# POSITIVITY PROPERTIES OF METRICS AND DELTA-FORMS

WALTER GUBLER AND KLAUS KÜNNEMANN

Abstract. In previous work, we have introduced $\delta$-forms on the Berkovich analytification of an algebraic variety in order to study smooth or formal metrics via their associated Chern $\delta$-forms. In this paper, we investigate positivity properties of $\delta$-forms and $\delta$-currents. This leads to various plurisubharmonicity notions for continuous metrics on line bundles. In the case of a formal metric, we show that many of these positivity notions are equivalent to Zhang's semipositivity. For piecewise smooth metrics, we prove that plurisubharmonicity can be tested on tropical charts in terms of convex geometry. We apply this to smooth metrics, to canonical metrics on abelian varieties and to toric metrics on toric varieties.

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## Contents



## 0. Introduction

Pluripotential theory studies plurisubharmonic functions and Monge-Ampère operators and constitutes a central area of the modern theory of complex analytic spaces. In recent years a number of authors have introduced ideas and concepts from pluripotential theory into the theory of non-archimedean analytic spaces. Let us mention here the work of Baker and Rumely [BR10] for the Berkovich projective line and the work of Rumely, Kani, Chinburg-Rumely, Zhang, and Thuillier [Rum89, Kan89, CR93, Zha93, Thu05] via reduction graphs or skeletons for potential theory on non-archimedean analytic curves. An axiomatic vision of a theory of plurisubharmonic functions on

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higher dimensional non-archimedean analytic spaces was formulated by Chinburg, Rumely and Lau in [CLR03, Sect. 4].

The higher dimensional theory started with Zhang's study of semipositive approximable metrics on line bundles in [Zha95a] and in [Zha95b]. Zhang realized that any model $\mathscr{L}$ of a line bundle $L$ induces a metric on $L$. Such an algebraic metric is called semipositive if $\mathscr{L}$ restricts to a nef line bundle on the special fibre. A semipositive approximable metric on $L$ is defined as a uniform limit of metrics on $L$ for which some positive tensor powers are semipositive algebraic. Bloch, Gillet and Soulé developed in [BGS97] a non-archimedean Arakelov theory based on the Chow groups of all regular proper models over the valuation ring assuming resolution of singularities. Using Zhang's metrics and this non-archimedean Arakelov theory, Boucksom, Favre and Jonsson gave an approach to plurisubharmonic functions via skeletons in the case of residue characteristic zero [BFJ12]. If the involved measure is supported on a skeleton and if a certain algebraicity condition is satisfied, then they prove the existence of a solution to the non-archimedean Calabi-Yau problem [BFJ15]. Uniqueness was shown before by Yuan–Zhang [YZ13] in complete generality. For proper toric varieties, Burgos, Philippon and Sombra gave in [BPS14] a complete characterization of semipositive toric metrics in terms of convex functions and they also described in [BMPS12] other positivity notions of toric metrics in terms of convex geometry.

The theory of plurisubharmonic functions in [BFJ12] satisfies the required axioms of [CLR03], except that it is not of analytic character. Such an analytic theory has recently been established by Chambert-Loir and Ducros [CLD12] introducing forms and currents on Berkovich spaces by using Lagerberg's superforms [Lag12] on tropical charts. Their theory has the additional advantage that it works without any hypotheses on the characteristic. Their notion of plurisubharmonicity is directly related to the positivity of currents. They have also transferred a part of the Bedford-Taylor theory to plurisubharmonic functions which are locally approximable by smooth plurisubharmonic functions leading to Monge–Ampère measures for such locally psh-approximable functions. It would be desirable to extend the theory to all plurisubharmonic continuous functions as in the complex Bedford–Taylor theory [BT82] and the monotone regularization theorem from the program in [CLR03, Sect. 4] is also missing.

The analytic theory of forms and currents in [CLD12] has been extended in [GK14] in order to treat smooth and algebraic metrics on line bundles in a more natural way. The new formalism of $\delta$-forms introduced in [GK14] allows to study such metrics in terms of the associated Chern $\delta$-forms and Chern $\delta$-currents. It is the aim of this paper to extend the central notions of plurisubharmonicity and positivity to the theory of $\delta$-forms and $\delta$-currents.

We recall that a $\delta$-preform on $\mathbb{R}^r$ is a supercurrent $\alpha$ in the sense of Lagerberg [Lag12] of the form

$$\alpha = \sum_{i=1}^{N} \alpha_i \wedge \delta_{C_i},$$

where $\alpha_i$ is a superform and $\delta_{C_i}$ is the supercurrent of integration over a tropical cycle $C_i$ with smooth weights. The $\delta$-preforms on an open subset $\Omega$ of a tropical cycle $C$ with constant non-negative weights are supercurrents of the form $\alpha \wedge \delta_C$. Similarly as in complex analysis, Lagerberg [Lag12] introduced weakly positive (resp. positive, resp. strongly positive) superforms and supercurrents on $\mathbb{R}^r$. We show in Section 1 that this leads to corresponding positivity properties for supercurrents and hence for $\delta$-preforms on $\Omega$.

Let $X^{\mathrm{an}}$ denote the non-archimedean analytification of a variety $X$ defined over an algebraically closed field $K$ complete with respect to a non-trivial non-archimedean absolute value $|\ |$. We apply the above to tropical charts $(V, \varphi_U)$ of $X^{\mathrm{an}}$, where $\varphi_U : U \to \mathbb{G}_m^r$ is the canonical closed embedding of a very affine open subset $U$ of $X$ and where $V := \mathrm{trop}_U^{-1}(\Omega)$ for an open subset $\Omega$ of the tropical variety $\mathrm{Trop}(U)$. We have seen in [GK14] that $\delta$-preforms on $\Omega$ represent generalized $\delta$-preforms on $V$ and by a sheafification process we get the bigraded algebra of generalized $\delta$-forms on any open subset $W$ of $X^{\mathrm{an}}$. The subalgebra of $\delta$-forms is characterized by a closedness condition with respect to natural differential operators $d', d''$ analogous to $\partial, \bar{\partial}$ in complex analytic geometry. As a topological dual, we have the space of $\delta$-currents on $W$ (see [GK14, §4] for details). The smooth forms from [CLD12] build a subalgebra of the algebra of $\delta$-forms. We will show in Section 2 that the positivity notions of $\delta$-preforms induce corresponding positivity notions of (generalized) $\delta$-forms requiring that these notions are functorial. By duality, we get corresponding positivity properties for $\delta$-currents. Recall from [CLD12, §5.5] that a continuous function $f$ on $W$ is plurisubharmonic if the current $d'd''[f]$ is positive. In Section 3, we investigate variants of this definition. For example, we define $f$ to be $\delta$-psh if a similar condition holds for $\delta$-currents. It is not clear that these notions behave functorially with respect to morphisms $\varphi : X' \to X$ of algebraic varieties over $K$ and so we introduce also functorial psh (resp. functorial $\delta$-psh) continuous functions. In the following, we consider a line bundle $L$ of $X$ and a continuous metric $\|\ \|$ on $L^{\mathrm{an}}$ over $W$. Then $\|\ \|$ is called plurisubharmonic if $-\log \|s\|$ is psh on $W$ for any local frame $s$ of $L$. Similarly, we transfer the other positivity notions to metrics.

