to prove the above identity on $\mathfrak{Y}_1'$. Using $\rho^*(X_1) = Y_1'$ and axiom (C2), we obtain

$$c \cap_{\psi\circ\varphi'} Y_1' = \rho^*(c \cap_\varphi X_1) \in CH^{\rho^{-1}\varphi^{-1}S}_{\dim(Y')-p}(\mathfrak{Y}_1', v).$$

Note that

$$c \cap_\varphi X_1 \in CH^{\varphi^{-1}S}_{\dim(X)-p}(\mathfrak{X}_1, v).$$

By dimensionality reasons, this local Chow group is zero. Hence $c \cap_\varphi X_1 = 0$ proving the claim. □

4.2.10 Remark. If the horizontal part of S is the whole generic fibre, then $CH^*_S(\mathfrak{X}, v)$ is called the *Chow cohomology ring* $CH^*(\mathfrak{X}, v)$ which is an associative ring with $1 = \mathrm{cyc}(X)$.

4.2.11 Example. We compute the local Chow cohomology groups for $\mathfrak{X} := \mathrm{Spf} K^\circ$. Note that any morphism $\psi : \mathfrak{X}' \to \mathfrak{X}$ is flat. By (C2), $c \in CH^p_S(\mathfrak{X}, v)$ is completely determined by the action on cycles of $\mathfrak{X}$ which are $1 = \mathrm{cyc}(X)$ and $v = \mathrm{cyc}(\tilde{\mathfrak{X}})$. Note that the local Chow groups are given by

$$CH^S_k(\mathfrak{X}, v) = \begin{cases} \mathbb{Z} \cdot 1 & \text{if } k = 0 \text{ and } S = 1 \cup v, \\ \log|K^\times| \cdot v & \text{if } k = -1 \text{ and } S = v, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that k denotes the relative dimension. We easily deduce that the map $c \mapsto c \cap 1$ induces an ismomorphism

$$CH^p_S(\mathfrak{X}, v) \cong CH^S_{-p}(\mathfrak{X}, v)$$

for all $p \in \mathbb{Z}$ and all supports S.

4.2.12 Proposition. *Let X be a proper scheme over K with K°-model $\mathfrak{X}$. Let Y be a closed subset of X, let S be a support on $\mathfrak{X}$ with horizontal part Y and let $c_{\mathfrak{X}} \in CH^p_S(\mathfrak{X}, v)$. For a proper scheme X' over K, a morphism $\psi : X' \to X$ and a closed analytic subset Y' of X', there is a unique operation*

$$\widehat{CH}^{Y'}_k(X', v) \to \widehat{CH}^{Y' \cap \psi^{-1}Y}_{k-p}(X', v), \quad \alpha' \mapsto \hat{c} \cap_\psi \alpha',$$

such that for each K°-model $\mathfrak{X}'$ of X' with an extension $\bar{\psi} : \mathfrak{X}' \to \mathfrak{X}$ of ψ, we have

$$(\hat{c} \cap_\psi \alpha')_{\mathfrak{X}'} = c_{\mathfrak{X}} \cap_{\bar{\psi}} \alpha'_{\mathfrak{X}'}.$$

Proof: We use the last equality to define the operation of $\hat{c}$ on suitable large K°-models of X' (Proposition 3.4.2). By (C1), this is compatible with push-forward, thus $\hat{c}$ is well-defined and uniqueness is obvious. □

4.2.13 We call $\hat{c}$ the *local Arakelov-Chow cohomology class on X with support in Y* induced by $c_{\mathfrak{X}}$. The local Arakelov-Chow cohomology classes with support in Y (for varying K°-models $\mathfrak{X}$) form a group denoted by $\widehat{CH}^p_Y(X, v)$. For $Y = \emptyset$, we use the notation $\widehat{CH}^p_{fin}(X, v)$.

4.2.14 Immediately, the properties of local Chow cohomology classes on K°-models induce corresponding results for local Arakelov-Chow cohomology classes on X. If $\hat{c} \in \widehat{CH}^p_Y(X, v)$ and $\hat{c}' \in \widehat{CH}^q_Z(X, v)$ are induced by $c_{\mathfrak{X}}$ and $c'_{\mathfrak{X}'}$, then let $\hat{c}^{an} := c^{an}_{\mathfrak{X}} \in CH^p_Y(X)$. To define the cup product, we may assume that $\mathfrak{X} = \mathfrak{X}'$ (Proposition 3.4.2) and then let

$$\hat{c} \cup \hat{c}' \in \widehat{CH}^{p+q}_{Y \cap Z}(X, v)$$

be induced by $c_{\mathfrak{X}} \cup c'_{\mathfrak{X}}$. To define the pull-back with respect to the morphism $\psi : X' \to X$ of proper schemes over K, we choose any extension $\bar{\psi} : \mathfrak{X}' \to \mathfrak{X}$ and then

$$\psi^*(\hat{c}) \in \widehat{CH}^p_{\psi^{-1}Y}(X', v)$$

is defined to be the class induced by $\bar{\psi}^*(c_{\mathfrak{X}})$. It is easy to see that these definitions do not depend on the choice of $c_{\mathfrak{X}}$, $c'_{\mathfrak{X}'}$ and the extension $\bar{\psi}$.

4.2.15 Proposition. *For the constructions above, we have the following properties:*

a) If $\varphi : X'' \to X'$ is a morphism and $\alpha'' \in \widehat{CH}^{Y''}_k(X'', v)$ for a closed subset Y'' of X'', then

$$\varphi_*(\hat{c} \cap_{\psi\circ\varphi} \alpha'') = \hat{c} \cap_\psi \varphi_*(\alpha'') \in \widehat{CH}^{\varphi Y'' \cap \psi^{-1}Y}_{k-p}(X', v).$$

b) If $\hat{c}'' \in \widehat{CH}^r_W(X, v)$, then we have associativity

$$\hat{c} \cup (\hat{c}' \cup \hat{c}'') = (\hat{c} \cup \hat{c}') \cup \hat{c}'' \in \widehat{CH}^{p+q+r}_{Y\cap Z\cap W}(X, v).$$

c) The pull-back is functorial with respect to composition and

$$\psi^*(\hat{c} \cup \hat{c}') = \psi^*(\hat{c}) \cup \psi^*(\hat{c}') \in \widehat{CH}^{p+q}_{\psi^{-1}(Y\cap Z)}(X', v).$$

d) We have $\widehat{CH}^p_Y(X, v) = 0$ for $p < 0$ or $p > \dim(X) + 1$.

Proof: This follows immediately from the corresponding properties of local Arakelov-Chow cohomology classes on K°-models. □

4.2.16 Example. If s is an invertible meromorphic section of a line bundle L on X with K°-model $\mathcal{L}$, then by Example 4.2.3 we get

$$\hat{c}_1^{\mathcal{L}}(s) \in \widehat{CH}^1_{|\mathrm{div}(s)|}(X, v)$$

induced by $c_1^{\mathcal{L}}(s)$.

4.2.17 Example. Note that $\mathrm{Spf}K^\circ$ is the only K°-model of $\mathrm{Spf}K$. By Example 4.2.11, we get easily

$$\widehat{CH}^p_Y(\mathrm{Spf}K, v) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0 \text{ and } Y = \mathrm{Spf}K, \\ \log|K^\times| & \text{if } p = 1 \text{ and } Y = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

4.3 Admissible Metrics

In this section, K denotes an algebraically closed field with a non-trivial non-archimedean complete absolute value $|\ |$. Let us fix a subfield R of $\mathbb{R}$ containing the value group $\log|K^\times|$. The assumption K algebraically closed is natural since we allow the vertical cycles to have coefficients in the field R.

The contribution of the infinite places to Arakelov theory is given in terms of smooth hermitian metrics on the line bundles. Zhang ([Zh2], section 2) noticed that there are analogous metrics at finite places induced by formal models over the discrete valuation rings. Similarly, every K°-model $\mathcal{L}$ of a line bundle L gives rise to a metric $\|\ \|_{\mathcal{L}}$ on L called a formal metric. For an invertible meromorphic section s of L, we prove that the operation $\widehat{\mathrm{div}}_{\mathcal{L}}(s)$ on the Arakelov-cycle group (cf. section 3.4) depends only on the formal metric $\| \ \|_{\mathcal{L}}$ but not on the choice of $\mathcal{L}$.

On a line bundle algebraically equivalent to 0, there is a canonical metric determined up to multiples. If our valuation ring is discrete, then the canonical metric is a root of a formal metric. But in general, this is not true (cf. Example 4.3.35 and Remark 4.3.36). This forces us to introduce a more general class of metrics called admissible. They are the tensor product of a formal metric on L and metrics on the trivial bundle locally given by R-tensor powers of formal metrics. It may be viewed as the analogue of introducing R-coefficients for vertical cycles.

For a smooth projective curve C over K, every K°-model is dominated by a semistable K°-model, so it is enough to restrict our attention to such models of C. We describe all admissible metrics on O_C where local formal metrics are living on a given semistable K°-model $\mathcal{C}$. Such admissible metrics are in one-to-one correspondence with continuous functions on the intersection graph of $\mathcal{C}$ which are linear on the edges. This is a generalization of the corresponding results of Chinburg-Rumely ([CR], section 2) and Zhang ([Zh1], section 2) obtained in the case of discrete valuations. Note that our approach using rigid and formal geometry gives a direct interpretation of the intersection graph and the piecewise linear functions above in terms of the coordinate functions on the formal fibres with respect to reduction.

Let L be a line bundle on the generic fibre of a reduced formal analytic variety $\mathfrak{X}$ and let s be an invertible meromorphic section of L. As remarked above, the order of $\mathrm{div}(s)$ in an irreducible component of the reduction $\tilde{\mathfrak{X}}$ is completely determined by the formal metric $\| \ \|_{\mathcal{L}}$ on L. We prove that it is equal to $-\log \|s(x)\|$ for all $x \in X$ with generic reduction in W. Moreover, we make very precise what generic means.

We use this result to define the Weil divisor associated to $\mathrm{div}(s)$ with respect to an admissible metric on L. As in section 3.3, this leads to a proper intersection product $\widehat{\mathrm{div}}_{\mathfrak{X}}(s).Z$ for horizontal cycles Z on an admissible formal scheme $\mathfrak{X}$.

Since admissibly metrized line bundles haven't a reduction on the special fibre, we can't define $\widehat{\mathrm{div}}_{\mathfrak{X}}(s).V$ for all vertical cycles V of $\mathfrak{X}$. But if $\mathfrak{X}$ is a formal analytic variety and V is a linear combination of irreducible components of $\tilde{\mathfrak{X}}$, then it is possible to define the product even as a cycle which couldn't be expected from refined intersection theory. A remarkable result is that we can deduce commutativity of admissibly metrized Cartier divisors as an identity of cycles.

For a smooth projective curve C, the divisor operation of an admissibly metrized line bundle is defined on all cycles of a K°-model $\mathcal{C}$ because the operation is zero on vertical cycles of codimension > 0 in $\tilde{\mathcal{C}}$. It induces a local Arakelov-Chow cohomology class on C playing the role of a Chern class. This is not so trivial to prove because we have to define Chern class operations on every proper scheme lying over C. For admissible metrics, we have to argue on the cycle level and so we have to use all the results above about proper intersection products. As a corollary of proof, we note that Chern classes with respect to admissibly metrized line bundles commute.

The Hodge index theorem in algebraic geometry says that the intersection pairing on a surface has signature $(1, -1, \ldots, -1)$. An analogue on arithmetic surfaces was proved by Faltings ([Fa2], Theorem 4) and Hriljac ([Hr], Theorem 3.4). Here, we deduce a local version of it showing that the intersection pairing on vertical divisors is a negative semidefinite bilinear form with kernel equal to the special fibre. The proof is standard using the properties of divisoral intersections. As an application, we show that the canonical metric on $L \in \mathrm{Pic}^\circ(C)$ is admissible and we give a similar description of the Néron pairing in terms of the intersection pairing as Hriljac ([Hr], Theorem 1.6) gave in the case of discrete valuations. At the end, we illustrate the methods for Tate's elliptic curve.

4.3.1 In the previous section, the vertical cycles were supposed to have coefficients in the value group. But without any change, we could have used coefficients of vertical cycles in R. For an admissible formal scheme $\mathfrak{X}$ over K° with generic fibre X, we denote by $Z(\tilde{\mathfrak{X}}, R)$ the cycles with coefficients in R and

$$Z(\mathfrak{X}, R) := Z(X) \oplus Z(\tilde{\mathfrak{X}}, R).$$

Furthermore, let

$$CH(\tilde{\mathfrak{X}}, R) := Z(\tilde{\mathfrak{X}}, R)/\langle \operatorname{div}(\mathbf{f}) \mid \mathbf{f}\ K_1\text{-chain on } \tilde{\mathfrak{X}}\rangle$$

and

$$\tilde{Z}(\mathfrak{X}, R) := Z(X) \oplus CH(\tilde{\mathfrak{X}}, R).$$

As usual, we gade the cycles by relative dimension. Similarly, we can perform all the other constructions from the previous sections. We replace always the v standing for the value group by R.

4.3.2 Let X be a rigid analytic variety over K and let L be a line bundle on X. If L is given by the admissible open covering $\{U_\alpha\}_{\alpha \in I}$ and the transition functions $g_{\alpha\beta}$, then a metric $\| \ \|$ on L is given by functions $\rho_\alpha : U_\alpha \to]0, \infty[$ such that

$$\rho_\alpha(x)|g_{\alpha\beta}(x)| = \rho_\beta(x)$$

for all $x \in U_\alpha \cap U_\beta$. If s_α is a section of L on U_α, then s_α corresponds to an analytic function γ_α on U_α with respect to the trivialization and we have

$$\|s_\alpha(x)\| = |\gamma_\alpha(x)|\rho_\alpha(x)$$

for all $x \in U_\alpha$.

4.3.3 Let $\mathcal{L}$ be a K°-model of L living on the K°-model $\mathfrak{X}$. Then we have a canonical metric $\| \ \|_{\mathcal{L}}$ on L defined in the following way: Let $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ be an open covering of $\mathfrak{X}$ trivializing $\mathcal{L}$ and let $s_\alpha \in L(\mathcal{U}_\alpha^{an})$. If $\gamma_\alpha \in \mathcal{O}_X(\mathcal{U}_\alpha^{an})$ corresponds to s_α with respect to the trivialization, then

$$\|s_\alpha(x)\|_{\mathcal{L}} = |\gamma_\alpha(x)|$$

for all $x \in \mathcal{U}_\alpha^{an}$. Such a metric is called *formal*. Note that it is given by functions $\rho_\alpha = 1$ with respect to the trivialization.

If X is reduced, then we may assume that $\mathcal{L}$ lives on a K°-model $\mathfrak{X}$ with reduced special fibre (3.3.5) and the formal metric determines the K°-model $\mathcal{L}$ up to isomorphisms, i.e. we have canonically

$$\mathcal{L}(\mathcal{U}) \cong \{s \in L(\mathcal{U}^{an}) \mid \|s\| \leq 1\}$$

for every formal open subset $\mathcal{U}$ of $\mathfrak{X}$. In other words, on reduced formal analytic varieties, the line bundles are determined up to isomorphisms by its formal metrics. If X is quasi-compact, quasi-separated and reduced, then a formal metric is characterized by the property that it may be represented by $\rho_\alpha = 1$ ([Gu3], Proposition 7.5). Hence the definition of a formal metric is local in this case.

4.3.4 Proposition. *Let $\mathcal{L}$ be a line bundle on the admissible formal scheme $\mathfrak{X}$ over K° and let s be an invertible meromorphic section of $\mathcal{L}$. If $\mathfrak{Z} \in Z(\mathfrak{X}, R)$ intersects $|\operatorname{div}(s)|$ properly in the generic fibre, then $\operatorname{div}_{\mathcal{L}}(s).\mathfrak{Z}$ depends only on the formal metric $\| \ \|_{\mathcal{L}}$ but not on the choice of $\mathcal{L}$.*

Proof: If $\mathfrak{Z}$ is horizontal, then the definition of proper intersection product uses push-forward from formal analytic varieties where the model is determined by its metric. So we may assume $\mathfrak{Z}$ prime and vertical. For an irreducible component X_j of $\mathfrak{X}^{an}$, let $\mathfrak{X}_j^{an}$ be the formal analytic structure on X_j induced by $\mathfrak{X}^{f-an}$. Then we have a finite map $\tilde{\iota_j}$ between special fibres of $\mathfrak{X}_j^{f-an}$ and $\mathfrak{X}$ (cf. 3.3.4). We conclude that there is a cycle $\mathfrak{Z}_j$ on the special fibre of some $\mathfrak{X}_j^{f-an}$ mapping to a non-zero multiple of $\mathfrak{Z}$. Again, the operation of $\mathcal{L}|_{\mathfrak{X}_j^{f-an}}$ on $\mathfrak{Z}_j$ is completely determined by the metric and the projection formula 3.3.13 proves the claim. $\square$

4.3.5 Remark. If X is a quasi-compact and quasi-separated rigid analytic variety, if s is an invertible meromorphic section of a formally metrized line bundle L on X and if $\operatorname{div}(s)$ intersects the horizontal part of $\alpha \in \hat{Z}(X,R)$ properly in X, then $\widehat{\operatorname{div}}_{\mathcal{L}}(s).\alpha$ is well-defined in $\hat{Z}(X,R)$ and we omit the reference to the formal model of the metric. As in Arakelov theory, the notation $\widehat{\operatorname{div}}(s)$ indicates that the product depends only on the metric.

4.3.6 Remark. Let L be a line bundle on X. A metric $\| \ \|$ is called a *root of a formal metric* if there is $n \in \mathbb{N} \setminus \{0\}$ such that $\| \ \|^{\otimes n}$ is a formal metric of $L^{\otimes n}$. Note that they are closed under $\otimes$, inverse and pull-back. Using the fact that n is invertible in R, we define for an invertible meromorphic section s of L and $\alpha \in \hat{Z}(X,R)$ the intersection product

$$\widehat{\operatorname{div}}(s).\alpha := \frac{1}{n}\widehat{\operatorname{div}}(s^{\otimes n}).\alpha \in \hat{Z}(X,R).$$

All the results for formal metrics proved in the context of intersection product hold trivially also for roots of formal metrics.

4.3.7 Definition. Let $\mathfrak{X}$ be an admissible formal scheme over K° with generic fibre X. A metric $\| \ \|$ on the trivial bundle O_X is called $\mathfrak{X}$-*admissible* if there is an open covering $\{\mathcal{U}\}$ of $\mathfrak{X}$, invertible analytic functions $\gamma_j \in \mathcal{O}_X(\mathcal{U}^{an})^\times$ and $\lambda_j \in R$ $(j = 1, \ldots, r)$ satisfying

$$\|1(x)\| = |\gamma_1(x)|^{\lambda_1} \cdots |\gamma_r(x)|^{\lambda_r}$$

for all $x \in \mathcal{U}^{an}$. A metric $\| \ \|$ on a line bundle L on X is called $\mathfrak{X}$-admissible if there is a K°-model $\mathcal{L}$ of L on $\mathfrak{X}$ such that $\| \ \|/\| \ \|_{\mathcal{L}}$ is a $\mathfrak{X}$-admissible metric on O_X.

4.3.8 Let X be a quasi-compact and quasi-separated rigid analytic variety over K. A metric on a line bundle of X is called *admissible* if it is $\mathfrak{X}$-admissible for some K°-model $\mathfrak{X}$ of X. Note that admissible metrics are closed under pull-back, tensor product and inverse. Every root of a formal metric is admissible.

4.3.9 Example. For a semistable model of a curve, we will describe the admissible metrics on O_C. Let C be an irreducible smooth projective curve over K and let $\mathcal{C}$ be a *semistable* K°-*model* of C, i.e. $\mathcal{C}$ is a K°-model of C such that the special fibre $\tilde{\mathcal{C}}$ is reduced with singularities at most ordinary double points. The reader should be aware that the reduction field $\tilde{K}$ is algebraically closed ([BGR], Lemma 3.4.1/4). For $\mathcal{C}$, we define the *intersection graph* $G(\mathcal{C})$ in the following way: The vertices of $G(\mathcal{C})$ correspond to the irreducible components of $\tilde{\mathcal{C}}$. For each double point $\tilde{x}$ of $\tilde{\mathcal{C}}$, we define an edge of $G(\mathcal{C})$ by connecting the vertices corresponding to the irreducible components through $\tilde{x}$. Let $\pi : C \to \tilde{\mathcal{C}}$ be the reduction map. Then the *formal fibre* $\pi^{-1}(\tilde{x})$ is an open annulus of height $r < 1$ ([BL1], Proposition 2.3), i.e. it is isomorphic to $\{\zeta \in \mathbb{B}^1 \mid r < |\zeta| < 1\}$ where the value $r \in |K^\times|$ is uniquely determined. If we require that the edge corresponding to the double point $\tilde{x}$ is isometric to a closed real interval of length $-\log r$ (or to a circle if there is only one irreducible component through $\pi(x)$), then $G(\mathcal{C})$ will be a metrized graph.

There is a canonical map $p : C \to G(\mathcal{C})$. If $x \in C$ has regular reduction $\pi(x) \in \tilde{\mathcal{C}}$, then $p(x)$ is the vertex corresponding to the irreducible component of $\tilde{\mathcal{L}}$ containing $\pi(x)$. If $\pi(x)$ is an ordinary double point $\tilde{x}$ in $\tilde{\mathcal{C}}$, then we identify the edge corresponding to $\tilde{x}$ with $[0, -\log r]$ (or the corresponding loop) and we set $p(x) := -\log|\zeta(x)|$ where ζ is the coordinate on the open annulus of height r isomorphic to $\pi^{-1}(\tilde{x})$ as above.

Since the special fibre of $\mathcal{C}$ is reduced, we may view $\mathcal{C}$ also as a formal analytic variety (cf. 3.3.5). This will be used frequently, for example in the proof of the following result.

4.3.10 Proposition. *Let $\| \ \|$ be a $\mathcal{C}$-admissible metric on O_C. Then the function $-\log \|1(x)\|$ on C factors through the intersection graph, i.e. there is a uniquely determined function $f : G(\mathcal{C}) \to \mathbb{R}$ with the property*

$$f \circ p(x) = -\log \|1(x)\| \tag{4.1}$$

for all $x \in C$. Moreover, f is continuous on $G(\mathcal{C})$ and linear on the edges with slopes and constant terms in R. Conversely, for every continuous real function on $G(\mathcal{C})$ which is linear on the edges with slopes and constant terms in R, there is a unique $\mathcal{C}$-admissible metric $\| \ \|$ on O_C with property (4.1).

Proof: Simultaneously, we prove that f is well-defined on $G(\mathcal{C})$ by (4.1) and satisfies the required properties. Let $\tilde{\mathcal{U}}$ be an irreducible open affine subscheme of $\tilde{\mathcal{C}}$. Then $\mathcal{U} := \pi^{-1}(\tilde{\mathcal{U}})$ is a formal open affinoid subspace of $\mathcal{C}$ with integral special fibre, hence the maximum norm on $\mathcal{U}$ is multiplicative. We may assume that the $\mathcal{C}$-admissible metric $\| \ \|$ is given on $\mathcal{U}$ by

$$\|1(x)\| = |\gamma_1(x)|^{\lambda_1} \cdots |\gamma_r(x)|^{\lambda_r} \tag{4.2}$$

for suitable $\gamma_i \in \mathcal{O}_C(\mathcal{U}^{an})^\times$ and $\lambda_i \in R$. Because of multiplicativity of the maximum norm, the absolute values $|\gamma_i(x)|$ are constant on $\mathcal{U}$ and hence the same holds for $\|1(x)\|$.

If t is a vertex of $G(\mathcal{C})$ corresponding to an irreducible component V of $\tilde{\mathcal{C}}$, then $x \in C$ maps to t if and only if the reduction $\pi(x)$ is a regular point of $\tilde{\mathcal{C}}$ contained in V. The set of these reductions is an irreducible open subset of $\tilde{\mathcal{C}}$ contained in V and may be covered by finitely many $\tilde{\mathcal{U}}$ as above. We conclude that $\|1(x)\|$ is constant on $p^{-1}(t)$.

If t is on a loop, then there is a unique irreducible component V through the corresponding double point and the reduction of $p^{-1}(t)$ is again contained in an irreducible open subset of $\tilde{\mathcal{C}}$ contained in V. As above, we conclude that $\|1(x)\|$ is constant on $p^{-1}(t)$, hence f is well-defined and constant on the loop.

It remains to consider an interior point t of an edge (which is not a loop). Then $p^{-1}(t) \subset \pi^{-1}(\tilde{x})$ for a double point $\tilde{x}$ of $\tilde{\mathcal{C}}$. We can identify this formal fibre with an open annulus with coordinate ζ. For every unit γ on a formal open neighbourhood of $\pi^{-1}(\tilde{x})$, there is $m \in \mathbb{Z}$ and $\alpha \in K^\times$ with

$$|\gamma(x)| = |\alpha \zeta(x)^m| \tag{4.3}$$

for all $x \in \pi^{-1}(\tilde{x})$ ([BGR], Lemma 9.7.1/1). By [BL1], Proposition 2.3ii), this identity extends to a formal open neighbourhood $\mathcal{U}$ of $\pi^{-1}(\tilde{x})$. We may assume that the metric on $\mathcal{U}$ is given as in (4.2). Applying (4.3) to every factor, we conclude that $f(z)$ is well-defined and linear on the whole edge with slope and constant term in R. Note that we also have proved continuity of f and so the proof of the first claim is complete.

Conversely, let f be a continuous real function on $G(\mathcal{C})$ which is linear on each edge with slope and constant term in R. We define the metric $\| \ \|$ on O_X by (4.1). We have to check $\mathcal{C}$-admissibility in a formal open neighbourhood of $x \in C$. If $p(x)$ is a vertex or on a loop, then $\| \ \|$ has to be constant on a formal open neighbourhood of x, hence admissibility is trivial. So we may assume that $\pi(x)$ reduces to an ordinary double point contained in two different irreducible components of $\tilde{\mathcal{C}}$. The formal fibre $\pi^{-1}(\pi(x))$ is an open annulus with coordinate ζ which again extends to a unit on a formal open neighbourhood $\mathcal{U}$ of x. We may assume that $\tilde{\mathcal{U}}$ contains no other double point than $\tilde{x}$. By assumption, there are $\lambda, \mu \in R$ with

$$f \circ p(u) = -\lambda \log |\zeta(u)| + \mu$$

for every $u \in \pi^{-1}(\tilde{x})$. In fact, the identity extends to $\mathcal{U}$ since $|\zeta(u)|$ has to be constant outside the double points. Let $\alpha \in K^\times$ with $|\alpha| < 1$. For $\lambda' := -\mu / \log |\alpha| \in R$, we get

$$\|\gamma(u)\| = |\zeta(u)|^\lambda |\alpha|^{\lambda'}$$

for all $u \in \mathcal{U}$. This proves $\mathcal{C}$-admissibility of $\| \ \|$. $\square$

4.3.11 Remark. We conclude that the $\mathcal{C}$-admissible metrics on O_C may be identified with the R-vector space of one dimensional cycles on $\tilde{\mathcal{C}}$ with coefficients in R. A $\mathcal{C}$-admissible metric $\| \ \|$ gives rise to the cycle $\sum m_V V$ where the multiplicity m_V in the irreducible component V of $\tilde{\mathcal{C}}$ is given by $-\log \|1(x)\|$ with $p(x)$ the vertex on $G(\mathcal{C})$ corresponding to V.

4.3.12 Remark. A blowing up of $\mathcal{C}$ in a double point leads to a refinement of the intersection graph $G(\mathcal{C})$ corresponding to a subdivision of the open annulus in two annuli. Hence we can construct admissible metrics with $-\log \|1(x)\|$ arbitrarily close to any continuous function on $G(\mathcal{C})$. If we blow up in a regular point $\tilde{x} = \pi(x)$ of $\tilde{\mathcal{C}}$, then we have to add a new vertex and a new edge connecting it with the vertex $p(x)$.

4.3.13 Let $\mathcal{L}$ be a line bundle on the reduced formal analytic variety $\mathfrak{X}$ over K (or equivalently on an admissible formal scheme over K° with reduced special fibre, cf. 3.3.5) with generic fibre X. For an invertible meromorphic section $\bar{s}$ of $\mathcal{L}$, we get a Cartier divisor $D = \operatorname{div}(\bar{s})$ on $\mathfrak{X}$. The following result shows how the formal metric $\| \ \|_{\mathcal{L}}$ is related to the multiplicities of D in the irreducible components of $\tilde{\mathfrak{X}}$. Let $\pi : X \to \tilde{\mathfrak{X}}$ be the reduction map and let $s = \bar{s}^{an}$.