In Theorem 4.1, we prove a crucial lifting result which enables us to lift closed subsets from the special fibre of a model over a valuation ring to the generic fibre. In Section 5, we recall the definition and some properties of a formal metric on a compact (reduced) strictly $K$-analytic space $V$. If $V$ is a strictly $K$-analytic domain in the analytification of a proper variety $X$ over $K$, then we show in Corollary 5.12 that any formal metric $\|\ \|$ on the restriction of $L$ to $V$ extends to an algebraic metric of the line bundle $L$ over $X$. In Section 6, we give a local variant of Zhang's semipositivity definition for a formal metric following a suggestion from Tony Yue Yu. Using our above lifting theorem, we prove in Theorem 6.10 the following result comparing the different notions of positivity for a formal metric:

**Theorem 0.1.** *Let $(L, \|\ \|)$ be a formally metrized line bundle on a proper variety $X$ and let $W$ be an open subset of $X^{\mathrm{an}}$. Then the following properties are equivalent:*

(1) *The formal metric* $\| \ \|$ *is semipositive over* $W$*.*
(2) *The restriction of the metric to* $W$ *is functorial* $\delta$*-psh.*
(3) *The restriction of the metric* $\| \ \|$ *to* $W$ *is functorial psh.*
(4) *The* $\delta$*-form* $c_1(L|_W, \| \ \|)$ *is positive on* $W$*.*
(5) *The restriction of* $\| \ \|$ *to* $W \cap C^{\mathrm{an}}$ *is psh for any closed curve* $C$ *of* $X$*.*

In Section 7, we will show that a formal metric on $L$ is semipositive as a formal metric if and only if it a semipositive approximable metric. In the case of a discretely value field with residue characteristic 0, this was first proved in [BFJ12, Remark after Theorem 5.12]. Here, this is an easy consequence of our lifting theorem. We show also that the restriction of a semipositive approximable metric to any closed curve is plurisubharmonic.

In Section 8, we characterize piecewise smooth metrics in terms of convex geometry on tropical charts.

**Theorem 8.4.** Let $L$ be a line bundle on the algebraic variety $X$ over $K$. Let $\| \ \|$ be a piecewise smooth metric on $L$ over an open subset $W$ of $X^{\mathrm{an}}$. Then the metric $\| \ \|$ is plurisubharmonic if and only if for each tropical frame $(V, \varphi_U, \Omega, s, \phi)$ of $\| \ \|$ we have

(i) the restriction of $\phi$ to each maximal face of $\mathrm{Trop}(U) \cap \Omega$ is a convex function and
(ii) the corner locus $\tilde{\phi} \cdot \mathrm{Trop}(U)$ is an effective tropical cycle on $\Omega$ with smooth weights.

Let us briefly explain the terminology used in the above theorem. The metric $\| \ \|$ is called piecewise smooth if there is a tropical chart $(V, \varphi_U)$ in $W$ and a frame $s$ of $L$ over $U$ such that $-\log \|s\| = \phi \circ \mathrm{trop}_U$ on $V$ for a piecewise smooth function $\phi$ on the open subset $\Omega = \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. If $\Omega$ is also convex and if $\phi$ extends to a piecewise smooth function $\tilde{\phi} : N_{\mathbb{R}} \to \mathbb{R}$, then we call it a tropical frame for $\| \ \|$. The corner locus $\tilde{\phi} \cdot \mathrm{Trop}(U)$ is a tropical cycle of codimension 1 in $N_{U,\mathbb{R}}$ which is supported in the non-differentiability locus of $\tilde{\phi}|_{\mathrm{Trop}(U)}$ and whose weights are defined in terms of the outgoing slopes (see [GK14, Definition 1.10]). In the remaining part of Section 8, we apply our results to compare different positivity notions in the following situations:

- $\delta$-metrics in Proposition 8.8;
- smooth metrics in Corollary 8.9;
- canonical metrics on abelian varieties in Example 8.10;
- canonical metrics on line bundles algebraically equivalent to zero in Remark 8.11;
- piecewise smooth toric metrics on toric varieties in Proposition 8.13.

0.1. **Terminology.** We use $A \subseteq B$ to denote subsets and $B \setminus A$ for the complement of $A$ in $B$. The zero is included in $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}_-$. In topology, compact means quasicompact and Hausdorff.

The group of multiplicative units in a ring $A$ is denoted by $A^\times$. An (algebraic) variety over a field is an irreducible and reduced scheme which is separated and of finite type. A curve is an algebraic variety of dimension one. A variety $U$ is called very affine if it has a closed immersion into a multiplicative torus. We refer to Section 2 for the canonical torus $T_U$ with cocharacter lattice $M_U$ and dual $N_U$, for the canonical closed embedding $\varphi_U : U \to T_U$ and for tropical charts. For Berkovich analytic spaces, we use the terminology from [Ber93] and [Ber90].

For the notation in convex geometry, we refer to [GK14, Appendix A]. We usually work with a finite dimensional real vector space $N_{\mathbb{R}}$ with an integral structure given by a lattice $N$. We recall that a polyhedron $\Delta$ is called integral $\mathbb{R}$-affine if the underlying linear space $\mathbb{L}_\Delta$ of the affine space $\mathbb{A}_\Delta$ generated by $\Delta$ is defined over $\mathbb{Z}$. An affine map $F$ is called integral $\mathbb{R}$-affine if the underlying linear map $\mathbb{L}_F$ is defined over $\mathbb{Z}$. For the terminology in tropical geometry, we refer to [GK14, §1]. We recall briefly that a tropical cycle $C = (\mathscr{C}, m)$ in $N_{\mathbb{R}}$ consists of an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ and smooth weight functions $m_\Delta : \Delta \to \mathbb{R}$ for each maximal face $\Delta \in \mathscr{C}$. We call $C$ *effective* on $\Omega \subseteq N_{\mathbb{R}}$ if $m_\Delta|_{\Delta\cap\Omega} \geq 0$ for each maximal $\Delta \in \mathscr{C}$.

In the whole paper, $K$ is an algebraically closed field endowed with a complete non-trivial non-archimedean absolute value. We use $K^\circ$ for the valuation ring, $K^{\circ\circ}$ for its maximal ideal and $\tilde{K} = K^\circ/K^{\circ\circ}$ for the residue field. For a (formal) scheme $\mathscr{X}$ over $K^\circ$, we use $\mathscr{X}_\eta$ for the generic fibre and $\mathscr{X}_s$ is the special fibre.

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## 1. Positive forms on tropical cycles

We recall some positivity notions for superforms from [Lag12] and [CLD12]. Then we introduce similar notions for $\delta$-preforms as defined in [GK14]. Let $N$ denote a free $\mathbb{Z}$-module of finite rank $r$ with dual lattice $M$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. In this section $C = (\mathscr{C}, m)$ always denotes an $n$-dimensional tropical cycle on $N_{\mathbb{R}}$ with positive *constant* weights.

**1.1.** We recall the positivity notions for superforms and supercurrents on an open subset $\tilde{\Omega}$ of $N_{\mathbb{R}}$ given in [Lag12, Sect. 2], [CLD12, Sect. 5.1].

(i) The canonical involution $J$ acts on the space $A(\tilde{\Omega})$ of superforms on $\tilde{\Omega}$ and maps $(p,q)$-superforms to $(q,p)$-superforms [CLD12, (1.2.5)]. A superform $\alpha \in A^{p,p}(\tilde{\Omega})$ is called *symmetric* if $J(\alpha) = (-1)^p\alpha$ holds and *anti-symmetric* if $J(\alpha) = (-1)^{p+1}\alpha$.

(ii) A *positive* $(r,r)$*-superform* on $\tilde{\Omega}$ is given by

$$f d'x_1 \wedge d''x_1 \wedge \cdots \wedge d'x_r \wedge d''x_r$$

for a non-negative function $f$ on $\tilde{\Omega}$, where $x_1, \ldots, x_r$ is a basis of $M$.

(iii) A symmetric superform $\alpha \in A^{p,p}(\tilde{\Omega})$ is called *weakly positive* if

$$\alpha \wedge \alpha_1 \wedge J(\alpha_1) \wedge \cdots \wedge \alpha_{r-p} \wedge J(\alpha_{r-p})$$

is a positive $(r,r)$-superform for all $\alpha_1, \ldots, \alpha_{r-p} \in A^{1,0}(\tilde{\Omega})$.

(iv) A symmetric superform $\alpha \in A^{p,p}(\tilde{\Omega})$ is called *positive* if

$$(-1)^{(r-p)(r-p-1)/2}\alpha \wedge \beta \wedge J(\beta)$$

is a positive $(r,r)$-superform for every $(r-p,0)$-superform $\beta$ on $\tilde{\Omega}$.

(v) A superform $\alpha \in A^{p,p}(\tilde{\Omega})$ is called *strongly positive* if

$$\omega = \sum_{k=1}^{l} f_k \alpha_{k1} \wedge J(\alpha_{k1}) \wedge \cdots \wedge \alpha_{kp} \wedge J(\alpha_{kp})$$

for non-negative smooth functions $f_k$ and $(1,0)$-superforms $\alpha_{ki}$ on $\tilde{\Omega}$. A strongly positive superform is automatically symmetric.

(vi) A supercurrent $\alpha \in D(\tilde{\Omega})$ is called *symmetric* iff it vanishes on anti-symmetric superforms with compact support in $\tilde{\Omega}$. A symmetric supercurrent $T \in D^{p,p}(\tilde{\Omega})$ is called *(strongly/weakly) positive* iff $\langle T, \alpha \rangle \geq 0$ for all (weakly/strongly) positive superforms $\alpha$ with compact support in $\tilde{\Omega}$.

(vii) Let $A^{p,p}_{+}(\tilde{\Omega})$ be the space of positive $(p,p)$-superforms on $\tilde{\Omega}$. Similarly, we use $A^{p,p}_{\mathrm{w}+}(\tilde{\Omega})$ (resp. $A^{p,p}_{\mathrm{s}+}(\tilde{\Omega})$) for the space of weakly positive (resp. strongly positive) $(p,p)$-superforms on $\tilde{\Omega}$. We denote by $D^{p,p}_{(\mathrm{s/w})+}(\tilde{\Omega})$ the corresponding spaces of supercurrents on $\tilde{\Omega}$.

We summarize some basic properties of positivity in the following statement.

**Proposition 1.2.** (a) *$A^{p,p}_{\mathrm{s}+}(\tilde{\Omega}) \subseteq A^{p,p}_{+}(\tilde{\Omega}) \subseteq A^{p,p}_{\mathrm{w}+}(\tilde{\Omega})$.*

(b) *For $p = 0, 1, r-1, r$, we have equality everywhere in (a).*

(c) *The pull-back of a (weakly/strongly) positive superform with respect to an affine map is a (weakly/strongly) positive superform.*

(d) *The wedge product of a (weakly/strongly) positive superform with a strongly positive superform is (weakly/strongly) positive.*

(e) *A symmetric superform $\alpha \in A^{p,p}(\tilde{\Omega})$ is (weakly/strongly) positive if and only if $\alpha \wedge \beta$ is positive for every (strongly/weakly) positive superform $\beta$ of type $(r-p, r-p)$.*

(f) *There is a natural morphism $A(\tilde{\Omega}) \to D(\tilde{\Omega})$ which maps a superform $\alpha$ to the associated supercurrent $[\alpha]$. A superform $\alpha \in A^{p,p}(\tilde{\Omega})$ is (weakly/strongly) positive if and only if $[\alpha]$ is (weakly/strongly) positive.*

*Proof.* See [Lag12, §2] and [CLD12, §5.1]. For (e) and (f) observe in particular [CLD12, (5.1.2)]. □

**1.3.** (i) The notions and results mentioned above carry immediately over to the situation where $\tilde{\Omega}$ is an open subset of an affine space under $N_{\mathbb{R}}$ [CLD12, 5.1].