4.3.14 Proposition. *For every irreducible component W of $\tilde{\mathfrak{X}}$, there is a number in $|K^\times|$ denoted by $\|s(W)\|_{\mathcal{L}}$ such that for all points $x \in X$ with $\pi(x)$ a generic point of W, we have $\|s(x)\|_{\mathcal{L}} = \|s(W)\|_{\mathcal{L}}$. Moreover, the order of D in W is $-\log \|s(W)\|_{\mathcal{L}}$. More precisely, generic means that the above identity holds for all $x \in X$ with $\pi(x) \notin \pi|D^{an}|$ and with $\pi(x)$ not contained in any other irreducible component of $\tilde{\mathfrak{X}}$ than W.*

Proof: By passing to irreducible components of X, we may assume that X is irreducible. Let $\tilde{\mathcal{U}}$ be the complement of $\pi|D^{an}|$ and all the other irreducible components of $\tilde{\mathfrak{X}}$ different from V. Then $\tilde{\mathcal{U}}$ is a non-empty irreducible open subset of $\tilde{\mathfrak{X}}$ contained in V, the corresponding formal open subset of $\mathfrak{X}$ is denoted by $\mathcal{U}$. For every formal open affinoid subspace $\operatorname{Spf}\mathcal{A}$ of $\mathcal{U}$, the maximum norm is a multiplicative norm ([BGR], Proposition 6.2.3/5). Since a local equation of D is given by a unit γ in $\mathcal{A}$, we conclude that $|\gamma(x)| = |\gamma|_{max}$ is constant on $\operatorname{Spf}\mathcal{A}$. This proves that $\|s(x)\|_{\mathcal{L}}$ is constant on $\mathcal{U}^{an}$, say equal to $\|s(W)\|_{\mathcal{L}}$. By 3.2.9 and 3.2.10, the order of D in W is $-\log \|s(W)\|_{\mathcal{L}}$. $\square$

4.3.15 Remark. Sometimes it is convenient to choose an $\alpha \in K^\times$ with $|\alpha| = \|s(W)\|_{\mathcal{L}}$. By total abuse of notation, we denote it by $s(W)$ if the context is clear.

4.3.16 Corollary. *Let $\mathfrak{X}$ be a reduced formal analytic variety with generic fibre X. Let s be an invertible meromorphic section of a line bundle L on X and let $\| \ \|$ be a $\mathfrak{X}$-admissible metric on L. For every irreducible component W of $\tilde{\mathfrak{X}}$, there is a number $\|s(W)\| \in \exp(R)$ with $\|s(x)\| = \|s(W)\|$ for all $x \in X$ with reduction neither in $\pi|\operatorname{div}(s)|$ nor in any other irreducible component of $\tilde{\mathfrak{X}}$ different from W.*

Proof: The claim is local, so we may assume that the metric is given by a product of R-powers of formal metrics on O_X. Then the claim follows from Proposition 4.3.14. $\square$

4.3.17 Definition. The *order* of s in W is $\operatorname{ord}(s, W) := -\log \|s(W)\|$. The *Weil divisor* of s is

$$\operatorname{cyc}\left(\widehat{\operatorname{div}}_{\mathfrak{X}}(s)\right) := \operatorname{cyc}(\operatorname{div}(s)) + \sum_W \operatorname{ord}(s, W) W$$

where $\operatorname{cyc}(\operatorname{div}(s))$ is horizontal and W ranges over all irreducible components of $\tilde{\mathfrak{X}}$.

4.3.18 Let $\mathfrak{X}$ be a reduced formal analytic variety over K with generic fibre X, let L be a line bundle on X with $\mathfrak{X}$-admissible metric $\| \ \|$ and let s be an invertible meromorphic section of L. A remarkable fact is that for an irreducible component V of $\tilde{\mathfrak{X}}$, we can define an *intersection product* $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).V$ well-defined as a vertical cycle with coefficients in R: There is a K°-model $\mathcal{L}$

of L on $\mathfrak{X}$ such that $\| \ \|/\| \ \|_{\mathcal{L}}$ is a $\mathfrak{X}$-admissible metric on O_X. We get a formal open covering $\{\mathcal{U}\}$ of $\mathfrak{X}$, trivializing L on the generic fibre, $\gamma_j \in \mathcal{O}_X(\mathcal{U}^{an})^\times$ and $\lambda_j \in R$ such that

$$\|s\| = |f||\gamma_1|^{\lambda_1} \cdots |\gamma_r|^{\lambda_r}$$

on $\mathcal{U}^{an}$ where f is the rational function corresponding to s with respect to the trivialization. Then we define the cycle $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).V$ on $\tilde{\mathcal{U}} \cap V$ by

$$\operatorname{div}\left(\widetilde{f/f(V)}\right) + \sum_j \lambda_j \operatorname{div}\left(\widetilde{\gamma_j/\gamma_j(V)}\right).$$

4.3.19 Proposition. *These cycles on $\tilde{\mathcal{U}} \cap V$ fit together to a cycle on V with coefficients in R called $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).V$ which depends only on $\| \ \|$, s and V.*

Proof: The rational functions $\widetilde{f/f(V)}$ on $\tilde{\mathcal{U}} \cap V$ form a Cartier divisor on V with corresponding line bundle isomorphic to $\mathcal{L}|_V$. So we may assume $L = O_X$ and $s = 1$. It is enough to show that the order of

$$\sum_j \lambda_j \operatorname{div}\left(\widetilde{\gamma_j/\gamma_j(V)}\right) \tag{4.4}$$

in an irreducible closed subset W of codimension 1 in V depends only on the metric. Using noetherian normalization with respect to W and base change, we reduce to the case of an irreducible and reduced formal analytic curve $\mathfrak{X}$ over a complete (stable) field Q (cf. [Gu3], Lemma 5.6 and proof of Theorem 5.9). By passing to formal open affinoid subspaces, we may assume that there is an analytic function g on X with maximum norm ≤ 1 such that W is an isolated zero of the reduction $\tilde{g}$ in $\tilde{\mathfrak{X}}$. By [BL1], Lemma 2.4, we know that for $r \in |K^\times|$, $r < 1$ sufficiently close to 1, the periphery

$$\{x \in X \mid \pi(x) = W, |g(x)| \geq r\}$$

of the formal fibre over W decomposes into n connected components $G_1, \ldots, G_n$. Here, π still denotes reduction. These components correspond to the points $\tilde{y}_1, \ldots, \tilde{y}_n$ in the normalization of $\tilde{\mathfrak{X}}$ lying over W. Moreover, G_k is isomorphic to the semi-open annulus

$$\{\zeta \in \mathbb{B}^1 \mid r^{1/\operatorname{ord}_{\tilde{y}_k}(g)} \leq |\zeta| < 1\}.$$

We may assume that the metric is given on the whole X by

$$\|1\| = |\gamma_1|^{\lambda_1} \cdots |\gamma_r|^{\lambda_r}. \tag{4.5}$$

Let $\tilde{y}_k$ be contained in the normalization V' of V and let γ be a unit on X, i.e. a nowhere vanishing analytic function. By [BL1], Lemma 2.5, we have

$$|\gamma(x)/\gamma(V)| = |\zeta(x)|^{\operatorname{ord}(\gamma/\gamma(V), \tilde{y}_k)}$$

for all $x \in G_k$. From (4.5), we get

$$\|1(x)\| = |\gamma_1(V)|^{\lambda_1} \cdots |\gamma_r(V)|^{\lambda_r} |\zeta(x)|^{\sum_j \lambda_j \operatorname{ord}(\gamma_j/\gamma_j(V), \tilde{y}_k)}$$

for all $x \in G_k$. We conclude that

$$\sum_j \lambda_j \operatorname{ord}(\gamma_j/\gamma_j(V), \tilde{y}_k) \tag{4.6}$$

depends only on the metric. By projection formula applied to the normalization morphism over $\widetilde{V}$ and to the divisor of $\gamma_j/\gamma_j(V)$ on V, we have

$$\operatorname{ord}(\gamma_j/\gamma_j(V), W) = \sum_k \operatorname{ord}(\gamma_j/\gamma_j(V), \tilde{y}_k) \tag{4.7}$$

where k ranges over all $\tilde{y}_k$ contained in V'. Now (4.6) and (4.7) imply (4.4) proving the claim. □

4.3.20 Remark. The product $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).V$ is linear in $(L, \| \ \|, s)$. By linearity, we extend it to all cycles of codimension 0 in $\tilde{\mathfrak{X}}$ with coefficients in R. By the same construction as in 3.2.15, the notion of Weil divisor leads to an *intersection product* $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).Z$ for every horizontal cycle Z on $\mathfrak{X}$ intersecting $|\operatorname{div}(s)|$ properly in X. Then $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).Z$ is a well-defined cycle on $\mathfrak{X}$ with horizontal part $\operatorname{div}(s).Z$. Both operations show that $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).D'$ is a well-defined cycle on $\mathfrak{X}$ if D' is a Weil-divisor on $\mathfrak{X}$ with $\operatorname{div}(s)$ intersecting D'^{an} properly in X.

4.3.21 Theorem. *Let $\mathfrak{X}$ be a reduced formal analytic variety over K with generic fibre X. Let L, L' be line bundles on X with $\mathfrak{X}$-admissible metrics $\| \ \|, \| \ \|'$ and invertible meromorphic sections s, s' such that* $\operatorname{div}(s)$ *and* $\operatorname{div}(s')$ *intersect properly in X. Then*

$$\widehat{\operatorname{div}}_{\mathfrak{X}}(s).\operatorname{cyc}\left(\widehat{\operatorname{div}}_{\mathfrak{X}}(s')\right) = \widehat{\operatorname{div}}_{\mathfrak{X}}(s').\operatorname{cyc}\left(\widehat{\operatorname{div}}_{\mathfrak{X}}(s)\right)$$

holds as an identity of cycles on $\mathfrak{X}$ whose vertical parts have coefficients in R.

Proof: The statement is proved for formal metrics in [Gu3], Theorem 5.9. The claim is local and then an admissible metric is a product of R-powers of formal metrics. □

4.3.22 Remark. Now let $\mathfrak{X}$ be an arbitrary formal scheme over K° with generic fibre X. Let s be an invertible meromorphic section of the $\mathfrak{X}$-admissible metrized line bundle $(L, \| \ \|)$ on X and let Z be a horizontal cycle intersecting $\operatorname{div}(s)$ properly in X. Using the same procedure as in 3.3.12 to reduce to the formal analytic structure of the components of Z, we can define the *proper intersection product* $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).Z$, well-defined as a cycle on $\mathfrak{X}$. For formal metrics, the product agrees with the Definition in 3.3.12.

We can't define the intersection product of $\widehat{\operatorname{div}}_{\mathfrak{X}}(s)$ with vertical cycles on $\mathfrak{X}$. However, using projection formula to reduce to the formal analytic situation, it is possible to define the terms in Theorem 4.3.21 as cycles on $\mathfrak{X}$. Then the result holds also for admissible formal schemes. We leave the details to the reader (compare with [Gu3], section 5).

4.3.23 Corollary. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism of admissible formal schemes with generic fibres X, X' and let s be an invertible meromorphic section of the $\mathfrak{X}$-admissible metrized line bundle $(L, \| \ \|)$ on X such that $\varphi^*\operatorname{div}(s)$ is well-defined as a Cartier divisor on X'.*

a) If Z' is a horizontal prime cycle on X' with $\varphi(Z')$ not contained in $|\operatorname{div}(s)|$, then the projection formula

$$\widehat{\operatorname{div}}_{\mathfrak{X}}(s).\varphi_* Z' = \varphi_*\left(\widehat{\operatorname{div}}_{\mathfrak{X}'}(s \circ \varphi).Z'\right)$$

holds as an identity of cycles on $\mathfrak{X}$.

b) If φ is flat and Z is a horizontal cycle on X intersecting $\operatorname{div}(s)$ *properly in X, then*

$$\varphi^*\left(\widehat{\operatorname{div}}_{\mathfrak{X}}(s).Z\right) = \widehat{\operatorname{div}}_{\mathfrak{X}'}(s \circ \varphi).\varphi^* Z$$

holds as an identity of cycles on $\mathfrak{X}'$.

Proof: These are identities known for formal metrics (Propositions 3.3.13, 3.3.15). Because both identities are local, we may assume that the admissible metric is a product of R-powers of formal metrics. This proves the claim. $\square$

4.3.24 Remark. Let C be a smooth projective curve over K. There is a semistable K°-model $\mathcal{C}$ of C such that $\tilde{\mathcal{C}}$ has smooth irreducible components. If necessary, we may choose $\mathcal{C}$ larger than any given K°-model of C ([BL1], Theorem 7.1 and its proof).

4.3.25 Theorem. Let $\mathcal{C}$ be as in Remark 4.3.24 and let L be a line bundle on C with $\mathcal{C}$-admissible metric $\| \ \|$. For every invertible meromorphic section s of L, there is a unique $\hat{c}_1(s)_{\mathcal{C}} \in CH^1_{|\mathrm{div}(s)| \cup \tilde{\mathcal{C}}}(\mathcal{C}, R)$ satisfying the following properties:

a) If $\mathrm{div}(s)$ intersects $\mathfrak{Z} \in Z(\mathcal{C}, R)$ properly in the generic fibre C, then

$$\hat{c}_1(s)_{\mathcal{C}} \cap \mathfrak{Z} = \widehat{\mathrm{div}}_{\mathcal{C}}(s).\mathfrak{Z} \in CH_*^{(|\mathrm{div}(s)| \cup \tilde{\mathcal{C}}) \cap |\mathfrak{Z}|}(\mathcal{C}, R).$$

b) If s' is another invertible meromorphic section of L, then

$$\hat{c}_1(s)_{\mathcal{C}} - \hat{c}_1(s')_{\mathcal{C}} = c_1^{O_{\mathcal{C}}}(s/s') \in CH^1_{|\mathrm{div}(s)| \cup |\mathrm{div}(s')| \cup \tilde{\mathcal{C}}}(\mathcal{C}, R)$$

where on the right hand side, we have the Chern class from Example 4.2.3.

Proof: First, we prove uniqueness. Let $\bar{\psi} : \mathfrak{X}' \to \mathcal{C}$ be a proper morphism of admissible formal schemes with generic fibre ψ. Let s be an invertible meromorphic section of L such that $\psi^*\mathrm{div}(s)$ is a well-defined Cartier divisor on X' intersecting a horizontal cycle Z' properly in X'. Then we claim that

$$\hat{c}_1(s)_{\mathcal{C}} \cap_{\bar{\psi}} Z' = \widehat{\mathrm{div}}_{\mathfrak{X}'}(s \circ \psi).Z' \in \tilde{Z}(\mathfrak{X}', R) \tag{4.8}$$

is necessary. Note that the right hand side is the intersection product from Remark 4.3.20 with respect to the $\mathfrak{X}'$-admissible metric $\psi^*\| \ \|$ of ψ^*L. To prove (4.8), we may assume Z' prime and $Z' = X'$. Let $\rho : X' \to Y := \psi X'$ be the map induced by ψ. By [Ha], Proposition 9.7, it is flat. To prove (4.8), we may replace $\mathcal{C}$ and $\mathfrak{X}'$ by sufficiently large K°-models for C and X' (use Corollary 4.3.23a) and (C1)). By Proposition 3.4.2, we may assume that ρ extends to a flat morphism $\bar{\rho} : \mathfrak{X}' \to \bar{Y}$, where $\bar{Y}$ is the closure of Y in $\mathcal{C}$. Note that $\bar{\rho}$ is induced by $\bar{\psi}$. By Corollary 4.3.23b), $\bar{\rho}^*Y = Z'$ and the required property from Theorem 4.3.25a), we deduce

$$\begin{aligned} \widehat{\mathrm{div}}_{\mathfrak{X}'}(s \circ \psi).Z' &= \bar{\rho}^* \left(\widehat{\mathrm{div}}_{\mathcal{C}}(s).Y \right) \\ &= \bar{\rho}^* \left(\hat{c}_1(s)_{\mathcal{C}} \cap Y \right) \\ &= \hat{c}_1(s)_{\mathcal{C}} \cap_{\bar{\rho}} Z' \\ &= \hat{c}_1(s)_{\mathcal{C}} \cap_{\bar{\psi}} Z' \in \tilde{Z}(\mathfrak{X}', R). \end{aligned}$$

This proves (4.8). Note that we already get uniqueness of $\hat{c}_1(s)$ on horizontal cycles because we may use b) to reduce to the proper intersection case.

Now let V be a vertical prime cycle on $\mathfrak{X}'$. Let $\rho : V \to W := \tilde{\psi}V$ be the morphism induced by the reduction $\tilde{\psi}$ of $\bar{\psi}$. Then ρ is again flat. By axiom (C4), $\hat{c}_1(s)_{\mathcal{C}}$ is induced by a Chow cohomology class $\tilde{c}$ on $\tilde{\mathcal{C}}$. Using (C2) for $\tilde{c}$ and $\rho^*W = V$, we get

$$\hat{c}_1(s)_{\mathcal{C}} \cap_{\bar{\psi}} V = \rho^*(\tilde{c} \cap W) = \rho^*(\hat{c}_1(s)_{\mathcal{C}} \cap W) \in CH_*(V, v).$$

By the required property from Theorem 4.3.25a), we conclude that

$$\hat{c}_1(s)_{\mathcal{C}} \cap_{\bar{\psi}} V = \rho^* \left(\widehat{\mathrm{div}}_{\mathcal{C}}(s).W \right) \in CH_*(V, v). \tag{4.9}$$

This proves uniqueness of $\hat{c}_1(s)_{\mathcal{C}}$ for vertical cycles and hence for all.

The proof of uniqueness indicates how to show existence. For a formal metric, existence is clear (Example 4.2.3). Since a $\mathfrak{X}$-admissible metric is the tensor product of a formal metric and a $\mathfrak{X}$-admissible metric on O_C, we may assume that $L = O_C$ and $s = 1$. First, we define the Chow cohomology class $\tilde{c}$ from axiom (C4) inducing the action on vertical cycles. Let $\mathcal{U}$ be a formal open covering of $\mathcal{C}$ such that

$$\|1(x)\| = |\gamma_1(x)|^{\lambda_1} \cdots |\gamma_r(x)|^{\lambda_r}$$

for suitable $\gamma_j \in \mathcal{O}_C(\mathcal{U}^{an})^\times$ and $\lambda_j \in R \setminus \{0\}$. Let us define the support of $\widehat{\operatorname{div}}_{\mathfrak{X}}(1)$ by

$$S := \bigcup_{\mathcal{U},j,W} \left|\operatorname{div}\left(\widetilde{\gamma_j/\gamma_j(W)}\right)\right|$$

where W ranges over all irreducible components of $\tilde{\mathcal{C}}$. Here, $\widetilde{\gamma_j/\gamma_j(W)}$ is viewed as a rational function on W. They are units on $W \setminus \{\text{double points}\}$ (cf. proof of Proposition 4.3.10), therefore S consists of double points. We define $\tilde{c} \in CH^1_S(\tilde{\mathcal{C}})$ by the following procedure: Let V be an integral scheme and let $\psi : V \to \tilde{\mathcal{C}}$ be a proper morphism. Inspired by (4.9), we consider the flat $\rho : V \to W := \psi V$ induced by ψ and we set

$$\tilde{c} \cap_\psi V = \rho^*\left(\widehat{\operatorname{div}}_{\mathcal{C}}(1).W\right) \in CH_{\dim V - 1}(\psi^{-1}S). \tag{4.10}$$

Note that it is a cycle if W is an irreducible component of $\tilde{\mathcal{C}}$ and 0 otherwise. By passing to prime components, this leads to an action $\tilde{c} \cap_\psi \cdot$ on all cycles of a proper scheme over $\tilde{\mathcal{C}}$.

We claim that $\tilde{c} \in CH^1_S(\tilde{\mathcal{C}})$. Axioms (C1) and (C2) follow immediately from the definition using projection formula and flat pull-back rule for cycles. We check axiom (C3) for $\tilde{c}$. Let $\psi : V' \to \tilde{\mathcal{C}}$ be a proper morphism over $\tilde{K}$, let V be a cycle on V' and let s' be an invertible meromorphic section of a line bundle L' on V'. We have to check

$$\operatorname{div}(s').(\tilde{c} \cap_\psi V) = \tilde{c} \cap_\psi (\operatorname{div}(s').V) \in CH_{\dim V - 2}(V \cap \psi^{-1}S \cap |\operatorname{div}(s')|). \tag{4.11}$$

We may assume V prime and $V = V'$ (replacing s' by an arbitrary invertible meromorphic section of L' if $V \subset |\operatorname{div}(s')|$). Let $W := \psi V$, then we may assume that W is an irreducible component of $\tilde{\mathcal{C}}$ otherwise both sides in (4.11) are zero. Let U be the complement of all non-proper intersection components of $\psi^{-1}(S) \cap |\operatorname{div}(s')|$ in V. On the open subset U of V, (4.11) is an identity of cycles. It may be checked locally where the operation $\tilde{c} \cap_\psi \cdot$ is an R-linear combination of Cartier divisors given by the reductions of the functions $\gamma_j/\gamma_j(W)$. Each Cartier divisor commutes with $\operatorname{div}(s')$, proving (4.11) on U. So let Y be a non-proper intersection component of $\psi^{-1}(S) \cap |\operatorname{div}(s')|$. Then Y maps to a point, hence there is an $\mathcal{U}$ from our formal open covering of $\mathcal{C}$ such that $Y \subset \psi^{-1}(\mathcal{U})$. On $\psi^{-1}(\mathcal{U})$, $\tilde{c}$ is an R-linear combination of Cartier divisors above, hence (4.11) holds also in a neighbourhood of Y. This proves (4.11) completely. Hence we have checked axioms (C1)-(C3) for cycles. By axiom (C3), $\tilde{c}$ passes to rational equivalence and hence $\tilde{c} \in CH^1_S(\tilde{\mathcal{C}})$.

Now we define the operations $\hat{c}_1(1)_{\mathcal{C}}$. Let X' be a proper scheme over K, let $\psi : \mathfrak{X}' \to \mathcal{C}$ be a morphism of K°-models and let $\alpha' \in Z(\mathfrak{X}', R)$. Then we define $\hat{c}_1(1)_{\mathcal{C}} \cap_\psi \alpha' \in \widehat{CH}_*^{|\alpha'| \cap \widetilde{\mathfrak{X}'}}(\mathfrak{X}', R)$ by the following procedure. If α' is vertical, then let

$$\hat{c}_1(1)_{\mathcal{C}} \cap_\psi \alpha' := \tilde{c} \cap_{\tilde{\psi}} \alpha'.$$

If α' is horizontal, we define

$$\hat{c}_1(1)_{\mathcal{C}} \cap_\psi \alpha' := \widehat{\operatorname{div}}_{\mathfrak{X}}(1).\alpha'$$

using the $\mathfrak{X}'$-admissible metric $\psi^*\| \ \|$ on $O_{X'}$. Axioms (C1)-(C3) are clear on vertical cycles using the corresponding axioms for $\tilde{c}$. For a horizontal cycle, axioms (C1) and (C2) follow from Corollary 4.3.23. Let us check axiom (C3) for a horizontal cycle α' on X', i.e. let $\mathcal{L}'$ be a line bundle on $\mathfrak{X}'$ with invertible meromorphic section $\overline{s'}$, then we claim

$$\operatorname{div}(\overline{s'}).\,(\hat{c}_1(1)_{\mathcal{C}} \cap_\psi \alpha') = \hat{c}_1(1)_{\mathcal{C}} \cap_\psi \left(\operatorname{div}(\overline{s'}).\alpha'\right) \in CH_*^{|\alpha'| \cap |\operatorname{div}(\overline{s'})| \cap \widetilde{\mathfrak{X}'}}(\mathfrak{X}', R). \tag{4.12}$$

We may assume α' prime and equal to X'. By Corollary 4.3.23 (or directly by definition), we may assume that $\mathfrak{X}'$ is formal analytic. Hence the claim follows from Theorem 4.3.21. Finally axiom (C4) is by definition. By axiom (C3), the operations $\hat{c}_1(1)_{\mathcal{C}}$ factor through rational equivalence and hence $\hat{c}_1(1)_{\mathcal{C}} \in \widehat{CH}^1_{\tilde{\mathcal{C}}}(\mathcal{C}, R)$. Clearly, it has properties a) and b). □

4.3.26 Remark. Let C be a smooth projective curve over K and let L be a line bundle on C with an admissible metric $\| \ \|$. There is a semistable K°-model $\mathcal{C}$ of C such that $\tilde{\mathcal{C}}$ has smooth irreducible components and such that $\| \ \|$ is $\mathcal{C}$-admissible. For an invertible meromorphic section s of L, let $\hat{c}_1(s) \in \widehat{CH}^1_{|\operatorname{div}(s)|}(C, R)$ be induced from $\hat{c}_1(s)_{\mathcal{C}}$ (Theorem 4.3.25). We claim that $\hat{c}_1(s)$ does not depend the choice of $\mathcal{C}$: Let $\mathcal{C}' \xrightarrow{\psi} \mathcal{C}$ be a morphism of such K°-models. It is enough to show that

$$\psi^* \hat{c}_1(s)_{\mathcal{C}} = \hat{c}_1(s)_{\mathcal{C}'}.$$

By Corollary 4.3.23, it is clear that condition a) of Theorem 4.3.25 is fulfilled for $\psi^* \hat{c}_1(s)_{\mathcal{C}}$ on $\mathcal{C}'$. Since condition b) is obvious, we get the claim. Note that if $\| \ \|$ is a formal metric $\| \ \|_{\mathcal{L}}$, then $\hat{c}_1(s)$ agrees with $\hat{c}_1^{\mathcal{L}}(s)$ from Example 4.2.16.

4.3.27 Corollary. *Let C, C' be smooth projective curves over K with invertible meromorphic sections s, s' of admissibly metrized line bundles L and L', respectively. If X is a proper scheme over K with morphisms $\varphi : X \to C$ and $\varphi' : X \to C'$, then*

$$\varphi^* \hat{c}_1(s) \cup \varphi'^* \hat{c}_1(s') = \varphi'^* \hat{c}_1(s') \cup \varphi^* \hat{c}_1(s) \in \widehat{CH}^2_{|\operatorname{div}(s)| \cap |\operatorname{div}(s')|}(X, R).$$

Proof: The argument is completely similar to the proof of axiom (C3) in Theorem 4.3.25. We can always reduce to a local question where the divisors are R-linear combinations of formal Cartier divisors. We omit the details. □

4.3.28 Remark. Let L be a line bundle on the smooth projective curve C and let $\mathcal{C}$ be a semistable K°-model of C. The space of $\mathcal{C}$-admissible metrics on O_C has a canonical basis $\{\| \ \|_W\}$ corresponding to the irreducible components W of $\tilde{\mathcal{C}}$ (cf. Remark 4.3.11). If s is an invertible meromorphic section of L and $\| \ \|$ is a $\mathcal{C}$-admissible metric on L, then the *vertical part* of $\hat{c}_1(s)$ is the class

$$\hat{c}_1^{ver} := \hat{c}_1^{ver}(1) \in \widehat{CH}^1_{fin}(C, R)$$

induced by the $\mathcal{C}$-admissible metric $\| \ \|_{ver} := \otimes_W \| \ \|_W^{\operatorname{ord}(s,W)}$ on O_C. The horizontal part of $\hat{c}_1(s)$ is the class

$$\hat{c}_1^{hor}(s) \in \widehat{CH}^1_{|\operatorname{div}(s)|}(C, R)$$

induced by the $\mathcal{C}$-admissible metric $\| \ \|_{hor} := \| \ \| / \| \ \|_{ver}$. Obviously, we have

$$\hat{c}_1(s) = \hat{c}_1^{hor}(s) + \hat{c}_1^{ver} \in \widehat{CH}^1_{|\operatorname{div}(s)|}(C, R).$$

For any morphism $\psi : X' \to C$ such that $\psi^* \operatorname{div}(s)$ is a well-defined Cartier divisor on X', it is clear that $\hat{c}_1^{hor}(s) \cap \operatorname{cyc}(X')$ is the horizontal cycle associated to $\operatorname{div}(s \circ \psi)$. This makes the analogy to horizontal and vertical parts of divisors on a K°-model perfect.

4.3.29 Definition. Let $\mathfrak{X}$ be an admissible formal scheme proper over K° and let $\alpha \in \tilde{Z}_{-1}(\mathfrak{X}, R)$, i.e. α is a vertical cycle class with coefficients in R and of dimension 0 in

$\tilde{\mathfrak{X}}$. If $\pi : \mathfrak{X} \to \mathrm{Spf}K^\circ$ is the morphism of structure, then $\pi_*(\alpha) \in \tilde{Z}_{-1}(\mathrm{Spf}K^\circ, R) \cong R$ (Example 4.2.11). We denote by $\int_{\mathfrak{X}} \alpha$ the corresponding number in R.