(ii) The above positivity notions for superforms are defined pointwise.

(iii) For an open subset $\Omega$ of a polyhedron $\Delta$ in $N_{\mathbb{R}}$, a superform $\alpha$ on $\Omega$ is defined as the restriction of a superform $\tilde{\alpha}$ on an open subset $\tilde{\Omega}$ in $N_{\mathbb{R}}$ with $\tilde{\Omega} \cap \Delta = \Omega$. We call $\alpha$ *(weakly/strongly) positive* if $\alpha$ is (weakly/strongly) positive at every point of $\Omega$ as a superform in the affine space $\mathbb{A}_{\Delta}$. An equivalent condition is that $\alpha|_{\Omega \cap \mathrm{relint}(\Delta)}$ is a (weakly/strongly) positive superform on the open subset $\mathrm{relint}(\Delta)$ of $\mathbb{A}_{\Delta}$. This uses the fact that the positivity loci of superforms are closed.

**Example 1.4.** Let $\alpha \in D(\tilde{\Omega})$ be a polyhedral supercurrent. By definition (see [GK14, Def. 2.3]) there exists an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}$ in $N_{\mathbb{R}}$

and a family $(\alpha_\Delta)_{\Delta\in\mathscr{D}}$ of superforms $\alpha_\Delta$ on $\tilde{\Omega}\cap\Delta$ such that

$$\alpha = \sum_{\Delta\in\mathscr{D}} \alpha_\Delta \wedge \delta_\Delta \text{ in } D(\tilde{\Omega}).$$

An easy support argument shows that the $\alpha_\Delta$ are uniquely determined once we have fixed the polyhedral complex $\mathscr{D}$. This implies that $\alpha$ is symmetric if and only if each superform $\alpha_\Delta$ is symmetric. It is furthermore a direct consequence of Proposition 1.2 (e), (f) that $\alpha$ is (weakly/strongly) positive if and only if each $\alpha_\Delta$ is (weakly/strongly) positive.

**Example 1.5.** Let $\phi : N_\mathbb{R} \to \mathbb{R}$ be a piecewise smooth function. By definition we can find an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ with support $N_\mathbb{R}$ and a family $(\phi_\sigma)_{\sigma\in\mathscr{C}}$ of smooth functions $\phi_\sigma : \sigma \to \mathbb{R}$ such that $\phi|_\sigma = \phi_\sigma$ holds for all $\sigma \in \mathscr{C}$. The following properties are equivalent for a convex open subset $\tilde{\Omega}$ of $N_\mathbb{R}$:

- (i) The function $\phi$ is convex on $\tilde{\Omega}$.
- (ii) The supercurrent $d'd''[\phi|_{\tilde{\Omega}}]$ is positive on $\tilde{\Omega}$.
- (iii) Each function $\phi_\sigma$ is convex on $\tilde{\Omega}$ and the corner locus $\phi\cdot N_\mathbb{R}$ is effective on $\tilde{\Omega}$.

Here, the corner locus $\phi\cdot N_\mathbb{R}$ is a tropical cycle on $N_\mathbb{R}$ of codimension 1 which might be viewed as the tropical Weil divisor associated to the piecewise smooth function $\phi$ (see [GK14, Definition 1.10] for details).

*Proof.* The equivalence of (i) and (ii) is [Lag12, Prop. 2.5]. We have

$$d'd''[\phi] = \sum_\sigma (d'd''\phi_\sigma)\wedge\delta_\sigma + \delta_{\phi\cdot N_{U,\mathbb{R}}}$$

by [GK14, Cor. 3.20] where $\sigma$ runs over the maximal polyhedra in $\mathscr{C}$. It follows that the supercurrent $d'd''[\phi]$ is polyhedral. Then the equivalence of (ii) and (iii) follows from Example 1.4. □

**1.6.** Let $\tilde{\Omega}$ be an open subset of $N_\mathbb{R}$. Recall from [GK14, Def. 2.9] that a $\delta$-preform on $\tilde{\Omega}$ is a supercurrent $\alpha \in D(\tilde{\Omega})$ of the form

$$\alpha = \sum_{i\in I} \alpha_i \wedge \delta_{C_i},$$

where $I$ is a finite set, $\delta_{C_i}$ is the supercurrent of integration over a tropical cycle $C_i$ in $N_\mathbb{R}$ with smooth weights and $\alpha_i$ is a superform on $\tilde{\Omega}$. The wedge product of superforms and the tropical intersection product makes $P^{\cdot,\cdot}(\tilde{\Omega})$ into a bigraded algebra [GK14, Proposition 2.12]. By definition, the bigrading and the positivity notions are induced by the corresponding notions in $D^{\cdot,\cdot}(\tilde{\Omega})$. Note that every $\delta$-preform is a polyhedral supercurrent (see Example 1.4).

Explicitly, the $\delta$-preform $\alpha$ has bidegree $(p,q)$ if and only if $\alpha_i \in A^{p_i,q_i}(\tilde{\Omega})$ and $C_i$ is a tropical cycle of codimension $l_i$ in $N_\mathbb{R}$ with $p_i + l_i = p$ and $q_i + l_i = q$ for all $i \in I$. We say that the $\delta$-preform $\alpha$ has *codimension* $l$ if there is a decomposition with all $C_i$ of codimension $l$. We define a trigrading $P^{s,t,l}(\tilde{\Omega})$ considering all $\delta$-preforms $\alpha$ as above with $p_i = s$, $q_i = t$ and $l_i = l$.

In the following, we consider an open subset $\Omega$ of the support $|\mathscr{C}|$ for the $n$-dimensional tropical cycle $C = (\mathscr{C}, m)$ in $N_{\mathbb{R}}$ with positive constant weights. The goal is to transfer the above notions to this relative situation.

**1.7.** We choose an open subset $\tilde{\Omega}$ in $N_{\mathbb{R}}$ with $\Omega = \tilde{\Omega} \cap |\mathscr{C}|$. The bigraded algebra of superforms on $\Omega$ is defined by $A(\Omega) := \{\alpha|_{|\mathscr{C}|} \mid \alpha \in A(\tilde{\Omega})\}$, the subalgebra of superforms with compact support in $\Omega$ is denoted by $A_c(\Omega)$ and the space of supercurrents $D(\Omega)$ is given by the linear functionals on $A_c(\Omega)$ which are induced by those supercurrents in $D(\tilde{\Omega})$ with support in $\Omega$.

A partition of unity argument shows that these definitions do not depend on the choice of $\tilde{\Omega}$ and the same holds for all definitions below (see [GK14, 3.1-3.3] for details). Note that on $A(\Omega)$ and dually on $D(\Omega)$, we have canonical differential operators $d', d''$ which are analogues of $\partial$ and $\bar{\partial}$ in complex analysis (see [CLD12, 1.4.7] or [Gub13a, 3.2]).

(i) Recall from [GK14, 3.4] that the space $P(\Omega)$ of *$\delta$-preforms on* $\Omega$ is the image of the natural map

$$P(\tilde{\Omega}) \longrightarrow D(\Omega) \subseteq D(\tilde{\Omega}), \ \tilde{\alpha} \mapsto \tilde{\alpha} \wedge \delta_C.$$

In fact, $\tilde{\alpha} \wedge \delta_C$ induces a polyhedral supercurrent on $\tilde{\Omega}$ with support in $\Omega$.

(ii) We give $P(\Omega)$ the unique structure as a bigraded algebra such that the surjective map $P(\tilde{\Omega}) \to P(\Omega)$ is a homomorphism of bigraded algebras. Similarly, we define the grading by *codimension* on $P(\Omega)$ and the trigrading $P^{s,t,l}(\Omega)$ with $l$ the codimension and $(p,q) = (s+l, t+l)$ the type of the $\delta$-preform.

(iii) A $\delta$-preform $\alpha \in P^{p,p}(\Omega)$ is called *(weakly/strongly) positive* if and only if $\alpha$ induces a (weakly/strongly) positive supercurrent in $D^{p,p}(\tilde{\Omega})$.

(iv) Note that $\delta$-preforms on $\Omega$ of codimension 0 are the same as superforms on $\Omega$. In particular, the above gives the corresponding positivity notions for superforms on $\Omega$.

**Proposition 1.8.** *For an open subset $\Omega$ of $|\mathscr{C}|$, we have the following properties:*

(a) $P^{p,p}_{\mathrm{s}+}(\Omega) \subseteq P^{p,p}_{+}(\Omega) \subseteq P^{p,p}_{\mathrm{w}+}(\Omega)$.

(b) *For $p = 0, 1, n-1, n$, equality holds everywhere in (a) and more precisely*

$$\begin{aligned} P^{t+l,t+l}_{\mathrm{s}+}(\Omega) \cap P^{t,t,l}(\Omega) &= P^{t+l,t+l}_{+}(\Omega) \cap P^{t,t,l}(\Omega) \\ &= P^{t+l,t+l}_{\mathrm{w}+}(\Omega) \cap P^{t,t,l}(\Omega) \end{aligned}$$

*for all $l$ and $t = 0, 1, n-l-1, n-l$.*

*If $\Omega$ is an open subset of $N_{\mathbb{R}}$ (and $C = N_{\mathbb{R}}$ with weight 1), then the following additional properties hold:*

(c) *The pull-back of a (weakly/strongly) positive $\delta$-preform on $\Omega$ with respect to an affine map is a (weakly/strongly) positive $\delta$-form.*

(d) *The wedge product of a strongly positive $\delta$-preform with a (weakly/strongly) positive $\delta$-preform on $\Omega$ is (weakly/strongly) positive.*

*Proof.* Properties (a) and (b) follow immediately from Proposition 1.2. If $\Omega$ is an open subset of $N_{\mathbb{R}}$, then the stable intersection product and the pull-back with respect to affine maps are well-defined as $\delta$-preforms. To prove (c) (resp. (d)), we use also (2.12.3) (resp. (2.12.5)) in [GK14]. $\square$

**Lemma 1.9.** *Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map and let $C'$ be an effective $n$-dimensional tropical cycle in $N'_{\mathbb{R}}$ with constant weights and $F_*(C') = C$. We consider an open subset $\tilde{\Omega}$ of $N_{\mathbb{R}}$ and $\tilde{\beta} \in P^{p,p}(\tilde{\Omega})$. For $\Omega := \tilde{\Omega} \cap |\mathscr{C}|$, the $\delta$-preform $\beta := \tilde{\beta} \wedge \delta_C$ is (weakly/strongly) positive on $\Omega$ if $\beta' := F^*(\tilde{\beta}) \wedge \delta_{C'}$ is (weakly/strongly) positive on $\Omega' := F^{-1}(\Omega) \cap |C'|$.*

*Proof.* We are here essentially in the setup of the projection formula given in [GK14, Proposition 2.14]. We may assume without loss of generality that $\tilde{\beta}$ is a $\delta$-preform of pure codimension $l$. Hence we can write

$$\tilde{\beta} = \sum_{i \in I} \alpha_i \wedge \delta_{C_i}$$

for suitable $\alpha_i \in A^{p-l,p-l}(\tilde{\Omega})$ and tropical cycles $C_i$ of codimension $l$ in $N_{\mathbb{R}}$. We write $C' = (\mathscr{C}', m')$. After suitable refinements we may assume that $F_*(\mathscr{C}')$ is a polyhedral subcomplex of $\mathscr{C}$ and that $\mathscr{C}$ and $\mathscr{C}'$ are polyhedral complexes of definition for $\tilde{\beta}$ and $\tilde{\beta}'$. We get polyhedral decompositions