Let X be a quasi-compact and quasi-separated rigid analytic variety proper over K. For $\alpha \in \hat{Z}_{-1}(X, R)$, we get a well-defined number

$$\int_X \alpha := \int_{\mathfrak{X}} \alpha_{\mathfrak{X}},$$

independent of the choice of $\mathfrak{X} \in M_X$. If $c_{\mathfrak{X}} \in CH_{fin}^{\dim(X)+1}(\mathfrak{X}, R)$ induces $\hat{c} \in \widehat{CH}_{fin}^{\dim(X)+1}(X, R)$, then we simply write

$$\int_X \hat{c} := \int_{\mathfrak{X}} c_{\mathfrak{X}} := \int_X \hat{c} \cap \mathrm{cyc}(X) = \int_{\mathfrak{X}} c_{\mathfrak{X}} \cap \mathrm{cyc}(X).$$

4.3.30 Let C be an irreducible smooth projective curce over K and let us choose a semistable K°-model $\mathcal{C}$. By Remark 4.3.11, we know there is a one-to-one correspondence between $\mathcal{C}$-admissible metrics on O_C and vertical divisors on $\mathcal{C}$ with coefficients in R. A one-dimensional cycle α on $\tilde{\mathcal{C}}$ with coefficients in R induces a $\mathcal{C}$-admissible metric $\| \ \|_\alpha$ on O_C. Let $\widehat{\mathrm{div}}_{\mathcal{C}}^{\alpha}(1)$ be the corresponding intersection operation, then we have

$$\alpha = \widehat{\mathrm{div}}_{\mathcal{C}}^{\alpha}(1).C.$$

For α, β one-dimensional cycles on $\tilde{\mathcal{C}}$ with coefficients in R, we get a pairing

$$\langle \alpha \mid \beta \rangle := \int_{\mathcal{C}} \widehat{\mathrm{div}}_{\mathcal{C}}^{\alpha}(1).\beta \in R.$$

For this pairing, we can prove a *local Hodge index theorem*:

4.3.31 Theorem. The above pairing is a symmetric negative semidefinite bilinear form on the R-vector space of one dimensional cycles on $\tilde{\mathcal{C}}$ with coefficients in R. The kernel of the pairing is $R\sum_V V$, where V is ranging over all irreducible components of $\tilde{\mathcal{C}}$.

Proof: Clearly, the pairing is bilinear and symmetry follows from Theorem 4.3.21. Let $\alpha = \sum_V m_V V$ and $\beta = \sum_W n_W W$ with V and W ranging over all irreducible components of $\tilde{\mathcal{C}}$. Then an easy calculation shows

$$\langle \alpha \mid \beta \rangle = -\frac{1}{2} \sum_{V,W} (m_V - n_W)^2 \langle V \mid W \rangle. \tag{4.13}$$

Let $V \neq W$. We compute the multiplicity of the cycle $\widehat{\mathrm{div}}_{\mathcal{C}}^{V}(1).W$ in $P \in W$. If P is regular, then the admissible metric $\| \ \|_V$ is constant on the formal fibre over P. Hence the multiplicity of $\widehat{\mathrm{div}}_{\mathcal{C}}^{V}(1).W$ in P is zero. If P is a double point, then the formal fibre is isomorphic to an open annulus $\{\zeta \in K \mid r < |\zeta| < 1\}$ of height $r \in |K^\times|, r < 1$. Let W' be the other irreducible component of $\tilde{\mathcal{C}}$ passing through P, it may happen $W' = W$. If $W' \neq V$, then the metric $\| \ \|_V$ is constant 1 on the formal fibre (Proposition 4.3.10) and again the multiplicity is 0. So we may assume that $W' = V$, i.e. $P \in V \cap W$. Then the metric $\| \ \|_V$ is given on the formal fibre over P by

$$\|1(x)\|_V = |\zeta(x)|^{-1/\log r}$$

(we allow the usual change of coordinates $\zeta \leftrightarrow r/\zeta$ if it is necessary). We conclude that the multiplicity of $\widehat{\mathrm{div}}_{\mathcal{C}}^{V}(1).W$ in P is $-1/\log r > 0$ (cf. proof of Proposition 4.3.19). Hence we have proved

$$\langle V \mid W \rangle \begin{cases} = 0 & \text{if } V \cap W = \emptyset, \\ > 0 & \text{if } V \cap W \neq \emptyset. \end{cases} \tag{4.14}$$

Using (4.13), we conclude that the bilinear form $\langle \mid \rangle$ is negative semidefinite. Note that the special fibre $\tilde{\mathcal{C}}$ is connected, otherwise C would be the disjoint union of two non-empty formal open subsets which contradicts the irreduciblility of C . Thus for $V \neq W$, there is a chain $V_0 = V, V_1, \ldots, V_r = W$ of irreducible components of $\tilde{\mathcal{C}}$ such that $V_{j-1} \cap V_j \neq \emptyset$. By (4.13) and (4.14), we conclude that α is in the kernel of $\langle \mid \rangle$ if and only if $\alpha \in R \sum_V V$. $\square$

4.3.32 Corollary. *Let C be a smooth projective curve over K and let $L \in \mathrm{Pic}^\circ(C)$. Then there is an admissible metric $\| \ \|$ on L such that for any invertible meromorphic section s of L and any $\alpha \in \widehat{CH}_0^{fin}(C, R)$, we have*

$$\int_C \hat{c}_1(s) \cap \alpha = 0.$$

If C is irreducible, then the metric is uniquely determined up to multiples in e^R.

Proof: We may assume C irreducible. Note that the operation of $\hat{c}_1(s)$ on $\widehat{CH}_*^{fin}(C, R)$ is independent of s. Let $\mathcal{C}$ be a semistable K°-model of C. Since we may compute the intersection number $\int_C \hat{c}_1(s) \cap \alpha$ on $\mathcal{C}$, we have to look for a $\mathcal{C}$-admissible metric $\| \ \|$ on L with

$$\int_{\mathcal{C}} \widehat{\mathrm{div}}_{\mathcal{C}}(s).\alpha = 0$$

for all cycles α on $\tilde{\mathcal{C}}$ of dimension 1. Let $\| \ \|_{\mathcal{L}}$ be any formal metric on L with K°-model $\mathcal{L}$ which we may assume to live on $\mathcal{C}$, then the corresponding action of $\widehat{\mathrm{div}}_{\mathcal{C}}^{\mathcal{L}}(s)$ on the R-vector space $Z_1(\tilde{\mathcal{C}}, R)$ is an R-linear form Φ. Let $\lambda \in K^\times, |\lambda| < 1$. Then λ may be viewed as constant section of O_C with the trivial metric. By commutativity, we get

$$\int_{\mathcal{C}} \widehat{\mathrm{div}}_{\mathcal{C}}^{\mathcal{L}}(s).\mathrm{cyc}\left(\widehat{\mathrm{div}}_{\mathcal{C}}(\lambda)\right) = \int_{\mathcal{C}} \widehat{\mathrm{div}}_{\mathcal{C}}(\lambda).\mathrm{cyc}\left(\widehat{\mathrm{div}}_{\mathcal{C}}^{\mathcal{L}}(s)\right).$$

Since $\mathrm{div}(s)$ is a divisor of degree 0 on C, the right hand side vanishes. As $\mathrm{cyc}\left(\widehat{\mathrm{div}}_{\mathcal{C}}(\lambda)\right)$ is a non-zero multiple of the special fibre $\sum_V V$, we conclude that Φ vanishes on $\sum_V V$. By the Hodge index theorem, there is $\beta \in Z_1(\tilde{\mathcal{C}}, R)$ with

$$\langle \beta \mid \alpha \rangle = \int_{\mathcal{C}} \widehat{\mathrm{div}}_{\mathcal{C}}^{\mathcal{L}}(s).\alpha$$

for all $\alpha \in Z_1(\tilde{\mathcal{C}}, R)$. Hence there is an admissible metric $\| \ \|_\beta$ on O_C with

$$\int_{\mathcal{C}} \widehat{\mathrm{div}}_{\mathcal{C}}^{\beta}(s).\alpha = \int_{\mathcal{C}} \widehat{\mathrm{div}}_{\mathcal{C}}^{\mathcal{L}}(s).\alpha.$$

We conclude that the $\mathcal{C}$-admissible metric $\| \ \|_0 / \| \ \|_\beta$ satisfies the claim. Moreover, as a $\mathcal{C}$-admissible metric, it is unique up to e^R multiples. Since the pull-back of such a metric to another model $\mathcal{C}' \geq \mathcal{C}$ satisfies the same property on $\mathcal{C}'$ (use projection formula), we get global uniqueness up to e^R-multiples. $\square$

4.3.33 Remark. Recall the notion of *Néron pairing* on a smooth projective curve C over K. There is a unique way to assign to divisors Y and Z of degree 0 and with disjoint support a number $\langle Y \mid Z \rangle_{N\acute{e}r}$ satisfying the following properties:

a) The pairing is bilinear and symmetric.

b) If $Y = \mathrm{div}(f)$ for a rational function f on C, then

$$\langle Y \mid Z \rangle_{N\acute{e}r} = -\log |f(Z)|$$

where on the right and in similar situations below, we extend the notion from the components to cycles by linearity.

c) For any $x_0 \in C$, the map $x \mapsto \langle Y \mid x - x_0 \rangle_{N\acute{e}r}$ is v-continuous and locally bounded on $X \setminus |Y|$.

Here continuity is with respect to the v-topology and may be omitted. Locally bounded is enough, i.e. the map is bounded on each bounded subset of $C \setminus |Y|$ (use Definition 5.2.14). This result is due to Néron (for details, cf. [La], Theorem 11.3.6). In the case of a discrete valuation, the pairing may be given in terms of intersection multiplicities (cf. [Fa2], Theorem 4, Hriljac, [Hr], Theorem 1.6). We have the same result here for arbitrary valuations:

4.3.34 Corollary. Let C be a projective smooth curve over K and $\| \ \|$ be an admissible metric on $L \in \mathrm{Pic}^\circ(C)$ of the same type as in Corollary 4.3.32. Then for any invertible meromorphic section s of L and any divisor Z of degree 0 with $|\mathrm{div}(s)| \cap |Z| = \emptyset$, we have

$$-\log \|s(Z)\| = \langle \mathrm{div}(s), Z \rangle_{N\acute{e}r}.$$

We call $\| \ \|$ a canonical metric.

Proof: It is obvious from the definitions that an admissible metric is always bounded. For divisors Y and Z of degree 0 with disjoint support, we define a pairing

$$\langle Y, Z \rangle := -\log \|s(Z)\|$$

where s is an invertible meromorphic section of $O(Y)$ with $Y = \mathrm{div}(s)$ and where $\| \ \|$ is the admissible metric on $O(Y) \in \mathrm{Pic}^\circ(C)$ from Corollary 4.3.32. Using that the divisors have degree 0, this pairing does neither depend on the choice of s nor on the choice of the metric because both are determined up to multiples. It is clear that the pairing is bilinear and satisfies b). Moreover, every admissible metric is continuous with respect to the v-topology and bounded (obvious from the definitions). This proves c) for our pairing. Note that

$$\langle Y, Z \rangle = \int_C \hat{c}_1(s) \cap Z = \int_C \hat{c}_1(s) \cup \hat{c}_1(t)$$

where $\mathrm{div}(t) = Y$. By Corollary 4.3.27, the pairing is symmetric. Thus we have proved a)-c) for the above pairing, hence it agrees with the Néron pairing. □

4.3.35 Example. Let $q \in K^\times, |q| < 1$, and let $C = \mathbb{G}_m / q^{\mathbb{Z}}$ be Tate's elliptic curve (cf. [BGR], Example 9.3.4/4 and 9.7.3). For $a \in K^\times$, $|q| < |a| < 1$, we are going to describe the canonical metric on $O([a] - [1]) \in \mathrm{Pic}^\circ(C)$. Let $\mathcal{C}$ be the formal analytic variety given by the formal open affinoid covering $\mathcal{U}_1 := \{\zeta \in \mathbb{G}_m \mid |a| \leq |\zeta| \leq 1\}$ and $\mathcal{U}_2 := \{\zeta \in \mathbb{G}_m \mid |q| \leq |\zeta| \leq |a|\}$ of C. Then $\mathcal{C}$ is a semistable K°-model of C with intersection graph equal to a circle of circumference $-\log |q|$ divided up in two arcs of length $-\log |a|$ and $\log |a| - \log |q|$, respectively. If we choose the local equations

$$\gamma_1 := \frac{a}{\zeta} \cdot \frac{\zeta - a}{\zeta - 1}$$

on U_1 and

$$\gamma_2 := \zeta \cdot \frac{\zeta - a}{\zeta - q}$$

on U_2, then we get a Cartier divisor D on $\mathcal{C}$ with generic fibre $D^{an} = [a] - [1]$. The formal metric $\| \ \|_0 := \| \ \|_{O(D)}$ is given by

$$\|s_{[a]-[1]}(\zeta)\|_0 := |\gamma_i(\zeta)| \qquad (\zeta \in U_i)$$

for the canonical section $s_{[a]-[1]}$ of $O([a]-[1])$. The reduction $\tilde{\mathcal{C}}$ consists of two components W_1, W_2 both isomorphic to $\mathbb{P}^1$ intersecting in two ordinary double points P_1, P_2. Here P_i is the reduction of the interior points of the annulus $\mathcal{U}_i$. On W_1 (resp. W_2), we use the affine coordinate $\widetilde{\zeta/a}$ (resp. $\tilde{\zeta}$). Computing the Weil divisor of $\{\widetilde{\mathcal{U}}_i \cap W_j, \gamma_i/\widetilde{\gamma_i(W_j)}\}_{i=1,2}$, we get easily

$$D.W_1 = \widehat{\operatorname{div}}^0(s).W_1 = [\tilde{a}] \quad , \quad D.W_2 = \widehat{\operatorname{div}}^0(s).W_2 = -[\tilde{1}]$$

as an identity of cycles (using the intersection product from 4.3.18). Hence we have

$$O(D)|_{W_1} \cong O_{\mathbb{P}^1}(1) \quad , \quad O(D)|_{W_2} \cong O_{\mathbb{P}^1}(-1).$$

We are looking for a $\mathcal{C}$-admissible metric $\| \ \|_{adm}$ on O_C such that $\| \ \| \otimes \| \ \|_0$ is a canonical metric for $O([a]-[1])$. It has the form

$$\|1(\zeta)\|_{adm} = r_i|\zeta^{\lambda_i}|$$

for $\zeta \in \mathcal{U}_i$ where $r_i \in \exp(R)$ and $\lambda_i \in R$. Compatibility on $\mathcal{U}_1 \cap \mathcal{U}_2$ leads to the conditions $r_1 = r_2|q|^{\lambda_2}$ and

$$\lambda_2 \log|q| = (\lambda_2 - \lambda_1)\log|a|. \tag{4.15}$$

Similarly as above, we get

$$\widehat{\operatorname{div}}^{adm}(1).W_1 = \lambda_2 P_2 - \lambda_1 P_1 \quad , \quad \widehat{\operatorname{div}}^{adm}(1).W_2 = \lambda_1 P_1 - \lambda_2 P_2.$$

Hence $\| \ \|_0 \otimes \| \ \|_{adm}$ is a canonical metric if and only if (4.15) and

$$\lambda_1 - \lambda_2 = 1$$

is satisfied. This leads to

$$\lambda_1 = 1 - \frac{\log|a|}{\log|q|} \quad , \quad \lambda_2 = -\frac{\log|a|}{\log|q|}.$$

Hence the canonical metric on $O([a]-[1])$ is given by

$$\|s_D(\zeta)\|_{can} = \begin{cases} r_2|\zeta^{\lambda_2}\frac{\zeta-a}{\zeta-1}| & \text{if } \zeta \in \mathcal{U}_1, \\ r_2|\zeta^{\lambda_1}\frac{\zeta-a}{\zeta-q}| & \text{if } \zeta \in \mathcal{U}_2 \end{cases}$$

for any $r_2 \in \exp(R)$.

4.3.36 Remark. Let $(L, \| \ \|)$ be an admissibly metrized line bundle on a smooth projective curve over K. Then there is a semistable K°-model $\mathcal{C}$ of X such that the metric is $\mathcal{C}$-admissible (cf. Remark 4.3.26). We choose any formal metric $\| \ \|_0$ on L, then $\| \ \|_{adm} := \| \ \|/\| \ \|_0$ is an admissible metric on O_C.

By Proposition 4.3.10, the function $-\log\|1(x)\|_{adm}$ may be interpretated as a continuous and piecewise linear function f on the intersection graph of $\mathcal{C}$. Note that units on an annulus with the coordinate ζ have a dominant term $\alpha\zeta^n$ in their Laurent series for some $n \in \mathbb{Z}$ ([BGR], Lemma 9.7.1/1). Thus the metric $\| \ \|$ is formal if and only if the slopes and the constant terms of f are contained in the value group $\log |K^\times|$. Note that it is allowed to work on the K°-model $\mathcal{C}$ since passing to a semistable K°-model $\mathcal{C}'$ over $\mathcal{C}$ would lead to an intersection graph containing the old one (Remark 4.3.12) and the condition on slopes and constant terms wouldn't change.

We conclude that the canonical metric $\| \ \|_{can}$ in Example 4.3.35 is a formal metric if and only if $\log |a| / \log |q| \in \mathbb{Z}$ and $r_2 \in |K^\times|$. If K is the algebraic closure of a complete field with discrete valuation, then the value group is $\mathbb{Q}$ and we conclude that there is always a canonical metric on $O([a]-[1])$ which is a root of a formal metric. But in general, there is $a \in K^\times$ with $\log |a| / \log |q|$ irrational and no canonical metric on $O([a]-[1])$ will be a root of a formal metric.

Moreover, it is clear that the canonical metric on $O([a]-[1])$ is not of the form

$$\| \ \|_{can} = \| \ \|_0 \otimes \bigotimes_{j=1}^{r} \| \ \|_j^{\otimes \lambda_j}$$

for a formal metric $\| \ \|_0$ on $O([a]-[1])$, formal metrics $\| \ \|_1, \ldots, \| \ \|_r$ on O_C and $\lambda_1, \ldots, \lambda_r \in R$. Otherwise, the quotient of the slopes of $\log \| \ \|_{can} / \| \ \|_0$ on the intersection graph would be rational. So it is really necessary to define admissible metrics locally.

4.4 Cohomological Models of Line Bundles

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $| \ |_v$. All spaces denoted by greek letters $X, X', \ldots$ are assumed to be proper schemes over K. The coefficients of vertical cycles on models are taken in a subfield R of $\mathbb{R}$ containing the value group $\log |K^\times|_v$.

In this section, we give a theory of Chern classes for line bundles L on X with values in Arakelov-Chow cohomology. We have already seen two examples. The first is for a formally metrized line bundle on a proper variety over K (Example 4.2.16) and the second is for an admissibly metrized line bundle on a projective smooth curve (Remark 4.3.26). In general, the difference from any first Arakelov-Chern class $\hat{c}_1$ to the class $\hat{c}_1^{\mathcal{L}}$ with respect to a formal metric $\| \ \|_{\mathcal{L}}$ on L is given by any class $\hat{c}_1^R \in \widehat{CH}_1^{fin}(X, R)$. Note that $\hat{c}_1(L)$ may be refined to $\hat{c}_1(L, s)$ for an invertible meromorphic section s of L using $\hat{c}_1^{\mathcal{L}}(s)$, but $\hat{c}_1^R$ does not depend on s. So we have no support problem working with $\hat{c}_1^R$ and the case of a formal metric is well understood. This explains why this decomposition is so handy, it splits the problem into the case of a formal metric considered before and into the cohomological case in $\widehat{CH}_1^{fin}(X, R)$.

There is a technical problem that commutativity is not known for Arakelov-Chow groups. But there is a strong need that our Arakelov-Chern classes commute. So we introduce the centralizator $\hat{B}^*(X, R)$ of the two examples above in $\widehat{CH}_1^{fin}(X, R)$ and its center $\hat{C}_1^{fin}(X, R)$. We deduce that $\hat{C}_1^{fin}(X, R)$ is commutative containing the Arakelov-Chern classes of formally merized line bundles (resp. of admissibly metrized line bundles if X is a projective smooth curve). We call a first Arakelov-Chern class admissible if $\hat{c}_1^R \in \hat{C}_1^{fin}(X, R)$.

Every admissible first Arakelov-Chern class gives rise to an associated metric $\| \ \|_{\hat{c}_1(L)}$ by taking the intersection number with points. In the two examples above, the associated metric is the formal and the admissible metric, respectively. In Proposition 4.4.13, we prove that the multiplicity of $\hat{c}_1(L, s) \cap X$ in an irreducible component V of the special fibre of a formal analytic model is $-\log \|s(x)\|_{\hat{c}_1(L)}$ for $x \in X$ with generic reduction in V. The proof is by induction on the dimension of X and uses axiom (C3) and the corresponding result for formal metrics.

In Proposition 4.4.14, we show that for a d-dimensional cycle Z algebraically equivalent to 0 on X, there is a vertical Arakelov cycle α^{fin} such that $\hat{C}_{fin}^{d+1}(X, R)$ is orthogonal to $Z + \alpha^{fin}$

with respect to the intersection pairing. We have seen this result already in the case of curves (Corollary 4.3.32). The general case is reduced to the case of curves by using a correspondence for Z induced by algebraic equivalence. This is a similar idea as in [Bei] and [Ku].

The main result is Theorem 4.4.15. We show that a line bundle L algebraically equivalent to 0 on a smooth proper variety X over K has an admissible first Arakelov-Chern class $\hat{c}_1(L)$ orthogonal to $\widehat{CH}_0^{fin}(X,R)$. Moreover, the associated metric $\| \ \|_{\hat{c}_1(L)}$ is unique up to multiples and we will show that it is a canonical metric on L. Uniqueness is a direct consequence of Proposition 4.4.14 and the proof of the latter may be viewed as a guideline to the proof of Theorem 4.4.15. Again, we use a correspondence for L to reduce to the case of curves. The additional technical difficulty is that to define the class $\hat{c}_1^R \in \hat{C}_1^{fin}(X,R)$, we have to distinguish between horizontal and vertical cycles, and that the flat push-forward of an Arakelov-Chow cohomology class is only well-defined on the level of K°-models.

4.4.1 Definition. Let $\hat{B}^*(X,R)$ be the set of all $\hat{c} \in \widehat{CH}^*_{fin}(X,R)$ such that for any projective smooth curve C', any admissible metric $\| \ \|$ on $O_{C'}$ and any morphisms $\psi : X' \to X, \phi : X' \to C'$, we have

$$\psi^*\hat{c} \cup \phi^*\hat{c}_1(1) = \phi^*\hat{c}_1(1) \cup \psi^*\hat{c} \in \widehat{CH}^*_{fin}(X',R).$$

4.4.2 Proposition.

a) For a formal metric $\| \ \|$ on O_X, we have $\hat{c}_1(1) \in \hat{B}^1(X,R)$.

b) If C is a projective smooth curve and $\| \ \|$ is an admissible metric on O_C, then

$$\hat{c}_1(1) \in \hat{B}^1(C,R).$$

c) Let $\varphi : X' \to X$ be a morphism and let $\hat{c} \in \hat{B}^*(X,R)$, then

$$\varphi^*(\hat{c}) \in \hat{B}^*(X',R).$$

d) Let $\bar{\varphi} : \mathfrak{X}' \to \mathfrak{X}$ be a flat morphism of relative dimension d extending the morphism φ above to K°-models and suppose that $c'_{\mathfrak{X}'} \in CH^p_{fin}(\mathfrak{X}',R)$ induces a class in $\hat{B}^p(X',R)$. Then $\bar{\varphi}_*(c'_{\mathfrak{X}'})$ induces a class in $\hat{B}^{p-d}(X,R)$.

e) If $\hat{c}, \hat{c}' \in \hat{B}^*(X,R)$, then $\hat{c} \cup \hat{c}' \in \hat{B}^*(X,R)$.

Proof: We have seen in Theorem 4.3.25 that the first Chern class of an admissibly metrized line bundle is a well-defined local Chow cohomology class. Thus a) follows immediately from axiom (C3) and b) is a consequence of Corollary 4.3.27. Properties c), e) are immediate from the definitions and d) follows from the fibre square rule and projection formula (Proposition 4.2.8). □.

4.4.3 Definition. Let $\hat{C}^*_{fin}(X,R)$ be the set of $\hat{c} \in \widehat{CH}^*_{fin}(X,R)$ satisfying

$$\hat{c}' \cup \psi^*\hat{c} = \psi^*\hat{c} \cup \hat{c}' \in CH^*_{fin}(X',R)$$

for all morphisms $\psi : X' \to X$ and all $\hat{c}' \in \hat{B}^*(X',R)$.

4.4.4 Proposition. *All properties in Proposition 4.4.2 hold also for $\hat{C}_{fin}$ instead of $\hat{B}$. Moreover, $\hat{C}^*_{fin}(X,R)$ is commutative with respect to the cup-product and*

$$\hat{C}^*_{fin}(X,R) \subset \hat{B}^*(X,R).$$

Proof: Again, a) follows from (C3), b) follows from the definitions and the proofs of c)-e) are the same as in Proposition 4.4.2. If $\hat{c} \in \hat{C}^*_{fin}(X,R)$, then in the notation of Definition 4.4.1, we know from Proposition 4.4.2 that $\phi^*\hat{c}_1(1) \in \hat{B}^1(X,R)$. We conclude that $\phi^*\hat{c}_1(1)$ and $\psi^*\hat{c}$ commute proving $\hat{c} \in \hat{B}^*(X,R)$. This proves $\hat{C}^*_{fin}(X,R) \subset \hat{B}^*(X,R)$. This implies commutativity of $\hat{C}^*_{fin}(X,R)$. □

4.4.5 Definition. Let L be a line bundle on X. An *admissible first Arakelov-Chern class* $\hat{c}_1(L)$ for L is a family of elements

$$\hat{c}_1(L,s') \in \widehat{CH}^1_{|\mathrm{div}(s')|}(X',R)$$

for every morphism $\psi : X' \to X$ and every invertible meromomorphic section s' of $\psi^*(L)$. It is required that all these operations are given by one formal metric $\| \ \|_{\mathcal{L}}$ of L and one $\hat{c}_1^R \in \hat{C}^1_{fin}(X,R)$ through

$$\hat{c}_1(L,s') = \hat{c}_1^{\mathcal{L}}(s') + \psi^*(\hat{c}_1^R) \tag{4.16}$$

where $\hat{c}_1^{\mathcal{L}}(s')$ is the first Arakelov-Chern class of s' with respect to the formal metric $\psi^*\| \ \|_{\mathcal{L}}$ on $\psi^*(L)$ constructed in Example 4.2.16.

4.4.6 Example. Every formally metrized line bundle L with corresponding K°-model $\mathcal{L}$ gives rise to an admissible first Arakelov-Chern class $\hat{c}_1(L)$ by

$$\hat{c}_1(L,s') := \hat{c}_1^{\mathcal{L}}(s').$$

More generally, this holds for roots of formal metrics. Similarly, every admissibly metrized line bundle on a smooth projective curve induces an admissible first Arakelov-Chern class (Remark 4.3.26).

4.4.7 Remark. Note that a formal metric $\| \ \|_{\mathcal{L}}$ and a class $\hat{c}_1^R \in \hat{C}^1_{fin}(X,R)$ induce always an admissible first Arakelov-Chern class by the rule (4.16).

4.4.8 Proposition. *Let $\hat{c}_1(L), \hat{c}_1(L')$ be admissible first Arakelov-Chern classes for line bundles L, L' on X. If s, s' are invertible meromorphic sections of L and L', then*

$$\hat{c}_1(L,s) \cup \hat{c}_1(L',s') = \hat{c}_1(L',s') \cup \hat{c}_1(L,s) \in \widehat{CH}^2_{|\mathrm{div}(s)| \cap |\mathrm{div}(s')|}(X,R).$$

Proof: If $\hat{c}_1(L)$ or $\hat{c}_1(L')$ are induced by a formal metric, then this follows from axiom (C3). By linearity, it remains to check commutativity for two classes in $\hat{C}^1_{fin}(X,R)$ which follows from Proposition 4.4.4. □

4.4.9 Remark. If $\hat{c}_1(L_1)$ and $\hat{c}_1(L_2)$ are admissible first Arakelov-Chern classes for line bundles L_1 and L_2 on X, then the *sum* is the admissible first Arakelov-Chern class $\hat{c}_1(L_1) + \hat{c}_1(L_2) = \hat{c}_1(L_1 \otimes L_2)$ for $L_1 \otimes L_2$ given by using the tensor product of formal metrics and the sum of the classes in $\hat{C}^1_{fin}(X,R)$. Hence the admissible first Arakelov-Chern classes form a group. Let $\psi : X' \to X$ be a morphism and let s_1', s_2' be invertible meromorphic sections of ψ^*L_1, ψ^*L_2, then

$$\hat{c}_1(L_1 \otimes L_2, s_1' \otimes s_2') = \hat{c}_1(L_1,s_1') + \hat{c}_1(L_2,s_2') \in \widehat{CH}^1_{|\mathrm{div}(s_1')| \cup |\mathrm{div}(s_2')|}(X',R).$$

4.4.10 Remark. Let $\varphi : Z \to X$ be a morphism and let $\hat{c}_1(L)$ be an admissible first Arakelov-Chern class for the line bundle L on X, then the *pull-back* is the admissible first Arakelov-Chern class $\varphi^*\hat{c}_1(L) = \hat{c}_1(\varphi^*L)$ given by $\varphi^*\| \ \|_{\mathcal{L}}$ and $\varphi^*\hat{c}_1^R$. If $\psi : Z' \to Z$ is a morphism and s' is an invertible meromorphic section of $\psi^*\varphi^*L$, then

$$\hat{c}_1(\varphi^*L,s') = \hat{c}_1(L,s') \in \widehat{CH}^1_{|\mathrm{div}(s')|}(Z',R).$$

4.4.11 Remark. To an admissible first Arakelov-Chern class $\hat{c}_1(L)$ on X, we have an *associated metric* $\| \ \|_{\hat{c}_1(L)}$ on L. For $x \in X$ and $s(x)$ in the fibre of L over x, we may view $s(x)$ as a section of $L_x = L|_{\{x\}}$ and we set

$$\|s(x)\|_{\hat{c}_1(L)} := \int_{\{x\}} \hat{c}_1(L, s(x)) \in R.$$

Here, we have used Definition 4.3.29. The definition is compatibel with cup-product and pull-back. If $\hat{c}_1(L)$ is the first Arakelov-Chern class of a formally metrized line bundle on X (or an admissibly metrized line bundle on a smooth projective curve), it is immediately clear from the definition that $\| \ \|_{\hat{c}_1(L)}$ is the original metric.