$$\beta = \tilde{\beta} \wedge \delta_C = \tilde{\beta} \wedge \delta_{F_* C'} = \sum_{\sigma \in \mathscr{C}_{n-l}} \alpha_\sigma \wedge \delta_\sigma,$$

$$\beta' = F^*(\tilde{\beta}) \wedge \delta_{C'} = \sum_{\sigma' \in \mathscr{C}'_{n-l}} \alpha_{\sigma'} \wedge \delta_{\sigma'}$$

as in [GK14, (2.14.3), (2.14.4)]. Given $\sigma \in \mathscr{C}_{n-l}$, we have as in *loc. cit.*

$$\alpha_\sigma = \sum_{\substack{\sigma' \in \mathscr{C}'_{n-l} \\ F(\sigma') = \sigma}} \left[N_\sigma : \mathbb{L}_F(N'_\sigma)\right] \cdot \tilde{\alpha}_{\sigma'} \tag{1.9.1}$$

where $\tilde{\alpha}_{\sigma'}$ denotes the unique superform in $A_\sigma(\sigma \cap \tilde{\Omega})$ such that $F^*(\tilde{\alpha}_{\sigma'}) = \alpha_{\sigma'}$ in $A_{\sigma'}(\sigma' \cap F^{-1}(\tilde{\Omega}))$. Now assume that $\beta'$ is positive. This implies by definition that all the $\alpha_{\sigma'}$ are positive. If $F(\sigma') = \sigma$ then $F$ induces an isomorphism from $\sigma'$ to $\sigma$ and $\tilde{\alpha}_{\sigma'}$ is positive. Then formula (1.9.1) implies that $\alpha_\sigma$ is positive as well. In the same way one proves the variants of the lemma for weak and strong positivity. □

**Definition 1.10.** Recall that the tropical cycle $C = (\mathscr{C}, m)$ has dimension $n$. Let $\Omega$ be an open subset of $|\mathscr{C}|$. A supercurrent $T \in D^{p,p}(\Omega)$ is called *(weakly/strongly) positive* if

$$\langle T, \alpha \rangle \geq 0$$

holds for all (strongly/weakly) positive superforms $\alpha \in A_c^{n-p,n-p}(\Omega)$. The corresponding spaces of (weakly/strongly) positive supercurrents are denoted by $D_+^{p,p}(\Omega)$, $D_{\mathrm{w}+}^{p,p}(\Omega)$ and $D_{\mathrm{s}+}^{p,p}(\Omega)$.

**Remark 1.11.** From Proposition 1.8, we obtain immediately

$$D_{\mathrm{s}+}^{p,p}(\Omega) \subseteq D_+^{p,p}(\Omega) \subseteq D_{\mathrm{w}+}^{p,p}(\Omega)$$

with equalities for $p = 0, 1, n-1, n$.

A superform on $\Omega$ is (weakly/strongly) positive if and only if its associated supercurrent is (weakly/strongly) positive in $D(\Omega)$. More generally, a $\delta$-preform on $\Omega$ is (weakly/strongly) positive if and only if its associated supercurrent has the same positivity property in $D(\Omega)$. The proof is similar to the proof of Proposition 2.13 and we leave the details to the reader.

**Remark 1.12.** (i) Let $\tilde{\Omega}$ be an open subset of $N_{\mathbb{R}}$ and $p \in \mathbb{N}$. If $\alpha \in A^{p,p}(\tilde{\Omega})$ is a superform such that $\alpha$ and $-\alpha$ are weakly positive, then $\alpha = 0$. This follows from the fact that every symmetric superform in $A^{p,p}(\tilde{\Omega})$ is the difference of two strongly positive elements in $A^{p,p}(\tilde{\Omega})$ [CLD12, Lemme 5.2.3] and from the duality in Proposition 1.2(e).

(ii) Recall that $\Omega$ is an open subset of $|\mathscr{C}|$ for the effective tropical cycle $C = (\mathscr{C}, m)$ with constant weights. We consider a $\delta$-preform $\alpha \in P^{p,p}(\Omega)$. After a subdivision of $\mathscr{C}$ we may write

$$\alpha = \sum_{\Delta \in \mathscr{C}} \alpha_\Delta \delta_\Delta \in D(\Omega) \tag{1.12.1}$$

with superforms $\alpha_\Delta$ on the open subsets $\Omega \cap \Delta$ of $\Delta$. The representation (1.12.1) is unique up to subdivision. Now assume that $\alpha$ and $-\alpha$ are weakly positive. Then each $\alpha_\Delta$ must be weakly positive and (i) implies $\alpha = 0$.

## 2. Positive delta-forms and delta-currents

Let $K$ be an algebraically closed field endowed with a complete non-trivial non-archimedean absolute value $|\ |$. In the following, we will always work in this setup except in Section 4. This is no restriction of generality as this situation can be always achieved by a base change.

Let $X$ be an algebraic variety of dimension $n$ over $K$. For an open subset $W$ of the Berkovich space $X^{\mathrm{an}}$, we use the bigraded algebra of generalized $\delta$-forms $P^{\cdot,\cdot}(W)$ and its bigraded subalgebra $B^{\cdot,\cdot}(W)$ of $\delta$-forms. The latter is a differential algebra with respect to differential operators $d', d''$ and behaves similarly as the algebra of differential forms on a complex manifold with respect to $\partial, \bar{\partial}$ (see [GK14, §4] for details). As a topological dual of $B_c(W)$, we have the space of $\delta$-currents $E(W)$ (see [GK14, §6]), where the subscript $c$ means always compact support. The smooth forms from [CLD12, §3] give a bigraded differential subalgebra $A^{\cdot,\cdot}(W)$ of $B^{\cdot,\cdot}(W)$ inducing a canonical linear map from $E(W)$ to the space of currents $D(W)$, where the latter is defined as a topological dual of $A_c(W)$ in [CLD12, §4].

The goal of this section is to transfer the positivity notions from Section 1 to the sheaves $B^{\cdot,\cdot}, P^{\cdot,\cdot}, E^{\cdot,\cdot}$ and to compare them with the positivity notions on $A^{\cdot,\cdot}, D^{\cdot,\cdot}$ introduced in [CLD12, §5]

**2.1.** We start with a *tropical chart* $(V, \varphi_U)$ on $X^{\mathrm{an}}$. Recall from [Gub13a, 4.15] that this is a very affine open subset $U$ of $X$ with a canonical closed embedding $\varphi_U : U \to T_U$ into the torus $T_U$ with the character lattice $M_U = \mathscr{O}(U)^\times / K^\times$ and an open subset $V := \mathrm{trop}_U^{-1}(\Omega)$ for an open subset of the tropical variety $\mathrm{Trop}(U) = \mathrm{trop}_U(U^{\mathrm{an}})$. Here, we have used the tropicalization map $\mathrm{trop}_U : U^{\mathrm{an}} \to N_{U,\mathbb{R}}$ for the cocharacter lattice $N_U = \mathrm{Hom}(M_U, \mathbb{Z})$. Note that tropical charts form a basis of topology for $X^{\mathrm{an}}$ [Gub13a, Proposition 4.16].

Recall from [GK14, §4] that the bigraded subalgebra $P^{\cdot,\cdot}(V, \varphi_U)$ of $P^{\cdot,\cdot}(V)$ is given by the generalized $\delta$-forms which are representable by some $\delta$-preform in $P^{\cdot,\cdot}(\tilde{\Omega})$ for an open subset $\tilde{\Omega}$ of $N_{U,\mathbb{R}}$ with $\Omega = \tilde{\Omega} \cap \mathrm{Trop}(U)$. In fact, every generalized $\delta$-form $\beta$ is locally given by elements $\beta_U$ in such a $P^{\cdot,\cdot}(V,\varphi_U)$. If $\beta_U$ is represented by a $\delta$-preform of codimension $l$, then we say that $\beta_U$ has *codimension* $l$. The grading by codimension $l$ leads to subspaces $P^{s,t,l}(U,\varphi_U)$ of $P^{s+l,t+l}(U,\varphi_U)$ and hence to subspaces $P^{s,t,l}(W)$ of $P^{s+l,t+l}(W)$.

**Definition 2.2.** For every morphism $f : X' \to X$ of varieties over $K$ and any pair of charts $(V',\varphi_{U'})$ on $X'^{\mathrm{an}}$ and $(V,\varphi_U)$ on $X^{\mathrm{an}}$ with $f(U') \subseteq U$ and $f(V') \subseteq V$, we have a pull-back $f^* : P^{p,p}(V,\varphi_U) \to P^{p,p}(V',\varphi_{U'})$ and a restriction map

$$P^{p,p}(V',\varphi_{U'}) \to P^{p,p}(\Omega'),\ \beta \mapsto f^*(\beta)|_{\Omega'}$$

to $\delta$-preforms on $\Omega' := \mathrm{trop}_{U'}(V')$.

(i) We say that $\beta \in P^{p,p}(V,\varphi_U)$ is *(weakly/strongly) positive* if $f^*(\beta)|_{\Omega'}$ is a (weakly/strongly) positive $\delta$-preform on $\Omega'$ for every $f : X' \to X$ and every $(V',\varphi_{U'})$ as above. These forms yield subspaces

$$P^{p,p}_{\mathrm{s}+}(V,\varphi_U) \subseteq P^{p,p}_{+}(V,\varphi_U) \subseteq P^{p,p}_{\mathrm{w}+}(V,\varphi_U) \tag{2.2.1}$$

of $P^{p,p}(V,\varphi_U)$ with equality in the cases $p = 0,1,n-1,n$. Obviously, all these positivity notions are stable with respect to pull-back $f^* : P^{p,p}(V,\varphi_U) \to P^{p,p}(V',\varphi_{U'})$ for any $f$ as above.

(iii) We say that $\beta \in P^{p,p}(V,\varphi_U)$ is *(weakly/strongly) positively representable* if there exists an open subset $\tilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \tilde{\Omega} \cap \mathrm{Trop}(U)$ and a (weakly/strongly) positive $\tilde{\beta} \in P^{p,p}(\tilde{\Omega})$ representing $\beta$. In this case we call $\tilde{\beta}$ a *positive representative of* $\beta$.

**2.3.** Using Proposition 1.8, the wedge-product of a strongly positively representable element with a (weakly/strongly) positively representable element is (weakly/strongly) positively representable. The notion of (weakly/strongly) positive representability in $P^{p,p}(V,\varphi_U)$ is closed under pull-back. It follows that a (weakly/strongly) positively representable element $\beta \in P^{p,p}(V,\varphi_U)$ is (weakly/strongly) positive.

**Lemma 2.4.** *Let $f : X' \to X$ be a generically finite dominant morphism of varieties over $K$ , let $(V,\varphi_U)$ be a tropical chart on $X^{\mathrm{an}}$ and let $U'$ be a very affine open subset of $X'$ with $f(U') \subseteq U$. Then $(V',\varphi_{U'})$ is a tropical chart on $X'^{\mathrm{an}}$ for $V' := f^{-1}(V) \cap U'^{\mathrm{an}}$. Moreover, $\beta \in P^{p,p}(V,\varphi_U)$ is (weakly/strongly) positive if and only if $f^*(\beta) \in P^{p,p}(V',\varphi_{U'})$ is (weakly/strongly) positive.*

*Proof.* Let $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ be the canonical integral $\mathbb{R}$-affine map induced by $f : U' \to U$. Then $\Omega' := F^{-1}(\Omega) \cap \mathrm{Trop}(U')$ is an open subset of $\mathrm{Trop}(U')$ and functoriality of tropicalizations shows that $V' = \mathrm{trop}_{U'}^{-1}(\Omega')$. We conclude that $(V',\varphi_{U'})$ is a tropical chart on $X'^{\mathrm{an}}$. The Sturmfels–Tevelev multiplicity formula shows

$$F_*(\mathrm{Trop}(U')) = \deg(f)\mathrm{Trop}(U) \tag{2.4.1}$$

(see [ST08], [BPR11] or [Gub13b, Thm. 13.17]). Using also that our positivity notions are stable under pull-back, the last claim follows from Lemma 1.9 □

**Remark 2.5.** Let $\alpha \in P^{p,p}(V,\varphi_U)$ such that $\alpha$ and $-\alpha$ are weakly positive. Then $\alpha = 0$. This follows from Remark 1.12 applied to all compatible pairs of charts as in 2.2.