4.4.12 Remark. It follows from Remark 4.4.9 that the operation of $\hat{c}_1(L, s')$ on $\widehat{CH}_*^{fin}(X', R)$ does not depend on the choice of s' and we simply write $\hat{c}_1(L)\cap$ for the action.

4.4.13 Proposition. *Let L be a line bundle on an integral proper scheme X with admissible first Arakelov-Chern class $\hat{c}_1(L)$ and invertible meromorphic section s. There is a formal analytic K°-model $\mathfrak{X}$ of X such that $\hat{c}_1(L)$ is given by a formal metric on $\mathfrak{X}$ and such that the corresponding $\hat{c}_1^R$ is induced by $\hat{c}_{1,\mathfrak{X}}^R \in \widehat{CH}^1_{fin}(\mathfrak{X}, R)$. Then the multiplicity of $\hat{c}_1(L, s) \cap X$ in an irreducible component V of $\tilde{\mathfrak{X}}$ is equal to*

$$-\log \|s(V)\| := -\log \|s(x)\|_{\hat{c}_1(L)}$$

for all $x \in X$ with reduction $\pi(x) \in V \cap \tilde{\mathfrak{X}}_{reg}$ and with $\pi(x)$ not contained in the reduction $\pi|\mathrm{div}(s)|$. In particular, the dependence of $\hat{c}_1(L, s) \cap X$ on $\hat{c}_1(L)$ is determined by the associated metric $\| \ \|_{\hat{c}_1(L)}$.

Proof: The result holds for the first Arakelov-Chern class of a formally metrized line bundle (Propostion 4.3.14), hence we may assume $L = O_X, s = 1$ and $\hat{c}_1(L, s) = \hat{c}_1^R$. By Proposition 3.4.2 and passing to the associated formal analytic variety, it is clear that such a $\mathfrak{X}$ exists.

The proof of the multiplicity claim is by induction on the dimension of X. If X is 0-dimensional, the claim is by definition of $\| \ \|_{\hat{c}_1(L)}$. So we may assume $\dim(X) > 0$. We choose $x \in X$ with $\tilde{x} := \pi(x) \in V_{reg}$ but not contained in any other irreducible component of $\tilde{\mathfrak{X}}$. By regularity, there is a regular function $\tilde{a}$ in an affine neighbourhood $\tilde{\mathcal{U}}$ of $\tilde{x}$ with $\tilde{a}(\tilde{x}) = 0$ and $d\tilde{a}(\tilde{x}) \neq 0$. Thus $\mathrm{div}(\tilde{a})$ is smooth in a neighbourhood of $\tilde{x}$. We may assume that there is a prime divisor W in V such that $\mathrm{div}(\tilde{a}) = \tilde{\mathcal{U}} \cap W$ is smooth. Let $\mathcal{U} = \mathrm{Spf}\mathcal{A} = \pi^{-1}\tilde{\mathcal{U}}$ be the corresponding formal affinoid open subspace of $\mathfrak{X}$. We choose a lift $a \in \mathcal{A}^\circ$ of $\tilde{a}$. Since the algebraic functions are dense in $\mathcal{A}$, we may assume that a is induced by a rational function on X also denoted by a. We may assume that $a(x) = 0$.

We consider the closed formal subscheme $\mathrm{Spf}\mathcal{A}^\circ/\langle a \rangle$ of $\mathrm{Spf}\mathcal{A}^\circ$. It is obvious that $\mathcal{A}^\circ/\langle a \rangle$ has no K°-torsion. Since the reduction $\tilde{\mathcal{A}}/\langle \tilde{a} \rangle$ is integral, we conclude that $\mathrm{Spf}\mathcal{A}^\circ/\langle a \rangle$ is an admissible formal scheme associated to the integral formal analytic variety $\mathrm{Spf}\mathcal{A}/\langle a \rangle$ (3.3.5 and [BGR], Proposition 6.2.3/5). In particular, the horizontal $\mathrm{div}(a)$ is prime on $\mathcal{U}$. Let Y be the irreducible component of $\mathrm{div}(a)$ passing through. Note that W is the intersection of $\mathcal{U}$ with the special fibre of $\bar{Y}$. Applying induction to $\bar{Y}$, we conclude that the multiplicity of $\hat{c}_1^R \cap Y$ in $\bar{W}$ is equal to $-\log \|1(x)\|_{\hat{c}_1(L)}$. If we use the trivial metric on O_X to define $\widehat{\mathrm{div}}(a)$, then axiom (C3) shows

$$\widehat{\mathrm{div}}_{\mathfrak{X}}(a).\left(\hat{c}_{1,\mathfrak{X}}^R \cap X\right) = \hat{c}_{1,\mathfrak{X}}^R \cap \left(\widehat{\mathrm{div}}_{\mathfrak{X}}(a).X\right) \in CH^{|\mathrm{div}_{\mathfrak{X}}(a)| \cap \tilde{\mathfrak{X}}}_{\dim(X)-2}(\mathfrak{X}, R). \tag{4.17}$$

Note that W is an irreducible component of the support $|\mathrm{div}_{\mathfrak{X}}(a)| \cap \tilde{\mathfrak{X}}$ and W has relative dimension $\dim(X) - 2$. The multiplicity of the left hand side of (4.17) in W is equal to the multiplicity of $\hat{c}^R_{1,\mathfrak{X}} \cap X$ in V (by construction of a and working on $\mathcal{U}$). The multiplicity of the right hand side of (4.17) in W is $-\log \|1(x)\|_{\hat{c}_1(L)}$. This proves the multiplicity claim.

Finally, it is clear from the proof of the existence of $\mathfrak{X}$ that we may choose $\mathfrak{X}$ sufficiently large. This proves the last claim since the horizontal part is $\operatorname{div}(s).X$ not depending on $\hat{c}_1(L)$ and the metric $\| \; \|_{\hat{c}_1(L)}$ anyway. $\square$

4.4.14 Proposition. *Let X be a smooth proper scheme over K and let Z be a d-dimensional cycle on X algebraically equivalent to 0. Then there is $\alpha^{fin} \in \widehat{CH}^{fin}_d(X,R)$ such that*

$$\int_X \hat{c} \cap \left(Z + \alpha^{fin}\right) = 0$$

for all $\hat{c} \in \hat{C}^{d+1}_{fin}(X,R)$.

Proof: There is a smooth projective curve C over K, a correspondence Γ on $X \times C$ and $t_0, t_1 \in C$ such that Z is rationally equivalent to $\Gamma_{t_1} - \Gamma_{t_0}$ ([Fu], Example 10.3.2), i.e. Γ is a cycle on $X \times C$ of dimension $d+1$ such that

$$Z = p_{1*}\left(p_2^* c_1\left(O(t_1 - t_0)\right) \cap \Gamma\right) \in CH_d(X)$$

where p_i are the projections of $X \times C$ onto the factors. Now we choose a canonical metric $\| \; \|$ on $O(t_1 - t_0)$ from Corollary 4.3.32. Let $\hat{c}_1(t_1 - t_0)$ be the first Arakelov-Chern class with respect to the canonical meromorphic section $s_{t_1 - t_0}$ of $O(t_1 - t_0)$. The generic fibre of

$$p_{1*}\left(p_2^* \hat{c}_1(t_1 - t_0) \cap \Gamma\right) \tag{4.18}$$

is rationally equivalent to Z. Using commutativity, we may replace Z by this generic fibre, hence we may assume that they agree and (4.18) has the form $Z + \alpha^{fin}$ for $\alpha^{fin} \in \widehat{CH}^{fin}_d(X,R)$. For $\hat{c} \in \hat{C}^d_{fin}(X,R)$, we get

$$\begin{aligned}
\int_X \hat{c} \cap \left(Z + \alpha^{fin}\right) &= \int_{X \times C} p_1^* \hat{c} \cap p_2^* \hat{c}_1(t_1 - t_0) \cap \Gamma \\
&= \int_{X \times C} p_2^* \hat{c}_1(t_1 - t_0) \cap p_1^* \hat{c} \cap \Gamma \\
&= \int_C \hat{c}_1(t_1 - t_0) \cap p_{2*}\left(p_1^* \hat{c} \cap \Gamma\right) \\
&= 0
\end{aligned}$$

where we have used (C1) in the first and third step, commutativity for $\hat{c}$ in the second step and that $p_{2*}\left(p_1^* \hat{c} \cap \Gamma\right)$ is vertical in the fourth step. $\square$

4.4.15 Theorem. *Let X be a smooth proper scheme over K and let $L \in \operatorname{Pic}^\circ(X)$. Then there is an admissible first Arakelov-Chern class $\hat{c}_1(L)$ for L such that*

$$\int_X \hat{c}_1(L) \cap \alpha = 0$$

for all $\alpha \in \widehat{CH}^{fin}_0(X,R)$. The corresponding metric $\| \; \|_{\hat{c}_1(L)}$ is uniquely determined up to multiples in e^R.

Proof: We may assume that X is irreducible. Since L is algebraically equivalent to 0, there is an irreducible smooth projective curve C over K with points $t_0, t_1 \in C$ and $M \in \operatorname{Pic}(X \times C)$ with $M_{t_0} = O_X$ and $M_{t_1} = L$. We endow the line bundle $O(t_1 - t_0)$ on C with a canonical admissible metric $\| \; \|$ from Corollary 4.3.32. There is a semistable K°-model $\mathcal{C}$ of C such that

$\| \ \|$ is $\mathcal{C}$-admissible and such that $\tilde{\mathcal{C}}$ has smooth irreducible components (4.3.24). By Theorem 4.3.25, there is

$$\hat{c}_1(t_1 - t_0)_{\mathcal{C}} \in CH^1_{\{t_0,t_1\}\cup\tilde{\mathcal{C}}}(\mathcal{C}, R)$$

such that $\hat{c}_1(t_1 - t_0)_{\mathcal{C}}$ agrees with $\widehat{\operatorname{div}}(s_{t_1-t_0})$ on $\mathcal{C}$ for the canonical meromorphic section $s_{t_1-t_0}$ of $O(t_1 - t_0)$. We have

$$\int_{\mathcal{C}} \hat{c}_1(t_1 - t_0)_{\mathcal{C}} \cap \beta = 0 \tag{4.19}$$

for all $\beta \in CH_0^{fin}(\mathcal{C}, R)$. We will use the decomposition

$$\hat{c}_1(t_1 - t_0)_{\mathcal{C}} = \hat{c}_1^{hor}(t_1 - t_0)_{\mathcal{C}} + \hat{c}_1^{ver}(t_1 - t_0)_{\mathcal{C}}$$

into horizontal and vertical part obtained by applying Remark 4.3.28 to the $\mathcal{C}$-component. We choose a K°-model $\mathcal{M}$ of M (Proposition 3.4.2). We may assume that $\mathcal{M}$ is a line bundle on the K°-model $\mathfrak{Y}$ of $X \times C$ and that the projections p_i of $X \times C$ to the factors extend to morphisms $\bar{p}_1 : \mathfrak{Y} \to \mathfrak{X}$ and $\bar{p}_2 : \mathfrak{Y} \to \mathcal{C}$ of K°-models. The formal metric $\| \ \|_{\mathcal{M}}$ induces formal metrics $\| \ \|_t$ on M_t for any $t \in C$. Replacing $\mathcal{M}$ by $\mathcal{M} \otimes p_1^*\mathcal{M}_{t_0}^{-1}$, we may assume that $\mathcal{M}_{t_0} = \mathcal{O}_{\mathfrak{X}}$ and hence $\| \ \|_{t_0}$ is the trivial metric on $M_{t_0} = O_X$. Replacing $\mathfrak{X}$ and $\mathfrak{Y}$ by suitable larger K°-models, we may assume that $\| \ \|_{t_1} = \| \ \|_{\mathcal{L}_1}$ for a K°-model $\mathcal{L}_1$ of $L = M_{t_1}$ on $\mathfrak{X}$ and that $\bar{p}_1$ is flat (Proposition 3.4.2).

We choose any invertible meromorphic section s_M of M. Its corresponding meromorphic section of $\mathcal{M}$ is denoted by $\bar{s}_M$. Note that we may assume that $\operatorname{div}(s_M)$ intersects any finite number of cycles properly in $X \times C$ ([Gu2], Lemma 3.6). so we may assume that the restriction of s_M to M_t is a well-defined invertible meromorphic section s_t for $t = 0, 1$. By the same argument as above, we may also suppose that s_{t_0} is the constant section 1 of $O_X = M_{t_0}$.

We define an admissible first Arakelov-Chern class $\hat{c}_1(L)$ for L by the formal metric $\| \ \|_{t_1}$ and by the following class $\hat{c}_1^R \in \hat{C}^1_{fin}(X, R)$: On horizontal cycles, the operation of $\hat{c}_1^R$ is induced by

$$\bar{p}_{1*}\left(c_1(\bar{s}_M) \cup \bar{p}_2^*\hat{c}_1^{ver}(t_1 - t_0)_{\mathcal{C}}\right) \tag{4.20}$$

and on vertical cycles, it is induced by

$$\bar{p}_{1*}\left(c_1(\bar{s}_M) \cup \bar{p}_2^*\hat{c}_1(t_1 - t_0)_{\mathcal{C}}\right) - c_1(\mathcal{L}_1). \tag{4.21}$$

This defines operations $\hat{c}_1^R$ on $\hat{Z}(X', R)$ for every proper morphism $\psi : X' \to X$ which do not depend on the choice of s_M. Note that $\hat{c}_1^R \cap_\psi \alpha'$ is always vertical for $\alpha' \in \hat{Z}(X', R)$. We have to show that $\hat{c}_1^R \in \hat{C}^1_{fin}(X, R)$. Let $c^R_{1,\mathfrak{X}} \cap_{\bar{\psi}}$ be the operation on $\tilde{Z}(\mathfrak{X}', R)$ given by (4.20) and (4.21) where $\bar{\psi} : \mathfrak{X}' \to \mathfrak{X}$ is a proper morphism of admissible formal schemes. We will prove that $c^R_{1,\mathfrak{X}}$ satisfies the axioms (C1)-(C4) on the cycle level. Hence the operations pass to rational equivalence and give a well-defined $c^R_{1,\mathfrak{X}} \in CH^1_{fin}(\mathfrak{X}, R)$ (compare with [Fu], Theorem 17.1). By definition, we conclude that $\hat{c}_1^R \in \widehat{CH}^1_{fin}(X, R)$ induced by $c^R_{1,\mathfrak{X}}$. Finally, we will prove that $\hat{c}_1^R \in \hat{C}^1_{fin}(X, R)$.

Axioms (C1) and (C2) may be checked on horizontal and vertical cycles separately, and axiom (C4) concerns only vertical cycles. So they follow from the corresponding axioms for (4.20) or (4.21). It remains to prove axiom (C3). Let $\bar{s}'$ be an invertible meromorphic section of a line bundle $\mathcal{L}'$ on $\mathfrak{X}'$ and let $\alpha' \in Z_k(\mathfrak{X}', R)$. We have to prove

$$c^R_{1,\mathfrak{X}} \cap_{\bar{\psi}} (\operatorname{div}(\bar{s}').\alpha') = \operatorname{div}(\bar{s}').\left(c^R_{1,\mathfrak{X}} \cap_{\bar{\psi}} \alpha'\right) \tag{4.22}$$

holds in $CH_{k-2}^{|\widetilde{\mathfrak{X}'}|\cap|\mathrm{div}(\bar{s}')|\cap|\alpha'|}(\mathfrak{X}', R)$. If α' is vertical, then this follows from (C3) for (4.21). So we may assume α' horizontal, prime and even that α' is the generic fibre X' of $\mathfrak{X}'$. Since $c_{1,\mathfrak{X}}^R$ does not depend on the choice of s_M, we may assume that $\mathrm{div}(s_M)$ intersects $\psi(Y') \times t$ properly in $X \times C$ for all horizontal components Y' of $\mathrm{div}(\bar{s}').\alpha'$ and $t = t_0, t_1$. By Remark 4.3.28, we get

$$\bar{p}_{1*}\left(c_1(\bar{s}_M) \cup \bar{p}_2^* \hat{c}_1^{hor}(t_1 - t_0)_{\mathcal{C}}\right) \cap_{\bar{\psi}} Y' = c_1(\bar{s}_{t_1}) \cap_{\bar{\psi}} Y' \in CH_{k-2}^{|Y'|\cap\bar{\psi}^{-1}|\mathrm{div}(\bar{s}_{t_1})|}(\mathfrak{X}', R)$$

where $\bar{s}_{t_1}$ is the extension of s_{t_1} to a meromorphic section of $\mathcal{L}_1$. We conclude that

$$c_{1,\mathfrak{X}}^R \cap_{\bar{\psi}} Y' = \left(\bar{p}_{1*}\left(c_1(\bar{s}_M) \cup \bar{p}_2^* \hat{c}_1(t_1 - t_0)_{\mathcal{C}}\right) - c_1(\bar{s}_{t_1})\right) \cap_{\bar{\psi}} Y' \tag{4.23}$$

holds in $CH_{k-2}^{fin}(\overline{Y}', R)$. Here, we have used again our assumptions on proper intersections to replace the support $|Y'|\cap\bar{\psi}^{-1}|\mathrm{div}(\bar{s}_{t_1})|$ by $|\tilde{\mathfrak{X}}'|$. By the same argument, we may also assume that (4.23) holds in $CH_{k-1}^{fin}(\mathfrak{X}', R)$ for X' instead of Y'. Note that (4.23) also holds in for vertical cycles instead of Y'. So we may replace $c_{1,\mathfrak{X}}^R$ in (4.22) by the class

$$\bar{p}_{1*}\left(c_1(\bar{s}_M) \cup \bar{p}_2^* \hat{c}_1(t_1 - t_0)_{\mathcal{C}}\right) - c_1(\bar{s}_{t_1}) \in CH_S^1(\mathfrak{X}, R) \tag{4.24}$$

where $S := p_1\left(|\mathrm{div}(s_M)| \cap (X \times \{t_0, t_1\})\right) \cup \tilde{\mathfrak{X}}$. We apply (C3) for this class to prove identity (4.22) in $CH_{k-2}^{\bar{\psi}^{-1}S\cap|\mathrm{div}(\bar{s}')|}(\mathfrak{X}', R)$. By our assumptions on proper intersections, we conclude that (4.22) holds in $CH_{k-2}^{|\mathrm{div}(\bar{s}'|\cap|\widetilde{\mathfrak{X}'}|}(\mathfrak{X}', R)$. This proves (C3) on the cycle level. As described above, this means that $c_{1,\mathfrak{X}}^R$ is well-defined in $CH_{fin}^1(\mathfrak{X}, R)$ inducing $\hat{c}_1^R \in \widehat{CH}_{fin}(X, R)$.

Next, we prove that $\hat{c}_1^R \in \hat{C}_{fin}(X, R)$. Let $\psi : X' \to X$ be a morphism and let $\hat{c}' \in \hat{B}^*(X', R)$. We have to prove that $\psi^* \hat{c}_1^R$ and $\hat{c}'$ commute. It is enough to check this on a cycle α' of a sufficiently large model $\mathfrak{X}'$ of X'. So we may assume that ψ extends to a morphism $\bar{\psi} : \mathfrak{X}' \to \mathfrak{X}$ and that $\hat{c}'$ is induced by $c'_{\mathfrak{X}'} \in CH_{fin}^*(\mathfrak{X}', R)$. We have to check the identity

$$c_{1,\mathfrak{X}}^R \cap_{\bar{\psi}} \left(c'_{\mathfrak{X}'} \cap \alpha'\right) = c'_{\mathfrak{X}'} \cap \left(c_{1,\mathfrak{X}}^R \cap_{\bar{\psi}} \alpha'\right) \in CH_*^{fin}(\mathfrak{X}', R). \tag{4.25}$$

It is clear that $\hat{c}'$ commutes with any first Arakelov-Chern class for a formally metrized line bundle (by axiom (C3)) and with any pull-back of a first Arakelov-Chern class of an admissibly metrized line bundle on a projective smooth curve (by dividing through a formal metric, we reduce to the case of the constant section 1 of the trivial line bundle and then it follows from Proposition 4.4.2). If α' is vertical, then working on the fibre square $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ and using projection formula as well as fibre square rule, we deduce (4.25) from (4.21). If α' is horizontal, then we may assume α' prime and even $\alpha' = X'$. Then as in the proof of axiom (C3) above, we can prove that $c_{1,\mathfrak{X}}^R$ in (4.25) may be replaced by (4.24). Then the same argument as in the vertical case proves (4.25).

The conclusion of the above consideration is that we have a first Arakelov-Chern class $\hat{c}_1(L)$ for L defined by the formal metric $\| \ \|_{t_1}$ and by $\hat{c}_1^R \in \hat{C}_{fin}^1(X, R)$. Let $\alpha \in \widehat{CH}_0^{fin}(X, R)$. By (4.21), we get

$$(\hat{c}_1(L) \cap \alpha)_{\mathfrak{X}} = \bar{p}_{1*}\left(\bar{p}_2^* \hat{c}_1(t_1 - t_0)_{\mathcal{C}} \cap c_1(\bar{s}_M) \cap \bar{p}_1^* \alpha_{\mathfrak{X}}\right)$$

and thus

$$\begin{aligned} \int_X \hat{c}_1(L) \cap \alpha &= \int_{\mathfrak{Y}} \bar{p}_2^* \hat{c}_1(t_1 - t_0)_{\mathcal{C}} \cap c_1(\bar{s}_M) \cap \bar{p}_1^* \alpha_{\mathfrak{X}} \\ &= \int_{\mathcal{C}} \hat{c}_1(t_1 - t_0)_{\mathcal{C}} \cap \bar{p}_{2*}\left(c_1(\bar{s}_M) \cap \bar{p}_1^* \alpha_{\mathfrak{X}}\right). \end{aligned}$$

Note that $\bar{p}_{2*}\left(c_1(\bar{s}_M) \cap \bar{p}_1^* \alpha_{\mathfrak{X}}\right)$ is vertical, hence (4.19) proves existence of $\hat{c}_1(L)$ with the required property.

Now we prove uniqueness. Suppose that admissible Arakelov-Chern classes $\hat{c}_1(L)$ and $\hat{c}_1'(L)$ for L satisfy the required property. Considering $\hat{c}_1'(L) - \hat{c}_1(L)$, it is enough to show that any admissible Arakelov-Chern class $\hat{c}_1(O_X)$ for O_X with

$$\int_X \hat{c}_1(O_X) \cap \beta = 0$$

for all $\beta \in \widehat{CH}_0^{fin}(X, R)$ has associated metric equal to a multiple of the trivial metric on O_X. Let us fix $x_0 \in X$ and let x be any point of X. Then $x - x_0$ is algebraically equivalent to 0. By Proposition 4.4.14, there is $\alpha^{fin} \in \widehat{CH}_0^{fin}(X, R)$ such that

$$0 = \int_X \hat{c}_1(O_X, 1) \cap (x - x_0 + \alpha^{fin}) = \log \|1(x_0)\|_{\hat{c}_1(O_X)} - \log \|1(x)\|_{\hat{c}_1(O_X)}.$$

Hence $\| \ \|_{\hat{c}_1(O_X)}$ is equal to $\|1(x_0)\|_{\hat{c}_1(O_X)}$ times the trivial metric on O_X. □

4.4.16 Remark. As in section 13, the above metrics are closely related to the *Néron-symbol*. It is given by the following result of Néron ([Ne2], Théorème 3). To every proper smooth scheme X over K and to every pair (D, Z) consisting of a divisor D on X algebraically equivalent to 0 and Z a zero-dimensional cycle on X of degree 0 with $|D| \cap |Z| = \emptyset$, there is a unique $\langle D, Z \rangle_{Nér} \in \mathbb{R}$ satisfying the following properties:

a) The Néron symbol is bilinear in D and Z.

b) If $D = \mathrm{div}(f)$ for a rational function f on X, then

$$\langle D, Z \rangle_{Nér} = -\log |f(Z)|$$

extending the right hand side linearly from components to Z.

c) If $\varphi : X' \to X$ is a morphism of smooth proper schemes over K, if Z' is a cycle of degree 0 and if D is a divisor on X such that $\varphi^* D$ is well-defined and $|Z'| \cap \varphi^{-1}|D| = \emptyset$, then

$$\langle \varphi^* D, Z' \rangle_{Nér} = \langle D, \varphi_* Z' \rangle_{Nér}.$$

d) For any $x_0 \in X \setminus |D|$, the map $X \setminus |D| \to \mathbb{R}$ given by

$$x \mapsto \langle D, x - x_0 \rangle_{Nér},$$

is continuous and locally bounded.

Here, continuity is with respect to the v-topology of X and may be omitted. Note that it has not the same importance as in analysis, since the image of a compact under a continuous map hasn't to be bounded. More important is locally bounded which is explained in Definition 5.2.14 for several absolute values.

4.4.17 Let X, X' be smooth proper schemes over K and let Γ be a divisor on $X \times X'$. For zero-dimensional cycles Y, Y' on X, X' with $(|Y| \times |Y'|) \cap |\Gamma| = \emptyset$, let

$$\Gamma(Y) := p_{2*}(\Gamma . p_1^* Y)$$

and

$${}^t\Gamma(Y') := p_{1*}(\Gamma . p_2^* Y')$$

where p_1, p_2 are the projections of $X \times X'$ onto the factors. Then the *reciprocity law* of Néron (cf. [La], Theorem 4.2) says

$$\langle \Gamma(Y), Y' \rangle_{Nér} = \langle {}^t\Gamma(Y'), Y \rangle_{Nér}.$$

For a projective smooth curve C, we may apply it to the diagonal on $C \times C$. We conclude that the Néron-symbol on C is symmetric, hence it agrees with the Néron pairing from Remark 4.3.33.

4.4.18 Corollary. *Let X be a smooth proper scheme over K and let $L \in \mathrm{Pic}^\circ(X)$ with an admissible first Arakelov-Chern class $\hat{c}_1(L)$ of type as in Theorem 4.4.15. For every invertible meromorphic section s of L and every 0-dimensional cycle Z of degree 0 with $|\mathrm{div}(s)| \cap |Z| = \emptyset$, we have*

$$-\log \|s(Z)\|_{\hat{c}_1(L)} = \langle \mathrm{div}(s), Z \rangle_{N\acute{e}r}.$$

We call $\| \ \|_{\hat{c}_1(L)}$ a canonical metric of L. In particular, it is bounded.

Proof: By Corollary 4.3.34, the result holds for curves. We use the notation of the proof of Theorem 4.4.15. We may choose s_M with $|\mathrm{div}(s_M)| \cap (|Z| \times \{t_0, t_1\}) = \emptyset$. As in (4.24), we get

$$\begin{aligned}
-\log \|s_{t_1}(Z)\|_{\hat{c}_1(L)} &= \int_X \hat{c}_1(L, s_{t_1}) \cap Z \\
&= \int_{\mathfrak{X}} \bar{p}_{1*}\left(c_1(\bar{s}_M) \cup \bar{p}_2^* \hat{c}_1(t_1 - t_0)_{\mathcal{C}}\right) \cap Z \\
&= \int_{\mathfrak{Y}} \bar{p}_2^* \hat{c}_1(t_1 - t_0)_{\mathcal{C}} \cap c_1(\bar{s}_M) \cap p_1^* Z \\
&= \int_{\mathcal{C}} \hat{c}_1(t_1 - t_0)_{\mathcal{C}} \cap \bar{p}_{1*}\left(c_1(\bar{s}_M) \cap p_1^* Z\right)
\end{aligned}$$

Using the case of curves and the correspondence $\Gamma = \mathrm{div}(s_M)$, we get

$$-\log \|s_{t_1}(Z)\|_{\hat{c}_1(L)} = \langle t_1 - t_0, \Gamma(Z) \rangle_{N\acute{e}r} = \langle \mathrm{div}(s_{t_1}), Z \rangle_{N\acute{e}r}.$$

The last step was by the reciprocity law and $s_{t_0} = 1$. By Remark 4.4.9, we have

$$\|s\|_{\hat{c}_1(L)} / \|s_{t_1}\|_{\hat{c}_1(L)} = |s/s_{t_1}|_v.$$

By property b) of the Néron symbol, we get the desired identity also for s. Together with property d) of the Néron symbol, we conclude that the metric is bounded. □

4.5 Local Heights over Non-Archimedean Fields

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $| \ |_v$. All spaces denoted by greek letters $X, Y, \dots$ are assumed to be proper schemes over K. The coefficients of vertical cycles on models are taken in a subfield R of $\mathbb{R}$ containing the value group $\log |K^\times|_v$.

In the last section, we have introduced admissible first Arakelov-Chern classes for line bundles on X. They generalize the intersection operation of $\widehat{\mathrm{div}}(s)$ for invertible meromorphic sections of formally metrized line bundles. In this section, we will introduce local heights of cycles with respect to admissible first Arakelov-Chern classes and invertible meromorphic sections of line bundles. They are similarly defined as in section 2.4 replacing the $*$-product by cap-product and they have the same properties. We will also show that the dependence of the local heights with respect to the admissible first Arakelov-Chern classes is given by the associated metrics.

4.5.1 Definition. Let $s_0, \dots, s_t$ be invertible meromorphic sections of line bundles $L_0, \dots, L_t$ with admissible first Arakelov-Chern classes $\hat{c}_1(L_0), \dots, \hat{c}_1(L_t)$ and let Z be a cycle on X of dimension t. Let $\hat{L}_j$ be the line bundle endowed with the associated metric $\| \ \|_{\hat{c}_1(L_j)}$. We say that the *local height* $\lambda(Z)$ of Z with respect to $(\hat{L}_0, s_0), \dots, (\hat{L}_t, s_t)$ is *well-defined* if

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset$$

and in this case, we set

$$\lambda(Z) := \lambda_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(Z) = \int_Z \hat{c}_1(L_0, s_0) \cup \cdots \cup \hat{c}_1(L_t, s_t).$$

By linearity, we extend the definition to not necessarily pure dimensional cycles using the convention $\lambda(Z_i) = 0$ for all components with $\dim(Z_i) \neq t$.

4.5.2 Proposition. *The dependence of the local height with respect to $\hat{c}_1(L_0), \dots, \hat{c}_1(L_t)$ is determined by the associated metrics. The local height is multilinear and symmetric in the variables $(\hat{L}_0, s_0), \dots, (\hat{L}_t, s_t)$, and linear in Z, under the hypothesis that all terms are well-defined.*

Proof: The multilinearity and symmetry follow from Remark 4.4.9 and Proposition 4.4.8. The first claim is a consequence of Proposition 4.4.13 and symmetry. □

4.5.3 A metric associated to an admissible first Arakelov-Chern class is called a *cohomological metric*. In the following, we denote the metrics of the cohomologically metrized line bundles L_j simply by $\| \ \|$. Every formal metric is a cohomological metric. On a projective smooth curve, every admissible metric is a cohomological metric. The next result shows how to reduce to $Z = X$. Its proof is obvious from the definitions.

4.5.4 Proposition. Let $s_0, \dots, s_t$ be invertible meromorphic sections of cohomologically metrized line bundles $\hat{L}_0, \dots, \hat{L}_t$ on X and let Z be a prime cycle of dimension t on X whose local height is well-defined. For $j = 0, \dots, t$, let $s_{j,Z} := s_j|_Z$ whenever this is possible. If $Z \subset |\mathrm{div}(s_j)|$, then $s_{j,Z}$ denotes a non-trivial meromorphic section of $\hat{L}_j|Z$. Then

$$\lambda_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(Z) = \lambda_{(\hat{L}_0|_Z,s_{0,Z}),\dots,(\hat{L}_t|_Z,s_{t,Z})}(Z).$$

4.5.5 Proposition. *Let $\varphi : X' \to X$ be a morphism and for $j = 1, \dots, t$, let s_j be an invertible meromorphic section of a cohomologically metrized line bundle $\hat{L}_j$ on X such that $\varphi^*(s_j)$ is well-defined. Let Z' be a cycle on X' such that*

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap \varphi(|Z'|) = \emptyset.$$

Then

$$\lambda_{\varphi^*(\hat{L}_0,s_0),\dots,\varphi^*(\hat{L}_t,s_t)}(Z') = \lambda_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(\varphi_* Z').$$

Proof: This follows immediately from projection formula for local Arakelov-Chow cohomology and cap product (Proposition 4.2.15a)) □

4.5.6 Proposition. Let X be smooth and let $s_0, \dots, s_t$ be invertible meromorphic sections of cohomologically metrized line bundles $\hat{L}_0, \dots, \hat{L}_t$ on X such that the local height $\lambda(Z)$ of the t-dimensional cycle Z is well-defined. We assume that $L_0 \in \mathrm{Pic}^\circ(X)$ is endowed with the canonical metric $\| \ \|$. Let Y be a cycle representing $\mathrm{div}(s_1) \dots \mathrm{div}(s_t).Z \in CH_0\left(|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z|\right)$. Then

$$\lambda(Z) = -\log \|s_0(Y)\|$$

where the right hand side is defined by linearity in the components of the zero-dimensional cycle Y.

Proof: First, we check that $\log \|s_0(Y)\|$ is well-defined. If $|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z|$ is empty, then Y is the zero cycle and hence $\log \|s_0(Y)\| = 0$. So we may assume that the intersection above is non-empty. Since $\lambda(Z)$ is well-defined, we conclude that $|\mathrm{div}(s_0)| \cap |Y| = \emptyset$, therefore $\log \|s_0(Y)\|$ makes sense. If Y' is another representative of $\mathrm{div}(s_1) \ldots \mathrm{div}(s_t).Z$, then there is a K_1-chain $\mathbf{f}$ on $|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z|$ with $\mathrm{div}(\mathbf{f}) = Y - Y'$. We have

$$\begin{aligned} \log \|s_0(Y')\| - \log \|s_0(Y)\| &= \int_{\mathrm{div}(\mathbf{f})} \hat{c}_1(L, s_0) \\ &= \int_X \hat{c}_1(L, s_0) \cap \widehat{\mathrm{div}}(\mathbf{f}). \end{aligned}$$

By refined intersection theory, we have

$$\hat{c}_1(L, s_0) \cap \widehat{\mathrm{div}}(\mathbf{f}) = 0 \in \widehat{CH}^{fin}_{-1}(X, R).$$

We conclude that $\log \|s_0(Y)\|$ is independent of the choice of the representative Y.

To prove the formula in the proposition, we may assume Z prime. By Proposition 4.5.4, we may compute the local height on Z, i.e.

$$\begin{aligned} \lambda(Z) &= \int_Z \hat{c}_1(L_0, s_{0,Z}) \cup \cdots \cup \hat{c}_1(L_t, s_{t,Z}) \\ &= \int_Z \hat{c}_1(L_0, s_{0,Z}) \cap Y \\ &= -\log \|s_{0,Z}(Y)\| \end{aligned}$$

If Z is not contained in $|\mathrm{div}(s_0)|$, then $s_{0,Z} = s_0$ proves the claim. But if $Z \subset |\mathrm{div}(s_0)|$, then $\lambda(Z)$ well-defined implies that $|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset$. Hence Y is the zero cycle and

$$\log \|s_{0,Z}(Y)\| = 0 = \log \|s_0(Y)\|$$

proves the claim. □

4.5.7 Corollary. *If X is smooth and if one L_i is in $\mathrm{Pic}^\circ(X)$ endowed with a canonical metric, then the local height $\lambda(Z)$ does not depend on the metrics of the other line bundles.*

4.5.8 Corollary. *Let f be an invertible meromorphic function on X, let $s_1, \ldots, s_t$ be invertible meromorphic sections of the cohomologically metrized line bundles $\hat{L}_1, \ldots, \hat{L}_t$ on X and let Z be a t-dimensional cycle on X. We assume that*

$$|\mathrm{div}(f)| \cap |\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$

If we endow the trivial bundle with the trivial metric, then

$$\lambda_{(\hat{O}_X, f), (\hat{L}_1, s_1), \ldots, (\hat{L}_t, s_t)}(Z) = -\log |f(Y)|$$

for any representative Y of $\mathrm{div}(s_1) \ldots \mathrm{div}(s_t).Z \in CH_0\left(|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z|\right)$.

4.5.9 Remark. There is also an analogue of the *induction formula.* We use the assumptions and notation of Proposition 4.5.4 and we may assume Z prime. There is a K°-model $\mathfrak{X}$ of X such that the formal metric and the class $\hat{c}_1^{R,j}$ in $\widehat{CH}^1_{fin}(X, R)$ corresponding to $\hat{c}_1(L_j)$ are defined on $\mathfrak{X}$ for all $j = 0, \ldots, t$ simultaneously. Let $\tilde{c}_1(L_j) \in CH^1_{fin}(\tilde{\mathfrak{X}}, R)$ inducing the action of $\hat{c}_1(L_j)$ on vertical cycles of $\mathfrak{X}$. By Proposition 4.4.13, we get

$$\lambda(Z) = \lambda(\mathrm{div}(s_{t,Z})) - \int_Z \sum_V \log \|s_{t,Z}(V)\| \cdot \tilde{c}_1(L_0) \cap \cdots \cap \tilde{c}_1(L_{t-1}) \cap V$$

where V ranges over the irreducible components of the special fibre of $(\bar{Z})^{f-an}$. If $|\mathrm{div}(s_0)| \cap Z = \emptyset$, then using symmetry, we get

$$\lambda(Z) = -\int_Z \sum_V \log \|s_0(V)\| \cdot \tilde{c}_1(L_1) \cap \cdots \cap \tilde{c}_1(L_t) \cap V.$$

This proves immediately

4.5.10 Proposition. *Let $s_0, \ldots, s_t$ be invertible meromorphic sections of cohomologically metrized line bundles $\hat{L}_0, \ldots, \hat{L}_t$ and let Z be a cycle such that the local height is well-defined. Let $\| \ \|'$ be a second cohomological metric on L_0 and let $\rho = \log(\|s_0\|/\|s_0\|')$. Then*

$$\begin{aligned}\lambda_{(\hat{L}_0',s_0),(\hat{L}_1,s_1),\ldots,(\hat{L}_t,s_t)}(Z) &- \lambda_{(\hat{L}_0,s_0),(\hat{L}_1,s_1),\ldots,(\hat{L}_t,s_t)}(Z) \\ &= \int_Z \sum_V \rho(V)\, \tilde{c}_1(L_1) \cap \cdots \cap \tilde{c}_1(L_t) \cap V.\end{aligned}$$

4.5.11 Definition. Let L be a line bundle on X. An admissible first Arakelov-chern class $\hat{c}_1(L)$ is called *semipositive* if it satisfies the following property: As in Remark 4.5.9, there is a K°-model $\mathfrak{X}$ of X and a $\tilde{c}_1(L)$ inducing the action of $\hat{c}_1(L)$ on vertical cycles. Then we assume that

$$\deg(\tilde{c}_1(L) \cap V) \geq 0$$

for all 1-dimensional prime cycles V on $\tilde{\mathfrak{X}}$. Note that the condition is independent of the choice of $\mathfrak{X}$ (use [Kl], Lemma I.4.1).

A metric on L is called *semipositive* if it is bounded and associated to a semipositive admissible first Arakelov-Chern class for L. For the definition of bounded metrics, we refer to section 5.2.

4.5.12 Example. Let L be a line bundle on X and let $\mathcal{L}$ be a K°-model of L generated by global sections. Then the formal metric $\| \ \|_{\mathcal{L}}$ is semipositive.

4.5.13 Remark. The concept of semipositivity is similar as numerically positive line bundles ([Kl]). A line bundle is called numerically positive if the degree of any prime cycle of dimension 1 with respect to the line bundle is non-negative. It is proved by Kleiman ([Kl], Corollary III.2.2) that any intersection number of numerically positive line bundles is non-negative.

4.5.14 Proposition.

a) The tensor product of semipositive metrics is a semipositive metric.

b) The pull-back of a semipositive metric with respect to a morphism of proper schemes is a semipositive metric.

c) If $\hat{c}_1(L_1), \ldots, \hat{c}_1(L_t)$ are semipositive admissible first Arakelov-Chern classes for line bundles $L_1, \ldots, L_t$ on X, then there is a K°-model $\mathfrak{X}$ of X such that for $j = 1, \ldots, t$, the operation of $\hat{c}_1(L_j)$ on vertical cycles is induced by $\tilde{c}_1(L_j) \in CH^1_{fin}(\tilde{\mathfrak{X}}, R)$. If V is an effective t-dimensional cycle on $\tilde{\mathfrak{X}}$, then

$$\deg(\tilde{c}_1(L_1) \cap \tilde{c}_1(L_2) \cap \cdots \cap \tilde{c}_1(L_t) \cap \mathrm{cyc}(V)) \geq 0.$$

d) If $\hat{c}_1(L_1), \ldots, \hat{c}_1(L_t)$ are semipositive admissible first Arakelov-Chern classes for line bundles $L_1, \ldots, L_t$ on X, then

$$\deg_{L_1,\ldots,L_t}(Z) \geq 0$$

for all t-dimensional effective cycles Z on X.

Proof: It is obvious that the sum and the pull-back of semipositive admissible first Arakelov-Chern classes are still semipositive. This proves a) and b).

The existence of the K°-model in c) is as in Remark 4.5.9. Then c) is just a statement about the Chow cohomology classes $\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_t)$ satisfying the positivity condition in Definition 4.5.11. So we may assume that $\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_t)$ are Chow cohomology classes on a proper integral scheme V over $\tilde{K}$ satisfying the positivity condition in Definition 4.5.11. By the alteration theorem of de Jong ([dJ], Theorem 4.1), there is a smooth scheme V' and a proper morphism $V' \to V$ which is the composition of a birational morphism with a finite surjective morphism. Replacing V by V', we may assume that V is smooth. By Proposition 4.1.10, the Chow cohomology classes $\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_t)$ are the first Chern classes of line bundles $\tilde{L}_1, \dots, \tilde{L}_t$ on V. The latter have to be numerically positive and we get c) by the result of Kleiman mentioned in Remark 4.5.13

To prove d), let $[v]$ be the special fibre of $\mathrm{Spf} K^\circ$, we may view it as an R-power of a formal Cartier divisor. If π denotes the morphism of structure of a K°-model $\mathfrak{X}$ as in c), then

$$\begin{aligned}\deg_{L_1,\dots,L_t}(X)[v] &= \pi_*\left(\pi^*[v] \cap \hat{c}_1(L_1) \cap \cdots \cap \hat{c}_1(L_t) \cap X\right) \\ &= \pi_*\left(\hat{c}_1(L_1) \cap \cdots \cap \hat{c}_1(L_t) \cap \pi^*[v] \cap X\right)\end{aligned}$$

Since the degree of the right hand side is non-negative, we get the claim. □

4.5.15 Corollary. *Let $s_0, \dots, s_t$ be invertible meromorphic sections of line bundles $L_0, \dots, L_t$. We endow $L_1, \dots, L_t$ with semipositive metrics. On L_0, we choose two arbitrary bounded cohomological metrics $\| \ \|$ and $\| \ \|'$. We denote the resulting line bundles by $\hat{L}_0, \hat{L}'_0, \hat{L}_1, \dots, \hat{L}_t$. Let Z be a t-dimensional effective cycle on X with non-negative multiplicities such that*

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$

Let us consider the constants

$$C_+ := \sup_{x \in |Z|} \log\left(\|s_0\|'/\|s_0\|\right)$$

and

$$C_- := \inf_{x \in |Z|} \log\left(\|s_0\|'/\|s_0\|\right).$$

Then the constants C_+, C_- are finite and we have

$$\begin{aligned}C_- \deg_{L_1,\dots,L_t}(Z) &\leq \lambda_{(\hat{L}_0,s_0),(\hat{L}_1,s_1),\dots,(\hat{L}_t,s_t)}(Z) - \lambda_{(\hat{L}'_0,s_0),(\hat{L}_1,s_1),\dots,(\hat{L}_t,s_t)}(Z) \\ &\leq C_+ \deg_{L_1,\dots,L_t}(Z).\end{aligned}$$

Proof: Since the metrics are bounded, we get finiteness of the constants. To prove the inequalities, we may assume Z prime, $Z = X$ (Proposition 4.5.4) and that the metrics on $L_0, \dots, L_t$ are associated to admissible first Arakelov-Chern classes whose corresponding formal metrics and classes in $\widehat{CH}^1_{fin}(X, R)$ are defined on the same K°-model $\mathfrak{X}$ of X. Let $[v]$ be the special fibre of $\mathrm{Spf} K^\circ$, we may view it as an R-power of a formal Cartier divisor as above. If π denotes the morphism of structure of $\mathfrak{X}$, then Proposition 4.5.10 gives

$$\begin{aligned}\lambda_{(\hat{L}_0,s_0),(\hat{L}_1,s_1),\dots,(\hat{L}_t,s_t)}(Z) &- \lambda_{(\hat{L}'_0,s_0),(\hat{L}_1,s_1),\dots,(\hat{L}_t,s_t)}(Z) \\ &\leq C_+ \int_{\mathfrak{X}} \tilde{c}_1(L_1) \cup \cdots \cup \tilde{c}_1(L_t) \cup \pi^*[v] \\ &= C_+ \int_{\mathfrak{X}} \pi^*[v] \cup \hat{c}_1(L_1, s_1)_{\mathfrak{X}} \cup \cdots \cup \hat{c}_1(L_t, s_t)_{\mathfrak{X}} \\ &= C_+ \int_{\mathrm{Spf} K^\circ} [v] \cap \pi_*\left(\hat{c}_1(L_1, s_1)_{\mathfrak{X}} \cap \cdots \cap \hat{c}_1(L_t, s_t)_{\mathfrak{X}} \cap X\right) \\ &= C_+ \deg_{L_1,\dots,L_t}(Z).\end{aligned}$$

Similarly, we get the lower bound. $\square$

4.5.16 Example. On a projective space $\mathbb{P}^n$ over K, we always fix a set of coordinates $\mathbf{x} = [x_0 : \cdots : x_n]$. Then the *standard metric* on $O_{\mathbb{P}^n}(1)$ is given by

$$\|s(x)\| = |s(x)| / \max_j |x_j|$$

for any global section s. It is the formal metric associated to $O_{\mathbb{P}^n_{K^\circ}}(1)$. We denote the corresponding metrized line bundle by $\bar{O}_{\mathbb{P}^n}(1)$. By pull-back and tensor product we get standard metrics on every line bundle L of a multiprojective space $\mathbb{P} := \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_t}$ and we denote the corresponding metrized line bundle by $\bar{L}$. We have seen in Remark 2.4.17 that the Chow form $F_Z(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t)$ of a t-dimensional cycle on $\mathbb{P}$ is a multihomogeneous polynomial in the the variables $\boldsymbol{\xi} = (\xi_{i0}, \ldots, \xi_{in_i})$ viewed as the dual coordinates on $\mathbb{P}^{n_i}$. We denote by $|F_Z|$ the Gauss norm of F_Z, i.e. the maximum of the absolute values of the coefficients. For $j = 0, \ldots, t$, let s_j be a global section of $O_{\mathbb{P}}(e_j)$ with dual coordinates $\mathbf{s_j}$. If

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset,$$

then

$$\lambda_{(\bar{O}_{\mathbb{P}}(e_0), s_0), \ldots, (\bar{O}_{\mathbb{P}}(e_t), s_t)}(Z) = \log |F_Z| - \log |F_Z(\mathbf{s_0}, \ldots, \mathbf{s_t})|.$$

This is proved in [Gu2], Proposition 1.12, for discrete valuations, but the proof generalizes to our situation.

Chapter 5

Canonical Local and Global Heights

5.1 Canonical Local Heights

Let K be an algebraically closed field with a complete absolute value $| \ |_v$.

First, we briefly summarize and extend the results of local heights of subvarieties obtained in the archimedean and non-archimedean case. We allow also uniform limits of semipositive metrics leading to a broader theory of local heights. It has the advantage to include also canonical local heights on abelian varieties. The latter were already defined in Propositions 2.4.8 and 4.5.6 if one line bundle is odd. If all line bundles are even, then we use Tate's limit argument to construct the canonical local height. We handle these canonical local heights in a more general dynamic situation. Note that we may assume that the absolute value is non-trivial, otherwise all local heights would be 0.

The assumption K complete and algebraically closed doesn't harm. For any field K with an absolute value, we just make base change to the field $\mathbb{K}_v$ which is the completion of the algebraic closure of the completion of K. By [BGR], 3.4.1, $\mathbb{K}_v$ is the smallest complete algebraically closed field containing K and extending the absolute value.

5.1.1 Let X be a proper scheme over K and let L be a line bundle on X. In the archimedean case, a metric on L was called semipositive if it is a smooth hermitian metric such that the pull-back metric to a smooth complex variety has always semipositive curvature (Definition 2.4.13). Note that we may always pass to X_{red} to apply the definitions and results of section 4.5. For non-archimedean v, a metric on L was called semipositive if it is a bounded cohomological metric associated to an admissible first Arakelov-Chern class acting numerically positive on vertical cycles (Definition 4.5.11). In this section, we always choose $R = \mathbb{R}$ for admissible first Arakelov-Chern classes.

Let $\mathfrak{g}_X^+$ be the set of isometry classes of line bundles with semipositive metrics. As an example, the reader may think of a line bundle generated by global sections endowed with the pull-back of the standard metric or the Fubini-Study metric from $O_{\mathbb{P}^n}(1)$ in the archimedean (resp. non-archimedean) case.

5.1.2 Definition. In various places, we already have used the following *distance* for two bounded metrics $\| \ \|, \| \ \|'$ on L. For $x \in X$, let s be a local non-vanishing section of L, let

$$(\| \ \|'/\| \ \|)(x) := \|s(x)\|'/\|s(x)\|$$

which does not depend on the choice of s. It is a bounded function on X and we set

$$d(\| \ \|, \| \ \|') := \max_{x \in X} |\log (\| \ \|'/\| \ \|)(x)| \, .$$

5.1.3 Definition. On the proper scheme X over K, let us denote by $\hat{\mathfrak{g}}_X^+$ the set of isometry classes of metrized line bundles $(L, \| \ \|)$ on X satisfying the following property: For all $n \in \mathbb{N}$, there is a proper surjective morphism $\varphi_n : X_n \to X$ and a metric $\| \|_n$ on $\varphi_n^* L$ with $(\varphi_n^* L, \| \|_n) \in \mathfrak{g}_{X_n}^+$ such that

$$\lim_{n \to \infty} d_{X_n}(\varphi_n^* \| \|, \| \|_n) = 0.$$

5.1.4 Definition. Let $\hat{\mathfrak{g}}_X$ be the set of isometry classes of metrized line bundles $\hat{L}$ on X such that there is a proper surjective morphism $\varphi : X' \to X$ and $\hat{M}, \hat{N} \in \hat{\mathfrak{g}}_{X'}^+$ with $\varphi^* \hat{L} \cong \hat{M} \otimes \hat{N}^{-1}$.

5.1.5 Proposition. *On a proper scheme X over K, the following properties hold:*

a) $\hat{\mathfrak{g}}_X^+$ (resp.$\hat{\mathfrak{g}}_X$) is a semigroup (resp. a group) with respect to $\otimes$.

b) Let $\varphi : X' \to X$ be a proper morphism. Then we have $\varphi^* \hat{L} \in \hat{\mathfrak{g}}_{X'}^+$ (resp.$\hat{\mathfrak{g}}_{X'}$) if $\hat{L} \in \hat{\mathfrak{g}}_X^+$ (resp.$\hat{\mathfrak{g}}_X$).

c) If φ is onto, then "if and only if" holds in b).

d) Let L be a line bundle on X with a sequence of bounded metrics $\| \|_n$ converging to $\| \|$. If all $(L, \| \|_n) \in \hat{\mathfrak{g}}_X^+$, then $(L, \| \|) \in \hat{\mathfrak{g}}_X^+$.

e) Every metric of a line bundle in $\hat{\mathfrak{g}}_X$ is bounded.

f) Let $\hat{L}$ be a metrized line bundle on X and $n \in \mathbb{N} \setminus \{0\}$. If $\hat{L}^{\otimes n} \in \hat{\mathfrak{g}}_X^+$, then $\hat{L} \in \hat{\mathfrak{g}}_X^+$.

g) If v is archimedean, then every line bundle with a smooth hermitian metric is in $\hat{\mathfrak{g}}_X$.

h) In the non-archimedean case, every line bundle with a formal metric is in $\hat{\mathfrak{g}}_X$.

i) Every line bundle L on X has a metric $\| \|$ such that $(L, \| \|) \in \hat{\mathfrak{g}}_X$.

Proof: Note that a),b) and f) hold for $\mathfrak{g}_X^+$ instead of $\hat{\mathfrak{g}}_X^+$. Thus properties a)-f) are immediate (cf. [Gu2], Proposition 1.18). For g), let $\| \|$ be a smooth hermitian metric on the line bundle L of X. By Chow's lemma, we may assume that X is projective. Let $\hat{H}$ be a very ample line bundle on X endowed with the pull-back of the Fubini-Study metric on $O_{\mathbb{P}^n}(1)$. Then $\hat{H}$ has positive curvature. We conclude that the tensor product of $(L, \| \|)$ with a large power of $\hat{H}$ is in $\hat{\mathfrak{g}}_X^+$. This proves g).

To prove h), let $\mathcal{L}$ be a line bundle on the formal K°-model $\mathfrak{X}$ of X with associated metric $\| \|_{\mathcal{L}}$. By Chow's lemma again, we may assume that X is projective. Using Proposition 5.1.6 below, $\mathfrak{X}$ is dominated by a projective K°-model of X. So we may assume that $\mathfrak{X}$ is projective. Let $\mathcal{H}$ be a very ample line bundle on $\mathfrak{X}$. We conclude that its reduction to $\mathfrak{X}$ is numerically positive. The same holds for the reduction of $\mathcal{H}^{\otimes n}$. We conclude that $\| \|_{\mathcal{L} \otimes \mathcal{H}^{\otimes n}}$ and $\| \|_{\mathcal{H}^{\otimes n}}$ are both in $\hat{\mathfrak{g}}_X^+$ proving that $\| \|_{\mathcal{L}}$ is in $\hat{\mathfrak{g}}_X$.

Note that i) is an immediate consequence of g), h) and Proposition 3.4.2 $\square$

The argument for proving the following result was given by Lütkebohmert.

5.1.6 Proposition. *Let X be a projective scheme over K with K°-model $\mathfrak{X}$. Then there is a projective K°-model $\mathfrak{X}_1$ of $\mathfrak{X}$ with $\mathfrak{X}_1 \geq \mathfrak{X}$, i.e. $\mathfrak{X}_1$ is the formal completion of a flat projective scheme over K° along the special fibre (see Example 3.3.6).*

Proof: We fix a closed embedding $X \subset \mathbb{P}^N$. There is a projective flat K°-model $\mathfrak{X}_0$ of X associated to a flat projective subscheme $\mathfrak{X}_0^{alg}$ of $\mathbb{P}^N_{K^\circ}$. Here, flatness is obtained as usual by passing to the closed subscheme given by the ideal of K°-torsion. By [BL3] §4, there is a K°-model $\mathfrak{X}_1$ of X with $\mathfrak{X}_1 \geq \mathfrak{X}_0, \mathfrak{X}_1 \geq \mathfrak{X}$. Moreover, $\mathfrak{X}_1$ is obtained as a formal blowing up of $\mathfrak{X}_0$ in an open coherent ideal $\mathcal{J}$. Using the GAGA-principle for projective schemes proved by Ullrich in the case of a valuation ring of height 1 ([Ullr], Theorem 6.8), $\mathcal{J}$ is algebraic and hence the blowing up of $\mathfrak{X}_0^{alg}$ in the corresponding algebraic ideal provides us with the underlying projective structure of $\mathfrak{X}_1$. For convenience of the reader, we give here a direct argument to prove that $\mathcal{J}$ is algebraic.

Since $\mathcal{J}$ is open, there is a $\pi \in K^\circ, |\pi|_v < 1$, with $\pi \in \mathcal{J}(\mathfrak{X}_0)$. Let $R_\pi := K^\circ/\pi K^\circ$. The coherent ideal sheaf $\mathcal{J} \otimes R_\pi$ is associated to a homogeneous ideal $\tilde{I}^{(\pi)}$ on the projective scheme $\mathfrak{X}_0 \otimes R_\pi$ over R_π (using quasicompactness, we may choose $\tilde{I}^{(\pi)}$ finitely generated ([EGA II], Corollaire 3.4.5).. Let I be the lift of $\tilde{I}^{(\pi)}$ to $K[x_0, \ldots, x_N]/I(\mathfrak{X}_0^{alg})$ and let $\mathcal{J}_{alg}$ be the coherent ideal sheaf on $\mathfrak{X}_0^{alg}$ associated to I. Using $\pi \in \mathcal{J}$ and $\mathcal{J}_{alg} \otimes R_\pi \cong \mathcal{J} \otimes R_\pi$, we easily deduce that

$$\mathcal{J} \cong \mathcal{J}_{alg} \otimes_{\mathcal{O}_{\mathfrak{X}_0^{alg}}} \mathcal{O}_{\mathfrak{X}_0}$$

proving the claim.

To finish the argument, note that the blowing up $\mathfrak{X}_1^{alg}$ of $\mathfrak{X}_0^{alg}$ in $\mathcal{J}_{alg}$ is projective, since $\mathfrak{X}_0^{alg}$ is quasicompact and $\mathcal{J}_{alg}$ is of finite type (use [EGA II], Proposition 8.1.7, Proposition 3.4.1, Théorème 5.5.3, Corollaire 5.3.3). It follows from the local description of admissible formal blowing ups ([BL3], Lemma 2.2) that $\mathfrak{X}_1$ is the formal completion of $\mathfrak{X}_1^{alg}$ along the special fibre. □

5.1.7 Remark. Note that we may always assume that the morphisms in Definitions 5.1.3 and 5.1.4 are from projective varieties (Chow lemma) with disjoint irreducible components and that the morphism is generically finite. The latter follows by intersection with generic hyperplanes until the right dimension is obtained. Thus the definitions of $\hat{\mathfrak{g}}_X^+$ and $\hat{\mathfrak{g}}_X$ agree with those in [Gu2], §1.

5.1.8 Theorem. *For a proper scheme X over K with invertible meromorphic sections $s_0, \ldots, s_t$ of $\hat{L}_0, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X$, there is a unique local height $\lambda(Z) = \lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z)$, well-defined on t-dimensional cycles Z of X with*

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset,$$

satisfying the following properties:

a) *$\lambda(Z)$ is multilinear and symmetric in the variables $(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)$, and linear in Z under the assumption that all terms are well-defined.*

b) *Let $\varphi : X' \to X$ be a morphism such that $\varphi^*\mathrm{div}(s_0), \ldots, \varphi^*\mathrm{div}(s_t)$ are well-defined Cartier-divisors on X' and let Z' be a t-dimensional cycle on X' such that*

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap \varphi(|Z'|) = \emptyset.$$

Then

$$\lambda_{\varphi^*(\hat{L}_0,s_0),\ldots,\varphi^*(\hat{L}_t,s_t)}(Z') = \lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(\varphi_* Z').$$

c) *If we replace s_0 by another invertible meromorphic section s_0' such that the corresponding local height $\lambda'(Z)$ is also well-defined, then*

$$\lambda(Z) - \lambda'(Z) = \log \left| \frac{s_0'}{s_0}(Y) \right|_v$$

for any representative Y of $\mathrm{div}(s_1) \ldots \mathrm{div}(s_t).Z \in CH_0\left(|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z|\right)$.

d) *Let $\hat{L}_1, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X^+$ and let $\| \ \|, \| \ \|'$ be two metrics on L_0 with $(L_0, \| \ \|), (L_0, \| \ \|') \in \hat{\mathfrak{g}}_X$ and corresponding local heights $\lambda(Z)$ and $\lambda'(Z)$. Then we have*

$$|\lambda(Z) - \lambda'(Z)| \leq d(\| \ \|, \| \ \|') \deg_{L_1,\ldots,L_t}(Z).$$

More precisely, if Z is an effective cycle, then we have

$$\begin{aligned}\min_{x\in|Z|(K)} \log\left(\| \ \|'/\| \ \|\right)(x)\deg_{L_1,\ldots,L_t}(Z) &\leq \lambda(Z)-\lambda'(Z)\\ &\leq \max_{x\in|Z|(K)} \log\left(\| \ \|'/\| \ \|\right)(x)\deg_{L_1,\ldots,L_t}(Z).\end{aligned}$$

e) *If $\hat{L}_0,\ldots,\hat{L}_t \in \mathfrak{g}_X^+$, then $\lambda(Z)$ agrees with the local heights defined in 2.4.3 and 4.5.1. In particular, for a t-dimensional cycle Z on the multiprojective space $\mathbb{P}=\mathbb{P}^{n_0}\times\cdots\times\mathbb{P}^{n_t}$, the local height of Z with respect to $\bar{O}_{\mathbb{P}}(e_0),\ldots,\bar{O}_{\mathbb{P}}(e_t)$ endowed with the standard (resp. Fubini-Study) metrics in the non-archimedean (resp. archimedean) case is given in terms of the Chow form F_Z by*

$$\lambda(Z)=\begin{cases}\int_S \log|F_Z(\xi)|_v\,dP(\xi)-\log|F_Z(\mathbf{s_0},\ldots,\mathbf{s_t})|_v+\frac{\log|e|_v}{2}\sum_{i=0}^t\delta_i(Z)\sum_{j=1}^{n_i}\frac{1}{j} & \text{if } v|\infty,\\ \log|F_Z|_v-\log|F_Z(\mathbf{s_0},\ldots,\mathbf{s_t})|_v & \text{else,}\end{cases}$$

where S is the product of the unit spheres in $\mathbb{C}^{N_i+1}$ $(i=0,\ldots,t)$ endowed with the Lebesque probability measure and δ_i is the degree with respect to $(O_{\mathbb{P}}(e_j))_{j\in\{0,\ldots,t\}\setminus\{i\}}$.

Proof: We have seen that our local heights on $\mathfrak{g}_X^+$ from Definitions 2.4.3 and 4.5.1 satisfy a)-e). For $j=0,\ldots,t$, let $\| \ \|_j, \| \ \|'_j$ be metrics on a line bundle L_j such that $(L_j,\| \ \|_j),(L_j,\| \ \|'_j)\in\mathfrak{g}_X^+$. Then

$$\begin{aligned}|\lambda_{(L_0,\| \ \|_0,s_0),\ldots,(L_t,\| \ \|_t,s_t)}(Z) &- \lambda_{(L_0,\| \ \|'_0,s_0),\ldots,(L_t,\| \ \|'_t,s_t)}(Z)|_v\\ &\leq \sum_{j=0}^t d(\| \ \|_j,\| \ \|'_j)\deg_{L_0,\ldots,L_{j-1},L_{j+1},\ldots,L_t}(Z)\end{aligned}\tag{5.1}$$

is an algebraic consequence of a) and d). Now we extend the definition of local heights to line bundles $\hat{L}_0=(L_0,\| \ \|_0),\ldots,\hat{L}_t=(L_t,\| \ \|_t)\in\hat{\mathfrak{g}}_X^+$. Clearly, we may assume that X is integral. Let Z be a t-dimensional prime cycle on X. There is a sequence $\varphi_n : Z_n\to Z$ of proper surjective morphisms and metrics $\| \ \|_{j,n}$ on $\varphi_n^*(L_j)$ $(j=0,\ldots,t)$ with $(\varphi_n^*(L_j),\| \ \|_{j,n})\in\mathfrak{g}_{Z_n}^+$ and

$$\lim_{n\to\infty} d(\varphi_n^*\| \ \|_j,\| \ \|_{j,n})=0.$$

By Remark 5.1.7, we may assume that Z_n is an integral t-dimensional scheme. Then we define the local height by

$$\lambda(Z):=\lim_{n\to\infty}\frac{1}{[Z_n:Z]}\lambda_{(\varphi_n^*L_0,\| \ \|_{0,n},s_0\circ\varphi_n),\ldots,(\varphi_n^*L_t,\| \ \|_{t,n},s_t\circ\varphi_n)}(Z_n)\tag{5.2}$$

where $[Z_n:Z]$ is the degree of φ_n. In fact, the pull-back $s_j\circ\varphi_n$ is only well-defined if $|Z|\subset|\mathrm{div}(s_j)|$. Otherwise, we replace s_j by any invertible meromorphic section of L_j whose restriction to Z makes sense (see [Gu2], Lemma 3.6) . Note that the limit exists (resp. is independent of our choices). To see it, note that the right hand side of (5.2) is a Cauchy sequence using (5.1) and projection formula on an integral proper t-dimensional scheme covering both Z_n and Z_m (resp. Z'_n). Applying c) for $\mathfrak{g}_{Z_n}^+$, we see that (5.2) doesn't depend on the choice of $s_{j,Z}$. By linearity, we extend the definition to all t-dimensional cycles. From a)-d) for $\mathfrak{g}_X^+$, it is easy to deduce the corresponding properties of $\hat{\mathfrak{g}}_X^+$.

If $\hat{L}_0,\ldots,\hat{L}_t\in\hat{\mathfrak{g}}_X$ and Z is a t-dimensional prime cycle on X, then there is a proper surjective morphism $\varphi: Z'\to Z$ and $\hat{M}_0,\ldots,\hat{M}_t,\hat{N}_0,\ldots,\hat{N}_t\in\hat{\mathfrak{g}}_{Z'}^+$ with

$$\varphi^*\hat{L}_j\cong\hat{M}_j\otimes\hat{N}_j^{-1}.$$

By Remark 5.1.7 again, we may assume Z' integral and t-dimensional. Using mulitlinearity and projection formula b), we define the local height of Z with respect to $(\hat{L}_0, s_0), \dots, (\hat{L}_t, s_t)$. If pull-back of sections is not well-defined, then we use the same procedure as above. By linearity, we extend the definition to all cycles. We deduce a)-d) for $\hat{\mathfrak{g}}_X$ from the corresponding properties of $\hat{\mathfrak{g}}_X^+$. Uniqueness is clear from the construction. □

5.1.9 Corollary. *Let $s_0, \dots, s_t$ be invertible meromorphic sections of $\hat{L}_0, \dots, \hat{L}_t \in \hat{\mathfrak{g}}_X$ and let Z be a prime cycle of dimension t on X whose local height is well-defined. For $j = 0, \dots, t$, let $s_{j,Z} := s_j|_Z$ whenever this is possible. If $Z \subset |\mathrm{div}(s_j)|$, then $s_{j,Z}$ denotes a non-trivial meromorphic section of $\hat{L}_j|Z$. Then*

$$\lambda_{(\hat{L}_0, s_0), \dots, (\hat{L}_t, s_t)}(Z) = \lambda_{(\hat{L}_0|_Z, s_{0,Z}), \dots, (\hat{L}_t|_Z, s_{t,Z})}(Z).$$

Proof: This follows immediately from Theorem 5.1.8b) and c). Note that it was already used in the course of proof of Theorem 5.1.8. □

5.1.10 Proposition. *Let $s_0, \dots, s_t$ be global sections of $\hat{L}_0, \dots, \hat{L}_t \in \hat{\mathfrak{g}}_X^+$ and let Z be an effective cycle of dimension t on X with*

$$|\mathrm{div}(s_0)| \cap \dots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset$$

and

$$\sup_{x \in |Z|(K)} \|s_j(x)\| \leq 1 \qquad (j = 0, \dots, t).$$

Then the local height of Z with respect to $(\hat{L}_0, s_0) \dots, (\hat{L}_t, s_t)$ is non-negative.

Proof: We proceed by induction on t. The 0-dimensional case is trivial, the local height of a point $x \in X(K)$ is given by $-\log \|s_0(x)\|$. Now we assume $t > 0$. We may assume Z prime. By symmetry, we may assume that Z is not contained in $|\mathrm{div}(s_t)|$. Then the induction step follows immediately from the induction formula (Propositions 2.4.6, Remark 4.5.9) and the assumption that our metrics are semipositive. □

5.1.11 Remark. The idea of using the induction formula to prove non-negativity of heights is from Faltings ([Fa3], Proposition 2.6).

We may apply the proposition to show that the local height of a t-dimensional effective cycle on $P := \mathbb{P}^{n_0} \times \mathbb{P}^{n_t}$ is non-negative with respect to $p_0^*(\bar{O}_{\mathbb{P}^{n_0}}(1), x_{j_0}^{(0)}), \dots, p_t^*(\bar{O}_{\mathbb{P}^{n_t}}(1), x_{j_t}^{(t)})$ if it is well-defined. Here, p_i is the projection onto $\mathbb{P}^{n_i}$ with coordinates $\mathbf{x}^{(i)}$ and we use the Fubini-Study metric (resp. the standard metric) in the archimedean (resp. non-archimedean) case.

5.1.12 We give a slight generalization of a result of Zhang ([Zh2], Theorem 2.2) constructing canonical metrics relative to dynamics. Let X be a proper scheme over K and let $\psi : X \to X$ be a morphism. We consider a line bundle L on X with an isomorphism

$$\theta : \psi^* L^{\otimes n} \xrightarrow{\sim} L^{\otimes m}$$

for some $n, m \in \mathbb{Z}, |m| > |n|$. By completeness, any other isomorphism has the form $\alpha\theta$ for some $\alpha \in K^\times$. For an introduction to bounded metrics, we refer to Definition 5.2.15 applied to one absolute value.

5.1.13 Theorem. *There is a unique bounded metric $\| \ \|_\theta$ on L satisfying*

$$\| \ \|_\theta^{\otimes m} \circ \theta = \psi^* \| \ \|_\theta^{\otimes n}.$$

If we replace θ by $\alpha\theta$, then we have

$$\| \ \|_{\alpha\theta} = |\alpha|_v^{\frac{1}{n-m}} \| \ \|_\theta.$$

Proof: The last claim is an obvious consequence of uniqueness. We consider the space of bounded metrics with the distance d. It is a complete metric space non canonically isometric to the space of bounded functions on X. To a metric $\| \ \|$ on L, we associate a metric

$$\Phi(\| \ \|) := \left((\psi^* \| \ \|^{\otimes n}) \circ \theta^{-1}\right)^{\frac{1}{m}}$$

on L. This induces a contractive endomorphism of the space of bounded metrics with contraction factor $\left|\frac{n}{m}\right|$. Since L has a formal or hermitian metric, there is at least one bounded metric $\| \ \|$ on L. By Banach's fixed point theorem, there is a unique bounded metric on L with $\Phi(\| \ \|) = \| \ \|$. Recall from its proof that

$$\| \ \|_\theta = \lim_{k \to \infty} \Phi^k(\| \ \|).$$

□

5.1.14 Example. Let us consider $\mathbb{P}^n$ with the endomorphism $\psi(\mathbf{x}) = \mathbf{x}^m$ for $m \in \mathbb{N}, |m| \geq 2$. For $L = O_{\mathbb{P}^n}(1)$, we have $\psi^* L = L^{\otimes m}$ and the standard metric

$$\|s(\mathbf{x})\| = \frac{|s(\mathbf{x})|_v}{\max_j |x_j|_v}$$

is the canonical metric.

5.1.15 Example. Let A be an abelian variety over K and let $\psi = [m]$ be multiplication with $m \in \mathbb{Z}, |m| \geq 2$. The theorem of the cube implies

$$[m]^* L \cong \begin{cases} L^{\otimes m^2} & \text{if } L \text{ is even,} \\ L^{\otimes m} & \text{if } L \text{ is odd.} \end{cases}$$

Any line bundle on A is isomorphic to the tensor product of an even and an odd line bundle and this factorization is unique up to 2-torsion in $\mathrm{Pic}(X)$. So we get canonical metrics on any line bundle, unique up to multiples in $|K^\times|_v$. In the same sense, they are not depending on the choice of m.

If v is archimedean, then there is a smooth hermitian metric $\| \ \|$ on L with harmonic Chern form $c_1(L, \| \ \|)$. This metric is unique up to multiples. Since the harmonic forms on A agree with the translation invariant forms, it is obvious that $[m]^*$ transforms harmonic forms to harmonic forms. We conclude that the canonical metrics above are smooth hermitian metrics with harmonic first Chern forms.

Now let v be non-archimedean and let L be odd. The latter is equivalent to $L \in \mathrm{Pic}^\circ(A)$ ([Mu]). By Corollary 4.4.18, we get a canonical metric on L unique up to multiples and we have seen that it may be given by the Néron symbol. From the basic properties of the Néron symbol (Remark 4.4.16), we easily deduce that these canonical metrics satisfy the characteristic property of Theorem 5.1.13, hence they agree with the canonical metrics above.

5.1.16 Remark. Let X be a smooth proper scheme over K and let $L \in \mathrm{Pic}^\circ(X)$. From the fundamental property of the Picard variety, there is an abelian variety A, $L' \in \mathrm{Pic}^\circ(A)$ and a morphism $\varphi : X \to A$ with $\varphi^* L' \cong L$. Then the pull-back of a canonical metric on L' is called a *canonical metric* on L. From property c) of the Néron symbol (Remark 4.4.16), we conclude that it is unique up to real multiples. The canonical metrics are compatible with pull-back, inverse and tensor product.

Note that the Néron symbol exists and has the same properties also in the archimedean case. Then the considerations in Example 5.1.15 show that the canonical metrics on L are smooth hermitian metrics characterized by the fact that the first Chern form is zero (since L is homologically trivial).

In the non-archimedean case, we recover the canonical metrics of Corollary 4.4.18. Note that in any case, the canonical metric is in $\mathfrak{g}_X^+$.

5.1.17 Proposition. *Let X be a smooth proper scheme over K and let $s_0, \ldots, s_t$ be invertible meromorphic sections of $\hat{L}_0, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X$. Let Z be a t-dimensional cycle on X such that the local height $\lambda(Z)$ with respect to $(\hat{L}_j, s_j)_{j=0,\ldots,t}$ is well-defined. Suppose that $L_0 \in \mathrm{Pic}^\circ(X)$ and that its metric is canonical. For a cycle Y representing the refined intersection $\mathrm{div}(s_1)\ldots\mathrm{div}(s_t).Z$, the identity*

$$\lambda(Z) = -\log \|s_0(Y)\|_{can}$$

holds, thus $\lambda(Z)$ does not depend on the metrics of $\hat{L}_1, \ldots, \hat{L}_t$.

Proof: This follows from Propositions 2.4.8 and 4.5.6. □

5.1.18 Corollary. *Let X be a smooth proper scheme over K and let $s_0, \ldots, s_t$ be invertible meromorphic sections of $\hat{L}_0, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X$. If $L_0 \in \mathrm{Pic}^\circ(X)$ and either $L_1 \in \mathrm{Pic}^\circ(X)$ or the t-dimensional cycle Z is algebraically equivalent to 0, then the local height is given by the Néron symbol:*

$$\lambda(Z) = \langle \mathrm{div}(s_0), Y\rangle_{N\acute{e}r},$$

where Y is a cycle representing the refined intersection product $\mathrm{div}(s_0)\ldots\mathrm{div}(s_t).Z$. In particular, $\lambda(Z)$ does not depend on the metrics of $\hat{L}_0, \ldots, \hat{L}_t$.

Proof: This follows from Example 5.1.15 , Remark 5.1.16 and Proposition 5.1.17 . □

5.1.19 Remark. Next, we are going to define canonical local heights in the dynamic situation. Let L be a line bundle as in 5.1.12 and assume that L has a metric $\| \ \|$ lying in $\hat{\mathfrak{g}}_X^+$. This holds in the following cases where there is even a metric in $\mathfrak{g}_X^+$:

a) L is generated by global sections (using pull-back of the standard or the Fubini-Study metric on $O_{\mathbb{P}^n}(1)$).

b) L is ample (using that a tensor power is very ample and then a), Proposition 5.1.5).

c) $L \in \mathrm{Pic}^\circ(X)$ (using Remark 5.1.16).

By Proposition 5.1.5, we conclude that the sequence $(\Phi^n\| \ \|)_{n\in\mathbb{N}}$ of metrics on L from the proof of Theorem 5.1.13 is in $\hat{\mathfrak{g}}_X^+$ and thus $\| \ \|_\theta$ is in $\hat{\mathfrak{g}}_X^+$.

5.1.20 Definition. Let $\psi : X \to X$ be a morphism of a proper scheme over K and we fix $m_j, n_j \in \mathbb{Z}, |m_j| > |n_j|$. For $j = 0, \ldots, t$, we consider a line bundle L_j on X with an invertible meromorphic section s_j and with isomorphisms

$$\theta_j : \psi^* L_j^{\otimes n_j} \xrightarrow{\sim} L_j^{\otimes m_j}. \tag{5.3}$$

We assume that every L_j has a metric in $\hat{\mathfrak{g}}_X^+$. By Remark 5.1.19, we conclude that the canonical metric $\| \ \|_{\theta_j}$ on L_j from Theorem 5.1.13 is also in $\hat{\mathfrak{g}}_X^+$. By Theorem 5.1.8, we get the local height of a t-dimensional cycle Z with respect to $(L_0, \| \ \|_{\theta_0}, s_0), \ldots, (L_t, \| \ \|_{\theta_t}, s_t)$, well-defined if

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$

We call it the *canonical local height* of Z with respect to $(L_0, \theta_0, s_0), \ldots, (L_t, \theta_t, s_t)$ denoted by $\hat{\lambda}(Z)$.

5.1.21 Proposition. *Under the hypothesis of Definition 5.1.20, the canonical local height has the following properties:*

a) It is multilinear and symmetric in the variables $(L_0, \theta_0, s_0), \dots, (L_t, \theta_t, s_t)$*, and linear in* Z *if all terms are well-defined.*

b) Let $\varphi : X' \to X$ *and* $\psi' : X' \to X'$ *be proper morphisms with a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\psi'"] \arrow[d, "\varphi"] & X' \arrow[d, "\varphi"] \\ X \arrow[r, "\psi"] & X \end{tikzcd} .$$

For $L_j' := \varphi^* L_j$*, the isomorphism* (5.3) *induces an isomorphism*

$$\theta_j' : \psi'^* L_j'^{\otimes n_j} \xrightarrow{\sim} L_j'^{\otimes m_j}.$$

If Z' *is a prime cycle on* X' *such that*

$$\varphi|Z'| \cap |\mathrm{div}(s_0)| \cap \dots \cap |\mathrm{div}(s_t)| = \emptyset,$$

then the canonical local height $\hat{\lambda}(Z')$ *with respect to* $(L_0', \theta_0', s_0'), \dots, (L_t', \theta_t', s_t')$ *is well-defined and we have*

$$\hat{\lambda}(Z') = \hat{\lambda}(\varphi_* Z').$$

Here, we use $s_j' := s_j \circ \varphi$ *if it is a well-defined invertible meromorphic section, otherwise we use any invertible meromorphic section of* $L_j'|_{Z'}$ *as in Corollary 5.1.9.*

c) If we replace s_0 *by another invertible meromorphic section* s_0' *of* L_0 *such that the corresponding canonical local height* $\hat{\lambda}'(Z)$ *is also well-defined, then*

$$\hat{\lambda}(Z) - \hat{\lambda}'(Z) = \log \left| \frac{s_0'}{s_0}(Y) \right|_v$$

for any representative Y *of* $\mathrm{div}(s_1) \dots \mathrm{div}(s_t).Z \in CH_0\left(|\mathrm{div}(s_1)| \cap \dots \cap |\mathrm{div}(s_t)| \cap |Z|\right)$.

d) Let $\| \ \|_0, \dots, , \| \ \|_t$ *be any metrics on* $L_0, \dots, L_t$ *lying in* $\hat{\mathfrak{g}}_X^+$ *and let* $\lambda(Z)$ *be the local height with respect to* $(L_0, \| \ \|_0, s_0), \dots, (L_t, \| \ \|_t, s_t)$*. If* Z *is an effective cycle, then*

$$|\hat{\lambda}(Z) - \lambda(Z)| \le \sum_{j=0}^{t} d(\| \ \|_{\theta_j}, \| \ \|_j) \deg_{L_0, \dots, L_{j-1}, L_{j+1}, \dots, L_t}(Z).$$

e) If Z *is a* t*-dimensional cycle on* X *such that*

$$\left(|\mathrm{div}(s_0)| \cup \psi^{-1}|\mathrm{div}(s_0)|\right) \cap \dots \cap \left(|\mathrm{div}(s_t)| \cup \psi^{-1}|\mathrm{div}(s_t)|\right) \cap |Z| = \emptyset,$$

then

$$m_0 \cdots m_t \hat{\lambda}(Z) - n_0 \cdots n_t \hat{\lambda}(\psi_* Z) = \sum_{j=0}^{t} \log \left| \frac{\theta_j \circ s_j^{\otimes n_j} \circ \psi}{s_j^{\otimes m}}(Y_j) \right|_v$$

where Y_j *is a representative of the refined intersection*

$$\mathrm{div}(s_0) \dots \mathrm{div}(s_{j-1}).\mathrm{div}(s_{j+1}) \dots \mathrm{div}(s_t).Z.$$

f) If we replace the isomorphisms θ_j *in* (5.3) *by other isomorphisms* θ_j' *inducing the canonical local height* $\hat{\lambda}'(Z)$*, then there are* $\alpha_j \in K^\times$ *with* $\theta_j' = \alpha_j \theta_j$ *and we have*

$$\hat{\lambda}(Z) - \hat{\lambda}'(Z) = \sum_{j=0}^{t} \frac{\log |\alpha_j|_v}{n_j - m_j} \deg_{L_0, \dots, L_{j-1}, L_{j+1}, \dots, L_t}(Z).$$

Proof: Properties a)-d) follow immediately from the corresponding properties in Theorem 5.1.8. By b) and c), we deduce e). Finally in f), note that

$$\| \ \|_{\theta'_j}/\| \ \|_{\theta_j} = |\alpha|_v^{\frac{1}{n-m}}$$

using Theorem 5.1.13. Since this is a canonical metric on O_X, we deduce d) from Proposition 5.1.17. □

5.1.22 Proposition. *Let A be an abelian variety over K with a line bundle L endowed with a canonical metric $\| \ \|_{can}$ as in Example 5.1.15. Then $(L, \| \ \|_{can}) \in \hat{\mathfrak{g}}_X$.*

Proof: The proof is based on the various properties in Proposition 5.1.5. We may assume that L is even or odd. For L odd, we know that $(L, \| \ \|_{can}) \in \mathfrak{g}_X^+$ (Remark 5.1.16). So we may assume that L is even. There is a very ample even line bundle H on A. By considering $L \otimes H^{\otimes n}$ for large powers n, we may assume that L is an even very ample line bundle on A. By Remark 5.1.19, we conclude that $(L, \| \ \|_{can}) \in \hat{\mathfrak{g}}_X^+$. □

5.1.23 Corollary. *For canonically metrized line bundles $\hat{L}_0, \ldots, \hat{L}_t$ on A with invertible meromorphic sections $s_0, \ldots, s_t$ and a t-dimensional cycle Z on A, we get a canonical local height*

$$\hat{\lambda}(Z) := \lambda_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z),$$

well-defined if

$$|\operatorname{div}(s_0)| \cap \cdots \cap |\operatorname{div}(s_t)| \cap |Z| = \emptyset.$$

It has the following properties:

a) $\hat{\lambda}(Z)$ is multilinear and symmetric in the variables $(\hat{L}_j, s_j)_{j=0,\ldots,t}$, and linear in Z under the assumption that all terms are well-defined.

b) If $\varphi : A' \to A$ is a homomorphism of abelian varieties over K and Z' is a cycle on A' such that

$$\varphi|Z'| \cap |\operatorname{div}(s_0)| \cap \cdots \cap |\operatorname{div}(s_t)| = \emptyset,$$

then the following identity of canonical local heights holds

$$\hat{\lambda}_{(\varphi^*\hat{L}_0,s'_0),\ldots,(\varphi^*\hat{L}_t,s'_t)}(Z') = \hat{\lambda}_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(\varphi_* Z').$$

Here, we use $s'_j := s_j \circ \varphi$ if it is a well-defined invertible meromorphic section, otherwise we use any invertible meromorphic section of $L'_j|_{Z'}$ as in Corollary 5.1.9.

c) If we replace s_0 by another invertible meromorphic section s'_0 of L_0 such that the corresponding canonical local height $\hat{\lambda}'(Z)$ is also well-defined, then

$$\hat{\lambda}(Z) - \hat{\lambda}'(Z) = \log \left| \frac{s'_0}{s_0}(Y) \right|_v$$

for any representative Y of $\operatorname{div}(s_1) \ldots \operatorname{div}(s_t).Z \in CH_0 (|\operatorname{div}(s_1)| \cap \cdots \cap |\operatorname{div}(s_t)| \cap |Z|)$.

d) If we replace the canonical metrics $\| \ \|_{j,can}$ on L_j by metrics $\| \ \|_j$ assumed to be in $\hat{\mathfrak{g}}_X^+$ for $j = 0, \ldots, t$ and if $\lambda(Z)$ denotes the corresponding local height of the effective cycle Z, then

$$|\hat{\lambda}(Z) - \lambda(Z)| \leq \sum_{j=0}^{t} d(\| \ \|_{j,can}, \| \ \|_j) \deg_{L_0,\ldots,L_{j-1},L_{j+1},\ldots,L_t}(Z).$$

e) If $m \in \mathbb{Z}, |m| \geq 2$, if

$$\big(|\mathrm{div}(s_0)| \cup [m]^{-1}|\mathrm{div}(s_0)|\big) \cap \cdots \cap \big(|\mathrm{div}(s_t)| \cup [m]^{-1}|\mathrm{div}(s_t)|\big) \cap |Z| = \emptyset,$$

if k line bundles of $L_0, \ldots, L_t$ are even and if the remaining line bundles are odd, then

$$m^{k+t+1}\hat{\lambda}(Z) - \hat{\lambda}([m]_* Z) = \sum_{j=0}^{t} \log \left| \frac{\theta_j \circ s_j \circ \psi}{s_j^{\otimes m_j}}(Y_j) \right|_v$$

where $\theta_j : [m]^* L_j \xrightarrow{\sim} L_j^{\otimes m^2}$ (resp. $L_j^{\otimes m}$) is an isometry with respect to $[m]^*\| \ \|_{j,can}$ and $\| \ \|_{j,can}^{\otimes m^2}$) (resp. $\| \ \|_{j,can}^{\otimes m}$) in the even (resp. odd) case and where Y_j is a representative of the refined intersection

$$\mathrm{div}(s_0) \ldots \mathrm{div}(s_{j-1}).\mathrm{div}(s_{j+1}) \ldots \mathrm{div}(s_t).Z.$$

f) If we replace the canonical metrics $\| \ \|_{j,can}$ on L_j by other canonical metrics $\| \ \|'_{j,can}$ for $j = 0, \ldots, t$, then there is $r_j \in \mathbb{R}$ with $\| \ \|'_{j,can} = r_j \| \ \|_{j,can}$ and we have

$$\hat{\lambda}(Z) - \hat{\lambda}'(Z) = \sum_{j=0}^{t} \log r_j \deg_{L_0,\ldots,L_{j-1},L_{j+1},\ldots,L_t}(Z).$$

Proof: Properties a)-d) follow from the corresponding properties in Theorem 5.1.8. For b), we have the additional fact that the left hand side is also a canonical local height since φ is a homomorphism. By b) and c), we deduce e). The existence of θ_j follows from the theorem of the cube and the definition of canonical metrics. The canonical metric is unique up to real multiples (Remark 5.1.16 and f) follows from Example 5.1.15. □

5.2 M-Fields

First, we introduce M-fields and we show that they include number fields, function fields and fields of meromorphic functions on $\mathbb{C}$ occuring in Nevanlinna theory. They are equipped with families of almost absolute values $(| \ |_v)_{v \in M}$ and with a positive measure on M. We do not require that the product formula is satisfied. In the next section, we use integration over all places to define global heights of subvarieties over M-fields.

Here, we recall the basic properties of schemes of finite type over an M-field. We introduce M-bounded subsets and functions. They have the same properties as known from the classical cases (cf. [La], chapter 11). Most important is that proper schemes are M-bounded. The additional difficulty here is that for $\alpha \in K$, the value $|\alpha|_v$ is only well-defined for almost all $v \in M$. So we have always to pass to finitely generated subfields of our M-field to get honest absolute values for almost all $v \in M$. This procedure is used frequently.

At the end, we introduce M-metrics on a line bundle L. They may be considered, up to the restriction mentioned above, as families $(\| \ \|_v)_{v \in M}$ of metrics on $L(\mathbb{K}_v)$ where $\mathbb{K}_v$ is the smallest complete algebraically closed field containing K with respect to v. We impose that the metrics are locally bounded. This has the advantage that the natural distance of two M-metrics on L is bounded by an integrable function on M. A fact used in the next section for estimates of global heights generalizing Weil's theorem.

With the exception of Moriwaki's example 5.2.5, all material is taken from [Gu3], §2. For technical details and proofs, the reader should consult this reference.

5.2.1 Definition. Let K be a field and let (M, μ) be a positive measure space. For every $\alpha \in K$, let

$$M \longrightarrow \mathbb{R}_+ \quad , \quad v \mapsto |\alpha|_v$$

be a μ almost everywhere (μ-ae) defined map. We assume the following axioms:

a) $|\alpha + \beta|_v \leq |\alpha|_v + |\beta|_v \quad \mu$-ae

b) $|\alpha\beta|_v = |\alpha|_v |\beta|_v \quad \mu$-ae

c) $\log |\gamma|_v \in L^1(M, \mu)$ and $|0|_v = 0 \quad \mu$-ae

for all $\alpha, \beta \in K$ and $\gamma \in K^\times$. Then K is called an *M-field.* For $\alpha \in K^\times$, we call $d_\alpha := \int_M \log |\alpha|_v \, d\mu(v)$ the *defect of product formula.* If $d_\alpha = 0$ for all $\alpha \in K^\times$, then K is said to satisfy the product formula.

5.2.2 Remark. Let K be a field with a set M of absolute values such that for all $\alpha \in K^\times$, there are only finitely many $v \in M$ with $|\alpha|_v \neq 1$. Let μ be a point measure on M, i.e. points are measurable and have finite positive measure. Obviously, K is an M-field. We assume that the product formula is satisfied. For simplicity, we assume that K is perfect. Let F be an algebraic extension of K (not necessarily finite dimensional) and let M_F be the set of absolute values on F extending those of M. We are going to construct a canonical M_F-structure on F satisfying the product formula. For details and also for the generalization to non-perfect ground fields, we refer to [Gu2], Remark 2.5. For a finite subextension L/K and $w \in M_L$, let

$$A_w := \{u \in M_F \mid u|w\}$$

where $u|w$ means that $| \ |_u$ extends $| \ |_w$.We denote the σ-algebra generated by A_w for all w and all L by $\mathcal{B}$. Then there is a unique positive measure μ on $(M_F, \mathcal{B})$ satisfying

$$\mu(A_w) = \frac{[L_w : K_v]}{[L : K]} \mu(\{v\})$$

where K_v, L_w are the completions and $v \in M_K$ is the restriction of w. Moreover, F is an M-field satisfying the product formula.

5.2.3 Example. Let $K = \mathbb{Q}$ and let M be the set of places of $\mathbb{Q}$. The corresponding absolute values are normalized in the standard way, i.e. we take the usual absolute value for the archimedean place and for a prime number p , we normalize by $|p|_p = \frac{1}{p}$. Together with the counting measure, $\mathbb{Q}$ is an M-field satisfying the product formula. Using Remark 5.2.2, any algebraic field extension $F/\mathbb{Q}$ has a canonical M_F-field structure satisfying the product formula.

5.2.4 Example. Let X be a projective variety over a field k and let deg be the degree with respect to an ample line bundle. We assume that X is regular in codimension 1. Let K be the function field of X and let M be the set of prime divisors on X. Again, we use the counting measure on M. We fix a constant $c > 1$. The order in $v \in M$ is a discrete valuation on K. The corresponding absolute value on K is given by

$$|f|_v := c^{-\operatorname{ord}(f,v) \deg(v)}$$

for $f \in K$. Then K is an M-field satisfying the product formula. The latter follows from the fact that the degree of a rational function on X is zero. By Remark 5.2.2, we get an M_F-field structure satisfying the product formula on every algebraic field extension F/K.

5.2.5 Example. Suppose that X is a projective variety of pure dimension t over a number field k. Then there is another M-field structure on the function field $K = k(X)$ due to Moriwaki

([Mori]). It is the arithmetic analogue of Example 5.2.4 using arithmetic intersection theory of Gillet-Soulé [GS2]. Let $\bar{X}$ be a projective flat model over the algebraic integers O_k. We may assume that $\bar{X}$ is regular in codimension 1, otherwise we pass to the normalization. Let $\hat{\mathcal{H}}$ be an ample line bundle on $\bar{X}$ endowed with a smooth hermitian metric with positive curvature. Let M_{fin} be the set of prime divisors on $\bar{X}$ and let $M_\infty := X(\mathbb{C})$ (defined as usual in Arakelov theory as the complex points of $X \otimes_{\mathbb{Q}} K$). We consider $M := M_{fin} \cup M_\infty$ endowed with the counting measure on M_{fin} and with the continuous positive measure $c_1(\hat{\mathcal{H}})^{\wedge t}$ on M_∞ where $c_1(\hat{\mathcal{H}})$ is the first Chern form of $\hat{\mathcal{H}}$. Let $\widehat{\deg}$ be the arithmetic degree with respect to $\hat{\mathcal{H}}$. For $v \in M_{fin}$, we consider the absolute value

$$|f|_v := e^{-\mathrm{ord}(f,v)\widehat{\deg}(v)}$$

on $K = k(X) = k(\bar{X})$. For $v \in M_\infty$ and $f \in K \setminus \{0\}$, let

$$|f|_v := |f(v)|$$

where $|\ |$ is the usual absolute value on $\mathbb{C}$. Note that this is only well-defined outside the poles of f which is a set of measure zero. We conclude that K is an M-field satisfying the product formula. To prove the latter, just note that

$$0 = \hat{c}_1(\hat{\mathcal{H}})^{\dim(X)}.\widehat{\mathrm{div}}(f) \in \widehat{CH}_{-1}(\bar{X})$$

and then divide this product up into the finite and the analytic part. More generally, we can use the arithmetic degree with respect to line bundles $\hat{\mathcal{H}}_0, \dots, \hat{\mathcal{H}}_t$ which are numerically positive on each finite fibre of $\bar{X}$ and which have semipositive curvature to get the same conclusions.

5.2.6 Example. The next example is coming from Nevanlinna theory and is due to Vojta. Let K be the field of meromorphic functions on $\mathbb{C}$ and let $R > 0$ with $M_R := \{v \in \mathbb{C} \mid |v| \leq R\}$. We define a positive measure μ in the following way. In the interior M_R°, we use the counting measure and on the boundary ∂M_R, μ is defined to be the Lebesque probability measure. For $v \in M_R^\circ$ and $f \in K$, we use the discrete absolute value

$$|f|_v := \begin{cases} e^{-\mathrm{ord}(f,v)\log(R/|v|)} & \text{if } v \neq 0 \\ e^{-\mathrm{ord}(f,v)\log R} & \text{if } v = 0. \end{cases}$$

For $v \in \partial M_R$, let

$$|f|_v := |f(v)|$$

which is only well-defined outside the poles of f. This gives an M_R-structure on K. It does not satisfy the product formula. Let c_f be the leading coefficient of the Laurent series in 0, then Jensen's formula shows that the defect of the product formula is

$$d_f = \log |c_f|.$$

It is also possible to define a canonical $M_{\bar{K}}$-field structure on the algebraic closure $\bar{K}$ of K using finite holomorphic coverings of Riemann surfaces (cf. [Gu2], Example 2.8 and 3.17).

5.2.7 Remark. For an M-field K, $|\ |_v$ has not to be an absolute value as it is seen in the example above. We can omit this problem by the following construction. We are always in a set up arising from algebraic geometry where finitely many proper schemes, line bundles and meromorphic sections are given over K. Hence there is a finitely generated subfield K' of K such that all these objects are defined over K'. Since K' is countable, there is $M' \subset M$ with $M \setminus M'$ a null-set such that the restriction of $|\ |_v$ to K' is an absolute value for all $v \in M$. Clearly, K' is an M'-field and we will work mostly with K'. We call

$$M_{fin} := \{v \in M \mid \log |2|_v \leq 0\}$$

the *non-archimedean part* of M, the complement M_∞ is called the *archimedean part*. Up to a null set, the restrictions to K correspond to the non-archimedean (resp. archimedean) absolute values of K'. In Example 5.2.6, the archimedean part is the boundary ∂M_R.

5.2.8 Let K be an M-field such that all $|\ |_v$ are absolute values. For $v \in M$, let $\mathbb{K}_v$ be the completion of the algebraic closure of the completion of K with respect to $|\ |_v$. The latter extends uniquely to an absolute value on $\mathbb{K}_v$ also denoted by $|\ |_v$. Then $\mathbb{K}_v$ is a complete algebraically closed field ([BGR] Proposition 3.4.1/3). Let X be a scheme over K. Then we set

$$X(M) := \coprod_{v \in M} X(\mathbb{K}_v) \times \{v\}.$$

5.2.9 Definition. Let X be an affine scheme of finite type over K. Then $E \subset X(M)$ is called *M-bounded in X* if and only if for any regular function a on X, there is an integrable function c on M with

$$|a(x)|_v \leq c(v)$$

for all $(x, v) \in E$.

5.2.10 Definition. Let X be a scheme of finite type over K. Then $E \subset X(M)$ is said to be *M-bounded in X* if there is an affine open covering $\{U_j\}_{j=1,\ldots,r}$ of X and a decomposition $E = \bigcup_{1=1}^{r} E_j$ with $E \subset U_j(M)$ such that E_j is M-bounded in U_j for all $j = 1, \ldots, r$ in the sense of Definition 5.2.9.

5.2.11 Remark. If K is an arbitrary M-field, then the above definitions can be generalized. The idea is to choose K' as in Remark 5.2.7 and then apply the definitions. To make sure that the definition does not depend on the choice of K', we have to assume that boundedness holds for all sufficiently large finitely generated subfields K' of K and for an integrable bound c not depending on K'. For details about this definition and for proofs of the following result, we refer to [Gu2], §2.

5.2.12 Proposition. *Let K be an M-field and let $\varphi : X' \to X$ be a morphism of finite type over K.*

a) The image of an M-bounded set in X' is M-bounded in X.

b) If φ is proper and E is an M-bounded set in X, then $\varphi^{-1}E$ is M-bounded in X'.

5.2.13 Corollary. *If X is a proper scheme over K, then $X(M)$ is M-bounded.*

5.2.14 Definition. Let X be a scheme of finite type over K and let f be a function on $X(M)$. Then f is called *locally M-bounded* if and only if for all M-bounded sets E in X, there is an integrable function c on M such that

$$\log |f(v, x)|_v \leq c(v) \tag{5.4}$$

on E. We say that f is *M-bounded* if (5.4) holds on $X(M)$.

5.2.15 Definition. Let L be a line bundle on the scheme X of finite type over K. An *M-metric* $\|\ \|$ on L is a real function on $L(M)$ such that the restriction to $L(\mathbb{K}_v) \times \{v\}$ is a metric $\|\ \|_v$ on the line bundle $L(\mathbb{K}_v)$ over $\mathbb{K}_v$ for almost every $v \in M$. We also assume that the metrics $\|\ \|_v$ are locally M-bounded, i.e. for every open subset U of X and every $s \in L(U)$, the function $\|s\|$ is locally M-bounded on U.

5.2.16 Remark. The definition is only precise if all $|\ |_v$ are absolute values. Otherwise, we have to replace $\mathbb{K}_v$ above by $\mathbb{K}'_v$ for sufficiently large subfields K' of K as in Remark 5.2.7

(cf. [Gu2], Definition 2.22) with c again not depending on K'. Furthermore, M-metrized line bundles are closed under tensor product, dual and pull-back.

5.2.17 Example. We consider the line bundle $O_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n_K$ for an M-field K. Then the standard metrics for $v \in M_{fin}$ and the Fubini-Study metrics for $v \in M_\infty$ induce an M-metric on $O_{\mathbb{P}^n}(1)$. Boundedness is an easy consequence of the definitions. By pull-back, we get M-metrics on every line bundle generated by global sections.

5.2.18 Let $\| \ \|, \| \ \|'$ be two M-metrics on a line bundle L. For almost every $v \in M$, we have the distance $d_v(\| \ \|, \| \ \|')$ of bounded metrics on $L(\mathbb{K}_v)$ introduced in 5.1.2. If X is a proper scheme, then M-boundedness (Corollary 5.2.13) implies that

$$d_v(\| \ \|, \| \ \|') \leq c(v)$$

for an integrable function c on M.

5.3 Global Heights

In this section, K denotes an M-field. All spaces considered are assumed to be proper schemes over K and are denoted by $X, X', \dots$.

In section 5.1, we have recalled and extended the theory of local heights and in section 5.2, we have introduced M-fields. Now we use both ingredients to define the global height of a t-dimensional cycle Z on X with respect to M-metrized line bundles $\hat{L}_0, \dots, \hat{L}_t$ and corresponding invertible meromorphic sections $s_0, \dots, s_t$. In these introductory remarks, we suppose for simplicity that all $| \ |_v, v \in M$, are absolute values. In general, we have to pass to sufficiently large finitely generated subfields, as already remarked in section 5.2. We have to assume that, for every $v \in M$, the local height $\lambda(Z, v)$ of Z with respect to $(\hat{L}_0^v, s_0), \dots, (\hat{L}_t^v, s_t)$ is well-defined in the sense of section 5.1. The boundedness condition on the metrics ensures that $\lambda(Z, v)$ is bounded by an integral function on M. But $\lambda(Z, v)$ has not to be measurable. If it is for all possible choices of $s_0, \dots, s_t$, we call Z integrable with respect to $(\hat{L}_0, s_0), \dots, (\hat{L}_t, s_t)$ and by integrating $\lambda(Z, v)$ over M, we get the global height $h(Z)$ with respect to $(\hat{L}_0, s_0), \dots, (\hat{L}_t, s_t)$.

The basic properties of local heights give immediately corresponding properties of global heights, i.e. we have multilinearity and symmetry, functoriality, the change of metrics estimates generalizing Weil's theorem and the global heights on multiprojective spaces given in terms of the Chow form.

Moreover, the change of section formula is now given by the defect of product formula, i.e. if the product formula is satisfied, then the global heights do not depend on the invertible meromorphic sections $s_0, \dots, s_t$.

If we consider only line bundles generated by global sections, then using pull-backs of standard or Fubini-Study metrics on $O_{\mathbb{P}^n}(1)$, every cycle is integrable. Hence we get a theory of global heights depending not on the metrics up to an error term bounded by a multiple of the degree. This is the generalization of Weil's height machine. All this can be generalized to line bundles with a positive power generated by global sections.

Then we define canonical global heights in the dynamic set up already considered in section 5.1. From the local case, we deduce canonical M-metrics and as above, we get easily the corresponding results for canonical global heights. If the product formula is satisfied, then they can be characterized by a homogenity property. As a special case, we get Néron-Tate heights for abelian varieties over K with respect to all line bundles. If the product formula is satisfied, then they do not depend on the meromorphic sections and the canonical metrics. Moreover, they are well-defined for all cycles and they are chracterized by an analogues homogenity property as in Tate's limit argument. On an arbitrary smooth X, we deduce canonical global heights if one line bundle is algebraically equivalent to 0. They may be described by the canonical height of a zero-dimensional cycle.

5.3.1 Definition. We denote by $\hat{\mathfrak{g}}_X$ the set of isometry classes of line bundles L on X endowed with an M-metric $\| \ \|$ such that the restriction $\| \ \|_v$ to $L(\mathbb{K}_v)$ is in $\hat{\mathfrak{g}}_{X \otimes \mathbb{K}_v}$ with respect to the place v for almost all $v \in M$ (using the notions introduced in 5.2.8 and Definition 5.1.4). Similarly, we define $\hat{\mathfrak{g}}_X^+$.

5.3.2 Remark. This is only precise if the $| \ |_v$ are absolute values for almost all $v \in M$. Otherwise, we have to replace K by sufficiently large finitely generated subfields (as in Remark 5.2.7, cf. also [Gu2], 3.1).

5.3.3 Let $s_0, \ldots, s_t$ be invertible meromorphic sections of M-metrized line bundles $\hat{L}_0, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X$. Let Z be a t-dimensional cycle on X. Then the local height is said to be *well-defined in* $v \in M$ if

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$

Then we get a *local height*

$$\lambda(Z, v) := \lambda_{(\hat{L}_0^v, s_0), \ldots, (\hat{L}_t^v, s_t)}(Z).$$

If v is not an absolute value, then we have to pass again to sufficiently large finitely generated subfields K' of K. For v non-archimedean, we additionaly assume that the local height does not depend on the choice of K'. It follows from 3.3.20 that this holds for formal metrics. If one metric is a canonical metric on a line bundle algebraically equivalent to 0, then the invariance of local heights under base change follows from the corresponding property of the Néron symbol and Proposition 5.1.17.

5.3.4 Definition. Let $D_1, \ldots, D_r$ be Cartier divisors on X and let Z be a prime cycle on X. We say that $D_1, \ldots, D_r$ *intersect* Z *properly* in X if for every $J \subset \{1, \ldots, r\}$, we have

$$\mathrm{codim}\left(\bigcap_{j \in J} |D_j| \cap Z, Z \right) \geq |J|.$$

By linearity, we extend the definition to all cycles Z.

5.3.5 Definition. A t-dimensional prime cycle Z on X is called *integrable* with respect to $\hat{L}_0, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X$ if there are invertible meromorphic sections $s_{0,Z}, \ldots, s_{t,Z}$ of $L_0|_Z, \ldots, L_t|_Z$ with $\mathrm{div}(s_{0,Z}), \ldots, \mathrm{div}(s_{t,Z})$ intersecting Z properly in Z such that the local height with respect to $(\hat{L}_0^v|_Z, s_{0,Z}), \ldots, (\hat{L}_t^v|_Z, s_{t,Z})$ is well-defined for almost all $v \in M$ and integrable on M. By linearity, we extend this definition to all cycles Z.

5.3.6 Remark. We have to restrict the line bundles above to the prime cycle Z to have enough meromorphic sections at hand. On X, a line bundle may not have any invertible meromorphic section.

5.3.7 Proposition. *We have the following properties:*

a) Let Z be a t-dimensional cycle on X integrable with respect to $\widehat{L_0}, \ldots, \widehat{L_t} \in \hat{\mathfrak{g}}_X$ and with respect to $\widehat{L_0'}, \widehat{L_1} \ldots, \widehat{L_t} \in \hat{\mathfrak{g}}_X$. Then Z is integrable with respect to $\widehat{L_0} \otimes \widehat{L_0'}, \widehat{L_1}, \ldots, \widehat{L_t}$. Moreover, integrability doesn't depend on the order of $\widehat{L_0}, \ldots, \widehat{L_t}$ and we may also replace any $\widehat{L_j}$ by $\widehat{L_j}^{-1}$.

b) Let $\varphi : X' \to X$ be a morphism and let Z' be a t-dimensional prime cycle on X'. Then $\varphi_* Z'$ is integrable with respect to $\hat{L}_0, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X$ if and only if Z' is integrable with respect to $\varphi^* \hat{L}_0, \ldots, \varphi^* \hat{L}_t$.

c) *Let Z be a t-dimensional cycle on X integrable with respect to $\hat{L}_0, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X$. Then for any invertible meromorphic sections $s_0, \ldots, s_t$ of $L_0, \ldots, L_t$ with*

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset,$$

the local height $\lambda_{(\hat{L}_0^v, s_0), \ldots, (\hat{L}_t^v, s_t)}(Z)$ is well-defined for almost all $v \in M$ and integrable on M.

d) *Let Z be a t-dimensional cycle on the multiprojective space $\mathbb{P} = \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_t}$. Then Z is integrable with respect to $\bar{O}_{\mathbb{P}}(e_0), \ldots, \bar{O}_{\mathbb{P}}(e_t)$ endowed with the standard (resp. Fubini-Study) metrics in the non-archimedean (resp. archimedean) case.*

Proof: Properties a)-d) follow from [Gu2], §3, with the exception that c) was only proved in the proper intersection case. To prove c) in general, we may assume $X = Z$ by the standard procedure of restricting sections to Z if possible and otherwise use any invertible meromophic section. By b) and de Jong's alteration theorem ([dJ], Theorem 4.1), we reduce to the case of a regular projective variety. By multilinearity a) and de Jong's alteration theorem again now applied to the divisors, we may assume that $\mathrm{div}(s_0), \ldots, \mathrm{div}(s_t)$ are all prime divisors and that a list of different representatives intersects properly. Using [Gu2], Lemma 3.6, we may replace repeated divisors by invertible meromorphic sections of L such that $\mathrm{div}(s_0), \ldots, \mathrm{div}(s_t)$ intersect properly in X. By assumption and property c) in the proper intersection case, the local height of $Z = X$ with respect to $(\hat{L}_0^v, s_0), \ldots, (\hat{L}_t^v, s_t)$ is well-defined for almost all $v \in M$ and integrable on M. Making the replacement step for step backwards and using the change of section formula (Theorem 5.1.8c)) together with axiom c) of an M-field, we conclude that our original local height $\lambda(Z, v)$ was well-defined for almost all $v \in M$ and integrable on M. □

5.3.8 Definition. Let $s_0, \ldots, s_t$ be invertible meromorphic sections of $\hat{L}_0, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X$. The *global height* of a t-dimensional cycle Z on X with respect to $(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)$ is *well-defined* if Z is integrable with respect to $\hat{L}_0, \ldots, \hat{L}_t$ and if

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$

Then it is denoted by

$$h(Z) := h_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(Z) := \int_M \lambda_{(\hat{L}_0^v, s_0), \ldots, (\hat{L}_t^v, s_t)}(Z)\, d\mu(v).$$

5.3.9 Theorem. *The basic properties of global heights are:*

a) *The global height $h(Z)$ is multilinear and symmetric in the variables $(\hat{L}_j, s_j)_{j=0,\ldots,t}$, and linear in Z under the assumption that all terms are well-defined.*

b) *Let $\varphi : X' \to X$ be a morphism such that $\varphi^*\mathrm{div}(s_0), \ldots, \varphi^*\mathrm{div}(s_t)$ are well-defined Cartier-divisors on X' and let Z' be a t-dimensional prime cycle on X' such that the global height of $\varphi_*(Z')$ with respect to $\hat{L}_0, \ldots, \hat{L}_t$ is well-defined. Then we have*

$$h_{\varphi^*(\hat{L}_0, s_0), \ldots, \varphi^*(\hat{L}_t, s_t)}(Z') = h_{(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)}(\varphi_* Z').$$

c) *Let Z be a t-dimensional cycle on X, integrable with respect to $\hat{L}_0, \ldots, \hat{L}_t \in \hat{\mathfrak{g}}_X$. For invertible meromorphic sections $(s_0, \ldots, s_t)$ and $(s_0', \ldots, s_t')$ of $(L_0, \ldots, L_t)$ satisfying*

$$(|\mathrm{div}(s_0)| \cup |\mathrm{div}(s_0')|) \cap \cdots \cap (|\mathrm{div}(s_t)| \cup |\mathrm{div}(s_t')|) \cap |Z| = \emptyset,$$

we have

$$h_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(Z) - h_{(\hat{L}_0,s_0'),\dots,(\hat{L}_t,s_t')}(Z) = \sum_{j=0}^{t} d_{f_j(Y_j)}$$

where d is the defect of product formula, $f_j := s_j'/s_j$ and Y_j is a representative of the refined intersection $\operatorname{div}(s_0') \dots \operatorname{div}(s_{j-1}').\operatorname{div}(s_{j+1}) \dots \operatorname{div}(s_t).Z$.

d) *Let $L_0, \dots, L_t$ be line bundles on X. On L_0, we consider M-metrics $\| \ \|, \| \ \|'$ in $\hat{\mathfrak{g}}_X$. Then there is an integrable function c on M with*

$$\log\left(\| \ \|_v'/\| \ \|_v\right) \leq c(v)$$

for almost all $v \in M$. Let $\hat{L}_1, \dots, \hat{L}_t \in \hat{\mathfrak{g}}_X^+$ and let Z be a t-dimensional effective cycle on X integrable with respect to $(L_0, \| \ \|), \hat{L}_1, \dots, \hat{L}_t$ and also integrable with respect to $(L_0, \| \ \|'), \hat{L}_1, \dots, \hat{L}_t$. If $s_0, \dots, s_t$ are invertible meromorphic sections of $L_0, \dots, L_t$ such that

$$|\operatorname{div}(s_0)| \cap \dots \cap |\operatorname{div}(s_t)| \cap |Z| = \emptyset,$$

then we have

$$h_{(L_0,\| \ \|,s_0),\dots,(\hat{L}_t,s_t)}(Z) - h_{(L_0,\| \ \|',s_0),\dots,(\hat{L}_t,s_t)}(Z) \leq C \deg_{L_1,\dots,L_t}(Z).$$

where $C := \int_M c \, d\mu$. Similarly, we get a lower bound (resp. a bound) for the difference of global heights using any integrable function c with

$$\log\left(\| \ \|_v'/\| \ \|_v\right) \geq c(v)$$

and

$$d(\| \ \|_v, \| \ \|_v') \leq c(v)$$

for almost all $v \in M$, respectively.

e) *Let Z be a t-dimensional cycle on the multiprojective space $\mathbb{P} = \mathbb{P}^{n_0} \times \dots \times \mathbb{P}^{n_t}$ and let $s_0, \dots, s_t$ be global sections of $O_{\mathbb{P}}(e_0), \dots, O_{\mathbb{P}}(e_t)$ such that*

$$|\operatorname{div}(s_0)| \cap \dots \cap |\operatorname{div}(s_t)| \cap |Z| = \emptyset.$$

Let $\bar{O}_{\mathbb{P}}(e_0), \dots, \bar{O}_{\mathbb{P}}(e_t)$ be endowed with the standard (resp. Fubini-Study) metrics in the non-archimedean (resp. archimedean) case. Then the corresponding global height $h(Z)$ is given in terms of the Chow form F_Z by

$$\begin{aligned} h(Z) = & - \int_M \log |F_Z(\boldsymbol{s}_0, \dots, \boldsymbol{s}_t)| + \int_{M_\infty} \int_S \log |F_Z(\boldsymbol{\xi})| \, dP(\boldsymbol{\xi}) \\ & + \int_{M_{fin}} \log |F_Z| \, d\mu(v) + \frac{1}{2} \sum_{i=0}^{t} \delta_i(Z) \sum_{j=1}^{n_i} \frac{1}{j} \int_{M_\infty} \log |e|_v \, d\mu(v) \end{aligned}$$

using the same notation as in Theorem 5.1.8

Proof: Using Proposition 5.3.7, the claims are obtained by integrating the corresponding formulas in Theorem 5.1.8. □

As in Corollary 5.1.9, we get the following immediate consequence of c) and d):

5.3.10 Corollary. *Let $s_0, \dots, s_t$ be invertible meromorphic sections of line bundles $L_0, \dots, L_t$ and let Z be a prime cycle of dimension t on X integrable with respect to $\hat{L}_0, \dots, \hat{L}_t \in \hat{\mathfrak{g}}_X$. We assume that*

$$|\operatorname{div}(s_0)| \cap \dots \cap |\operatorname{div}(s_t)| \cap |Z| = \emptyset.$$

For $j = 0, \dots, t$, let $s_{j,Z} := s_j|_Z$ whenever this is possible. If $Z \subset |\mathrm{div}(s_j)|$, then $s_{j,Z}$ denotes a non-trivial meromorphic section of $\hat{L}_j|Z$. Then

$$h_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(Z) = h_{(\hat{L}_0|_Z,s_{0,Z}),\dots,(\hat{L}_t|_Z,s_{t,Z})}(Z).$$

5.3.11 Remark. As noticed in 2.4.19, property e) of Theorem 5.3.9 determines the global height of any cycle on a multiprojective space in terms of the Chow form.

We have seen in Example 5.2.17 that any line bundle on X generated by global sections has an M-metric. It was obtained by pull-back of $\bar{O}_{\mathbb{P}^n}(1)$. More generally, let L be a line bundle on X such that there is $k \in \mathbb{N}, k \geq 1$, with $L^{\otimes k}$ generated by global sections. This includes also all ample line bundles. Then we endow L with the k-th root of a pull-back metric considered above. We denote by $\mathfrak{b}_X^+$ the set of such isometry classes. Using the Segre embedding, it is obvious that $\mathfrak{b}_X^+$ is a submonoid of $\hat{\mathfrak{g}}_X^+$. By Theorem 5.3.9b) and e), every t-dimensional cycle Z on X is integrable with respect to $\hat{L}_0, \dots, \hat{L}_t \in \mathfrak{b}_X^+$. If $Z = \sum_Y n_Y Y$ is the decomposition into components, then let

$$\delta_{L_0,\dots,L_t}(Z) := \sum_{j=0}^{t} \sum_{Y} |n_Y| \deg_{L_0,\dots,L_{j-1},L_{j+1},\dots,L_t}(Y).$$

As a generalization of Weil's theorem for heights of points, we can remove the dependence of the global heights on the metrics if we pass to classes modulo $\delta_{L_0,\dots,L_t}$:

5.3.12 Theorem. *For line bundles $L_0, \dots, L_t$ on X with a positive tensor power generated by global sections and with invertible meromorphic sections $s_0, \dots, s_t$, there is a real function $h_{(L_0,s_0),\dots,(L_t,s_t)}$, well-defined on all t-dimensional cycles Z of X with*

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset$$

and determined up to $O(\delta_{L_0,\dots,L_t})$ by

$$h_{(\hat{L}_0,s_0),\dots,(\hat{L}_t,s_t)}(Z) - h_{(L_0,s_0),\dots,(L_t,s_t)}(Z) = O(\delta_{L_0,\dots,L_t})(Z) \tag{5.5}$$

for every $\hat{L}_0, \dots, \hat{L}_t \in \mathfrak{b}_X^+$. It has the following properties:

a) It is multilinear and symmetric in the variables $(L_j, s_j)_{j=0,\dots,t}$, and linear in Z under the assumption that all terms are well-defined and always up to the corresponding error terms.

b) Let $\varphi : X' \to X$ be a morphism such that $\varphi^*\mathrm{div}(s_0), \dots, \varphi^*\mathrm{div}(s_t)$ are well-defined Cartier-divisors on X' and let Z' be a t-dimensional prime cycle on X' such that

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap \varphi(Z') = \emptyset,$$

then we have

$$h_{\varphi^*(L_0,s_0),\dots,\varphi^*(L_t,s_t)}(Z') - h_{(L_0,s_0),\dots,(L_t,s_t)}(\varphi_* Z') = O(\delta_{L_0,\dots,L_t})(\varphi_* Z').$$

c) If we replace s_0 by another invertible meromorphic section s_0' of L_0, then

$$h_{(L_0,s_0),\dots,(L_t,s_t)}(Z) - h_{(L_0,s_0'),\dots,(L_t,s_t)}(Z) = d_{f(Y)} + O(\delta_{L_0,\dots,L_t})(Z)$$

where d is the defect of product formula, $f := s_0'/s_0$ and Y is a representative of the refined intersection $\mathrm{div}(s_1) \dots \mathrm{div}(s_t).Z$.

d) For Z integrable with respect to $\hat{L}_0, \dots, \hat{L}_t \in \hat{\mathfrak{g}}_X^+$, estimate (5.5) still holds.

e) If $s_0, \dots, s_t$ are global sections, then we may assume $h_{(L_0,s_0),\dots,(L_t,s_t)}(Z) \geq 0$ for all effective cycles Z.

If the product formula is satisfied for the M-field K, then the global heights are independent of the invertible meromorphic sections, are defined for all t-dimensional cycles and may be assumed to be non-negative on effective cycles.

Proof: This follows immediately from Theorem 5.3.9, Remark 5.3.11 and Remark 5.1.11. □

5.3.13 Theorem. *Let $\psi : X \to X$ be a morphism and let L be a line bundle on X with an isomorphism*

$$\theta : \psi^* L^{\otimes n} \xrightarrow{\sim} L^{\otimes m}$$

for some $m, n \in \mathbb{N}, m > n$. We assume that L has an M-metric $\| \ \|$. Then there is a unique M-metric $\| \ \|_\theta$ on L satisfying

$$\| \ \|_\theta^{\otimes m} \circ \theta = \psi^* \| \ \|_\theta^{\otimes n}.$$

Moreover, if $\| \ \|$ is contained in $\hat{\mathfrak{g}}_X^+$, then the same holds for $\| \ \|_\theta$.

Proof: For almost all $v \in M$, we may assume that $| \ |_v$ is an absolute value (by passing to sufficiently large finitely generated subfields of K, cf. Remark 5.2.7). By Theorem 5.1.13, we get a canonical metric $\| \ \|_{\theta,v}$ on L satisfying the required identity locally. The proof of Theorem 5.1.13 shows

$$\| \ \|_{\theta,v} = \lim_{k \to \infty} \Phi^k(\| \ \|_v)$$

for almost all $v \in M$. Note that $\Phi^k(\| \ \|)$ is an M-metric, even in $\hat{\mathfrak{g}}_X^+$, if the same holds for $\| \ \|$. This is clear since the endomorphism Φ is contractive and from the fact that $d_v(\| \ \|, \Phi \| \ \|)$ is bounded by an integrable function (Remark 5.2.18). □

5.3.14 Remark. By the proof above, the sequence $\| \ \|_k := \Phi^k(\| \ \|)$ satisfies

$$\lim_{k \to \infty} d_v \left(\| \ \|_{k,v}, \| \ \|_{\theta,v} \right) = 0$$

for almost all $v \in M$ and the convergence is dominated by an integrable function on M. The last property follows again because Φ is contractive.

If a positive tensor power of L is generated by global sections, then L has even a metric $\| \ \|$ in $\mathfrak{b}_X^+$ (Remark 5.3.11) and thus the sequence $\| \ \|_k$ is also lying in $\mathfrak{b}_X^+$.

5.3.15 Let $\psi : X \to X$ be a morphism of a proper scheme over K and we fix $m_j, n_j \in \mathbb{N}, m_j > n_j$. For $j = 0, \dots, t$, we consider a line bundle L_j on X with an invertible meromorphic section s_j and with isomorphisms

$$\theta_j : \psi^* L_j^{\otimes n_j} \xrightarrow{\sim} L_j^{\otimes m_j}. \tag{5.6}$$

We assume that every L_j has a positive tensor power generated by global sections. By Theorem 5.3.13 and Remark 5.3.14, every L_j has a canonical M-metric $\| \ \|_{\theta_j}$ in $\hat{\mathfrak{g}}_X^+$ given as the limit of metrics $\| \ \|_{j,k}$ on L_j lying in $\mathfrak{b}_X^+$. From Theorem 5.1.8 and the dominated convergence theorem, we obtain:

5.3.16 Proposition. *Every t-dimensional cycle on X is integrable with respect to $(L_0, \| \ \|_{\theta_0}), \dots, (L_t, \| \ \|_{\theta_t})$.*

5.3.17 Definition. Let $s_0, \dots, s_t$ be invertible meromorphic sections of $L_0, \dots, L_t$. The global height of Z with respect to $(L_0, \| \ \|_{\theta_0}, s_0), \dots, (L_t, \| \ \|_{\theta_t}, s_t)$ is called the *canonical height* of Z with respect to $(L_0, \theta_0, s_0), \dots, (L_t, \theta_t, s_t)$ and is denoted by $\hat{h}(Z)$. It is well-defined if

$$|\mathrm{div}(s_0)| \cap \dots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$

5.3.18 Theorem. *Under the hypothesis above, the canonical height has the following properties:*

a) *It is multilinear and symmetric in the variables $(L_0, \theta_0, s_0), \ldots, (L_t, \theta_t, s_t)$, and linear in Z if all terms are well-defined.*

b) *Let $\varphi : X' \to X$ and $\psi' : X' \to X'$ be proper morphisms with a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\psi'"] \arrow[d, "\varphi"] & X' \arrow[d, "\varphi"] \\ X \arrow[r, "\psi"] & X \end{tikzcd} .$$

For $L'_j := \varphi^* L_j$, the isomorphism (5.6) induces an isomorphism

$$\theta'_j : \psi'^* L_j'^{\otimes n_j} \xrightarrow{\sim} L_j'^{\otimes m_j}.$$

If Z' is a prime cycle on X' such that

$$\varphi Z' \cap |\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| = \emptyset,$$

then the canonical height $\hat{h}(Z')$ with respect to $(L'_0, \theta'_0, s'_0), \ldots, (L'_t, \theta'_t, s'_t)$ is well-defined and we have

$$\hat{h}(Z') = \hat{h}(\varphi_* Z').$$

Here, we use $s'_j := s_j \circ \varphi$ if it is a well-defined invertible meromorphic section, otherwise we use any invertible meromorphic section of $L'_j|_{Z'}$ as in Corollary 5.1.9.

c) *If we replace s_0 by another invertible meromorphic section s'_0 of L_0 such that the corresponding canonical height $\hat{h}'(Z)$ is also well-defined, then*

$$\hat{h}(Z) - \hat{h}'(Z) = d_{f(Y)}$$

for any representative Y of $\mathrm{div}(s_1) \ldots \mathrm{div}(s_t).Z \in CH_0\left(|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z|\right)$ and $f := s'_0/s_0$.

d) *Let $\| \ \|_0, \ldots, , \| \ \|_t$ be any M-metrics on $L_0, \ldots, L_t$ lying in $\hat{\mathfrak{g}}_X^+$. Assume that Z is an effective cycle integrable with respect to $(L_0, \| \ \|_0), \ldots, (L_t, \| \ \|_t)$ and let $h(Z)$ be the global height with respect to $(L_0, \| \ \|_0, s_0), \ldots, (L_t, \| \ \|_t, s_t)$. Then*

$$|\hat{h}(Z) - h(Z)| \leq \sum_{j=0}^{t} C_j \deg_{L_0, \ldots, L_{j-1}, L_{j+1}, \ldots, L_t}(Z)$$

where $C_j = \int_M c_j \, d\mu_j$ for an integrable function c_j on M bounding $d_v(\| \ \|_{\theta_j, v}, \| \ \|_{j,v})$ for almost all $v \in M$.

e) *If Z is a t-dimensional cycle on X such that*

$$\left(|\mathrm{div}(s_0)| \cup \psi^{-1}|\mathrm{div}(s_0)|\right) \cap \cdots \cap \left(|\mathrm{div}(s_t)| \cup \psi^{-1}|\mathrm{div}(s_t)|\right) \cap |Z| = \emptyset,$$

then

$$m_0 \cdots m_t \hat{h}(Z) - n_0 \cdots n_t \hat{h}(\psi_* Z) = \sum_{j=0}^{t} d_{\beta_j}$$

where $\beta_j := \frac{\theta_j \circ s_j^{\otimes n_j} \circ \psi}{s_j^{\otimes m_j}}(Y_j)$ and where Y_j is a representative of the refined intersection

$$\mathrm{div}(s_0) \ldots \mathrm{div}(s_{j-1}).\mathrm{div}(s_{j+1}) \ldots \mathrm{div}(s_t).Z.$$

f) *If we replace the isomorphisms θ_j in (5.6) by other isomorphisms θ'_j inducing the canonical local height $\hat{h}'(Z)$, then there are $\alpha_j \in K^\times$ with $\theta'_j = \alpha_j \theta_j$ and we have*

$$\hat{h}(Z) - \hat{h}'(Z) = \sum_{j=0}^{t} \frac{d_{\alpha_j}}{n_j - m_j} \deg_{L_0,\ldots,L_{j-1},L_{j+1},\ldots,L_t}(Z).$$

Proof: Integrating the corresponding properties for canonical local heights in Proposition 5.1.21, this follows immediately from the above considerations. □

5.3.19 Corollary. *If the product formula is satisfied for the M-field K, then the canonical height $\hat{h}(Z)$ is independent of $(s_j, \theta_j)_{j=0,\ldots,t}$ and well-defined for all t-dimensional cycles. It has the following properties:*

a) *It is multilinear and symmetric in the variables $L_0, \ldots, L_t$, and linear in Z.*

b) *If φ is a morphism as in Theorem 5.3.18, then*

$$\hat{h}_{\varphi^* L_0,\ldots,\varphi^* L_t}(Z') = \hat{h}_{L_0,\ldots,L_t}(\varphi_* Z')$$

for all t-dimensional cycles Z' on X'.

c) *In particular, we have*

$$m_0 \cdots m_t \hat{h}(Z) = n_0 \cdots n_t \hat{h}(\psi_* Z).$$

d) *Let $\| \ \|_0, \ldots, , \| \ \|_t$ be any M-metrics on $L_0, \ldots, L_t$ lying in $\hat{\mathfrak{g}}_X^+$. Assume that Z is a cycle integrable with respect to $(L_0, \| \ \|_0), \ldots, (L_t, \| \ \|_t)$ and let $h(Z)$ be the corresponding global height. Then*

$$|\hat{h}(Z) - h(Z)| = O(\delta_{L_0,\ldots,L_t})(Z).$$

e) *For a t-dimensional effective cycle Z on X, we have $\hat{h}(Z) \geq 0$.*

Moreover, the canonical height $\hat{h}(Z)$ is uniquely characterized by c) and d).

Proof: Properties a)-d) are an immediate consequence of Theorem 5.3.18. To prove e), we may assume that the line bundles are generated by global sections and that Z is prime. If K is finite, all global heights are zero. So we may assume K infinite. Then there are global sections $s_0, \ldots, s_t$ such that $\operatorname{div}(s_0), \cdots, \operatorname{div}(s_t)$ intersect Z properly. We can construct an M-metric $\| \ \|$ on L_j lying in $\mathfrak{b}_X^+$ such that

$$\sup_{x \in X(K)} \|s_j(x)\|_v \leq 1$$

for almost all $v \in M$ (use a morphism to projective space such that s_j is a coordinate and take the restriction of the Fubini-Study or the standard metric). Then all the M-metrics $\| \ \|_{j,k}$ from Tate's limit argument have this property. By Proposition 5.1.10, we deduce $\hat{h}(Z) \geq 0$.

Finally, we prove uniqueness. Suppose that $\bar{h}$ is another real function on t-dimensional cycles satisfying c) and d). For any t-dimensional cycle Z on X and any $k \in \mathbb{N}$, we get

$$\begin{aligned} |\hat{h}(Z) - \bar{h}(Z)| &= \left(\frac{n_0 \cdots n_t}{m_0 \cdots m_t}\right)^k |\hat{h}(\psi_*^k Z) - \bar{h}(\psi_*^k Z)| \\ &\leq C \left(\frac{n_0 \cdots n_t}{m_0 \cdots m_t}\right)^k \delta_{L_0,\ldots,L_t}(\psi_*^k Z) \\ &= C \sum_{j=0}^{t} \left(\frac{n_j}{m_j}\right)^k \deg_{L_0,\ldots,L_{j-1},L_{j+1},\ldots,L_t}(Z) \end{aligned}$$

where the constant C is independent of Z and k. First, we have used c), then d) and in the last step, we have applied projection formula to the degrees occuring in the definition of $\delta_{L_0,\ldots,L_t}$. By $k \to \infty$, we get $\hat{h}(Z) = \bar{h}(Z)$. □

5.3.20 Example. By a theorem of Serre ([Ha], Theorem 5.17), every line bundle L on a projective variety is isomorphic to $L_+ \otimes L_-^{-1}$ for line bundles L_+, L_- generated by global sections. We conclude that every line bundle on a projective variety has an M-metric lying in the group $\mathfrak{b}_X$ generated by $\mathfrak{b}_X^+$..

Let A be an abelian variety over K and let $\psi = [m]$ be multiplication with $m \in \mathbb{N}, m \geq 2$. The theorem of the cube gives ridifications

$$[m]^* L \xrightarrow{\sim} \begin{cases} L^{\otimes m^2} & \text{if } L \text{ is even,} \\ L^{\otimes m} & \text{if } L \text{ is odd.} \end{cases}$$

Any line bundle on A is isomorphic to the tensor product of an even and an odd line bundle. By Theorem 5.3.13 and Example 5.1.15, we get *canonical M-metrics* $\| \ \|_{can}$ on any line bundle of A and the metric does neither depend on the decomposition into even and odd part nor on the choice of m. Moreover, $\| \ \|_{can,v}$ is a canonical metric for almost all $v \in M$ and thus $\| \ \|_{can}$ is contained in $\hat{\mathfrak{g}}_X$. A canonical M-metric is unique up to multiplication with the function $v \mapsto |\alpha|_v$, for some $\alpha \in K^\times$. Canonical M-metrics are closed under tensor product and pull-back with respect to homomorphisms of abelian varieties.

5.3.21 Proposition. *Let A be an abelian variety over K and let $\hat{L}_0, \ldots, \hat{L}_t$ be line bundles on A endowed with canonical metrics. Then every t-dimensional cycle on A is integrable with respect to $\hat{L}_0, \ldots, \hat{L}_t$.*

Proof: By multilinearity (Theorem 5.3.9), we may assume that every L_j is either even or odd. First, we handle the case where one line bundle is odd, say L_0. Then the local heights do not depend on the choice of the metrics on $L_1, \ldots, L_t$ (Proposition 5.1.17). Using multilinearity and the theorem of Serre mentioned above, we may assume that $\hat{L}_1, \ldots, \hat{L}_t \in \mathfrak{b}_A^+$. Clearly, L_0 has a metric $\| \ \|_0$ lying in $\mathfrak{b}_A$. Then the sequence $\| \ \|_{0,k} := \Phi^k(\| \ \|_0)$ of metrics on L_0 considered in Remark 5.3.14 is also lying in $\mathfrak{b}_A$ and it converges against the canonical metric of L_0. Moreover, the convergence is bounded by an integrable function on M. By Remark 5.3.11, every t-dimensional cycle on A is integrable with respect to $(L_0, \| \ \|_{0,k}), \hat{L}_1, \ldots, \hat{L}_t$. Using Theorem 5.1.8d) and the dominated convergence theorem, we conclude that the same holds for $\hat{L}_0, \ldots, \hat{L}_t$. By symmetry, this proves the case of at least one odd line bundle.

So we may assume that all line bundles are even. Note that an even line bundle is isomorphic to a line bundle of the form $L_+ \otimes L_-^{-1}$ with L_+, L_- even line bundles generated by global sections. By multilinearity, we may assume that all line bundles are even and generated by global sections. Every canonical M-metric on L_j may be approximated as above by metrics in $\mathfrak{b}_A^+$ with convergence dominated by integrable functions. It follows from Theorem 5.1.8d) using the same trick as in (5.1) and from the dominated convergence theorem that every cycle is integrable with respect to $\hat{L}_0, \ldots, \hat{L}_t$. □

5.3.22 Definition. Let $\hat{L}_0, \ldots, \hat{L}_t$ be line bundles on an abelian variety A over K endowed with canonical metrics. For invertible meromorphic sections $s_0, \ldots, s_t$ of $L_0, \ldots, L_t$ and a t-dimensional cycle Z on A, we have a global height with respect to $(\hat{L}_0, s_0), \ldots, (\hat{L}_t, s_t)$, well-defined if

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$

This follows from the Proposition 5.3.21. It is called the *Néron-Tate height* of Z and it is denoted by $\hat{h}(Z)$.

5.3.23 Theorem. *The Néron Tate height has the following properties:*

a) $\hat{h}(Z)$ *is multilinear and symmetric in the variables* $(\hat{L}_j, s_j)_{j=0,\ldots,t}$*, and linear in* Z *under the assumption that all terms are well-defined.*

b) If $\varphi : A' \to A$ *is a homomorphism of abelian varieties over* K *and* Z' *is a cycle on* A' *such that*

$$\varphi|Z'| \cap |\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| = \emptyset,$$

then the following identity of canonical local heights holds

$$\hat{h}_{(\varphi^*\hat{L}_0, s_0'),\ldots,(\varphi^*\hat{L}_t, s_t')}(Z') = \hat{h}_{(\hat{L}_0, s_0),\ldots,(\hat{L}_t, s_t)}(\varphi_* Z').$$

Here, we use $s_j' := s_j \circ \varphi$ *if it is a well-defined invertible meromorphic section, otherwise we use any invertible meromorphic section of* $L_j'|_{Z'}$ *as in Corollary 5.3.10.*

c) If we replace s_0 *by another invertible meromorphic section* s_0' *of* L_0 *such that the corresponding Néron-Tate height* $\hat{h}'(Z)$ *is also well-defined, then*

$$\hat{h}(Z) - \hat{h}'(Z) = d_f(Y)$$

for any representative Y *of* $\mathrm{div}(s_1)\ldots\mathrm{div}(s_t).Z \in CH_0\left(|\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z|\right)$ *and* $f := s_0'/s_0$.

d) If we replace the canonical metrics $\|\ \|_{j,can}$ *on* L_j *by* M*-metrics* $\|\ \|_j$ *assumed to be in* $\hat{\mathfrak{g}}_X^+$ *for* $j = 0,\ldots,t$ *such that the effective cycle* Z *is integrable with respect to* $(L_0, \|\ \|_0), \ldots, (L_t, \|\ \|_t)$ *and if* $h(Z)$ *denotes the corresponding global height, then*

$$\sum_{j=0}^{t} C_j^- \deg_{L_0,\ldots,L_{j-1},L_{j+1},\ldots,L_t}(Z) \leq \hat{h}(Z) - h(Z) \leq \sum_{j=0}^{t} C_j^+ \deg_{L_0,\ldots,L_{j-1},L_{j+1},\ldots,L_t}(Z) \quad (5.7)$$

for finite constants C_+, C_- *given explicitly as in Theorem 5.1.8d).*

e) If $m \in \mathbb{Z}, |m| \geq 2$*, if*

$$\left(|\mathrm{div}(s_0)| \cup [m]^{-1}|\mathrm{div}(s_0)|\right) \cap \cdots \cap \left(|\mathrm{div}(s_t)| \cup [m]^{-1}|\mathrm{div}(s_t)|\right) \cap |Z| = \emptyset,$$

if k *line bundles of* $L_0, \ldots, L_t$ *are even and if the remaining line bundles are odd, then*

$$m^{k+t+1}\hat{h}(Z) - \hat{h}([m]_* Z) = \sum_{j=0}^{t} d_{\beta_j}$$

where $\beta_j := \frac{\theta_j \circ s_j \circ \psi}{s_j^{\otimes m_j}}(Y_j)$*, where* $\theta_j : [m]^*L_j \xrightarrow{\sim} L_j^{\otimes m^2}$ *(resp.* $L_j^{\otimes m}$*) is an isometry with respect to* $[m]^*\|\ \|_{j,can}$ *and* $\|\ \|_{j,can}^{\otimes m^2}$ *(resp.* $\|\ \|_{j,can}^{\otimes m}$*) in the even (resp. odd) case and where* Y_j *is a representative of the refined intersection*

$$\mathrm{div}(s_0)\ldots\mathrm{div}(s_{j-1}).\mathrm{div}(s_{j+1})\ldots\mathrm{div}(s_t).Z.$$

f) If we replace the canonical metrics $\|\ \|_{j,can}$ *on* L_j *by other canonical metrics* $\|\ \|_{j,can}'$ *for* $j = 0, \ldots, t$*, then there is* $\alpha_j \in K$ *with* $\|\ \|_{j,can,v}' = |\alpha_j|_v \|\ \|_{j,can,v}$ *for almost all* $v \in M$ *and we have*

$$\hat{h}(Z) - \hat{h}'(Z) = \sum_{j=0}^{t} d_{\alpha_j} \deg_{L_0,\ldots,L_{j-1},L_{j+1},\ldots,L_t}(Z).$$

Proof: Using Theorem 5.1.8, this is an easy consequence of Corollary 5.1.23 by integrating the corresponding properties. $\square$

5.3.24 Theorem. *Let A be an abelian variety over an M-field K satisfying the product formula. Then the Néron-Tate height $\hat{h}_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z)$ is independent of the meromorphic sections $s_0,\ldots,s_t$ and the canonical metrics on $L_0,\ldots,L_t$. It is well-defined for all t-dimensional cycles Z on X and is denoted by $\hat{h}(Z)$. It has the following properties:*

a) It is multilinear and symmetric in the variables $L_0,\ldots,L_t$, and linear in Z.

b) If $\varphi : A' \to A$ is a homomorphism of abelian varieties, then

$$\hat{h}_{\varphi^* L_0,\ldots,\varphi^* L_t}(Z') = \hat{h}_{L_0,\ldots,L_t}(\varphi_* Z')$$

for all t-dimensional cycles Z' on A'.

c) If $m \in \mathbb{Z}, |m| \geq 2$, if k line bundles of $L_0,\ldots,L_t$ are even and if the remaining line bundles are odd, then we have

$$m^{k+t+1}\hat{h}(Z) = \hat{h}([m]_* Z).$$

d) Let $\| \ \|_0,\ldots,,\| \ \|_t$ be any M-metrics on $L_0,\ldots,L_t$ lying in $\hat{\mathfrak{g}}_X^+$. Assume that Z is a cycle integrable with respect to $(L_0,\| \ \|_0),\ldots,(L_t,\| \ \|_t)$ and let $h(Z)$ be the corresponding global height. Then

$$|\hat{h}(Z) - h(Z)| = O(\delta_{L_0,\ldots,L_t})(Z).$$

e) The Néron-Tate height of an effective cycle with respect to even ample line bundles is non-negative. More generally, this holds for even line bundles with a positive tensor power generated by global sections.

Moreover, the Néron-Tate height $\hat{h}(Z)$ is uniquely characterized by multilinearity a), homogenity for one m in c) and d).

Proof: Properties a)-d) follow immediately from Theorem 5.3.23 and e) follows from Corollary 5.3.19. Note that every line bundle on A is isomorphic to $L_+ \otimes L_-^{-1}$ for even line bundles L_+, L_- generated by global sections. Thus to prove uniqueness, we may assume that every L_j is either an even line bundle generated by global sections or $L_j \in \mathrm{Pic}^\circ(A)$. This implies $\hat{L}_0,\ldots,\hat{L}_t \in \hat{\mathfrak{g}}_A^+$. Then uniqueness follows similarly as in Corollary 5.3.19. $\square$

5.3.25 Let X be a proper smooth scheme over K and let $L \in \mathrm{Pic}^\circ(X)$. We assume that X has a K-rational point which usually may be achieved by base change. In accordance with Remark 5.1.16, an M-metric $\| \ \|$ on L is called *canonical* if there is a morphism to an abelian variety A over K with $L' \in \mathrm{Pic}^\circ(A)$ such that $L \cong \varphi^* L'$ and $\| \ \|$ is the pull-back of a canonical M-metric on L'. For almost all $v \in M$, we get a canonical metric $\| \ \|_v$ in the sense of Remark 5.1.16. From Example 5.3.20, we conclude that a canonical M-metric on L is unique up to multiplication by the function $v \mapsto |\alpha|_v$ on M for some $\alpha \in K$. Clearly, canonical metrics are closed under tensor product and pull-back.

5.3.26 Proposition. *Let X be as in 5.3.25 and let $s_0,\ldots,s_t$ be invertible meromorphic sections of $\hat{L}_0,\ldots,\hat{L}_t \in \hat{\mathfrak{g}}_X$. Suppose that $L_0 \in \mathrm{Pic}^\circ(X)$ and that its M-metric is canonical. Then every t-dimensional cycle Z on X is integrable with respect to $\hat{L}_0,\ldots,\hat{L}_t$. We assume that*

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$

a) Then the identity

$$h_{(\hat{L}_0,s_0),\ldots,(\hat{L}_t,s_t)}(Z) = h_{(\hat{L}_0,s_0)}(Y)$$

holds for any representative Y *of the refined intersection* $\mathrm{div}(s_1)\ldots\mathrm{div}(s_t).Z$ *. In particular, the global height does not depend on the metrics of* $\hat{L}_1,\ldots,\hat{L}_t$.

b) If Z *or at least one of the line bundles* $L_1,\ldots,L_t$ *is also algebraically equivalent to* 0*, then the global height is also independent of the metric on* L_0.

c) If the product formula is satisfied, then $h(Z)$ *is independent of the choice of invertible meromorphic sections* $s_0,\ldots,s_t$ *and also of the canonical metric on* L_0*. It is well-defined for all* t*-dimensional cycles on* X.

Proof: By Proposition 5.1.17, we have

$$\lambda_{(\hat{L}_0^v,s_0),\ldots,(\hat{L}_t^v,s_t)}(Z) = -\log\|s_0(Y)\|_{can,v}$$

for almost all $v \in M$. Since the metric is a pull-back metric from an abelian variety, we conclude that the local heights are integrable (Proposition 5.3.7 and that the identity for the global height holds. Claim b) is a consequence of Corollary 5.1.18. In c), independence of the sections follows from Theorem 5.3.9c). Using a) and 5.3.25, the independence of the canonical metric is a direct consequence of the product formula. □

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