In the following, $W$ is an open subset of $X^{\mathrm{an}}$. We introduce the above positivity notions on the space $P^{p,p}(W)$ of generalized $\delta$-forms on $W$.

**Definition 2.6.** A generalized $\delta$-form $\beta \in P^{p,p}(W)$ is called *(weakly/strongly) positive* if at any given point of $W$ there exists a tropical chart $(V, \varphi_U)$ such that $V \subseteq W$ and $\beta|_V = \mathrm{trop}_U^*(\beta_U)$ for a (weakly/strongly) positive element $\beta_U \in P^{p,p}(V, \varphi_U)$. Note that such a $\beta_U$ is uniquely determined by $(V, \varphi_U)$ [GK14, Proposition 4.18]. These generalized $\delta$-forms define subspaces

$$P^{p,p}_{\mathrm{s}+}(W) \subseteq P^{p,p}_{+}(W) \subseteq P^{p,p}_{\mathrm{w}+}(W) \tag{2.6.1}$$

of $P^{p,p}(W)$. For $p = 0, 1, n-1, n$, we have equalities in the above chain.

Similarly, we define *(weakly/strongly) positively representable* generalized $\delta$-forms in $P^{p,p}(W)$. For $p = 0, 1$, these three positivity notions agree again. All six positivity notions are closed under pull-back.

**Proposition 2.7.** *Let $\beta \in P^{p,p}(W)$ be a (weakly/strongly) positive generalized $\delta$-form on an open subset $W$ of $X^{\mathrm{an}}$. Let $(V, \varphi_U)$ be a tropical chart of $X^{\mathrm{an}}$ such that $V \subseteq W$ and $\beta|_V$ is induced by $\beta_U \in P^{p,p}(V, \varphi_U)$. Then $\beta_U$ is (weakly/strongly) positive.*

*Proof.* First, we note that if $U'$ is a very affine open subset of $X$ and if $\beta$ is given on tropical charts $(V'_j, \varphi_{U'})_{j \in J}$ in $W$ by a (weakly/strongly) positive $\beta_j \in P^{p,p}(V'_j, \varphi_{U'})$, then $\beta$ is given on the tropical chart $(V' := \bigcup_{j \in J} V'_j, \varphi_{U'})$ by a unique (weakly/strongly) positive $\beta_{U'} \in P^{p,p}(V', \varphi_{U'})$. Existence and uniqueness follow from [GK14, Proposition 4.12]. Positivity follows from the fact that the positivity notions of $\delta$-preforms are defined locally using that we always have the same tropicalization map $\mathrm{trop}_{U'}$.

Using this property and properness of $\mathrm{trop}_U$, we may assume that $V$ is relatively compact. Therefore $V$ may be covered by finitely many tropical charts $(V_i, \varphi_{U_i})$, $i = 1, \ldots, s$, such that $\beta$ is given on any $V_i$ by a (weakly/strongly) positive $\beta_i \in P^{p,p}(V_i, \varphi_{U_i})$. Then $U' := U \cap U_1 \cap \cdots \cap U_s$ is a very affine open subset of $X$ [Gub13a, 4.13]. For $V' := U'^{\mathrm{an}} \cap \bigcup_{i=1}^s V_i$, we get a tropical chart $(V', \varphi_{U'})$ as in the proof of [Gub13a, Proposition 5.13]. Again using the property at the beginning of the proof, we deduce that $\beta$ is given on $(V', \varphi_{U'})$ by the unique (weakly/strongly) positive $\beta_{U'} \in P^{p,p}(V', \varphi_{U'})$ which agrees with $\beta_i$ on $U'^{\mathrm{an}} \cap V_i$. The restriction of $\beta_{U'}$ to the tropical subchart $(V \cap U'^{\mathrm{an}}, \varphi_{U'})$ remains (weakly/strongly) positive. Since $U' \subseteq U$, Lemma 2.4 proves the claim. □

**Remark 2.8.** Chambert-Loir and Ducros have introduced subspaces

$$A^{p,p}_{\mathrm{s}+}(W) \subseteq A^{p,p}_{+}(W) \subseteq A^{p,p}_{\mathrm{w}+}(W) \tag{2.8.1}$$

of strongly-positive, positive, and weakly positive smooth forms. Since they use analytic moment maps, we would like to rephrase their definition in terms of tropical charts and the language of [Gub13a]. A smooth form $\alpha \in A^{p,p}(W)$ is (weakly/strongly) positive if whenever $(V, \varphi_U)$ is a tropical chart of $X^{\mathrm{an}}$ such that $V \subseteq W$ and $\alpha|_V$ is induced by $\alpha_U \in A^{p,p}(\Omega)$ for $\Omega = \mathrm{trop}_U(V)$ then $\alpha_U$ is (weakly/strongly) positive, i.e. the restriction of the superform $\alpha_U$ to any face of $\mathrm{Trop}(U)$ is (weakly/strongly) positive. This is a direct consequence of [Gub13b, Proposition 7.2]. We see in particular that we get (2.8.1) as the intersection of (2.6.1) with $A^{p,p}(W)$.

**Proposition 2.9.** *On an open subset $W$ of $X^{\mathrm{an}}$, the following holds:*

(a) *The product of a strongly positively representable generalized $\delta$-form with a (weakly/strongly) positively representable generalized $\delta$-form is a (weakly/strongly) positively representable generalized $\delta$-form.*
(b) *The product of a strongly positively representable generalized $\delta$-form with a (weakly/strongly) positive generalized $\delta$-form is a (weakly/strongly) positive generalized $\delta$-form.*
(c) *The product of a (weakly/strongly) positively representable generalized $\delta$-form of type $(p,p)$ with a (strongly/weakly) positive generalized $\delta$-form of type $(n-p,n-p)$ is a positive generalized $\delta$-form.*

*We can replace positively representable by positive in (b) and (c) if at least one of the two factors is a smooth form.*

*Proof.* Let $(V,\varphi_U)$ be a tropical chart in $W$. It is enough to show the properties (a) and (b) for $\alpha \wedge \beta$ with $\alpha \in P^{p,p,l}(V,\varphi_U)$ and $\beta \in P^{p',p',l'}(V,\varphi_U)$. Then (a) follows from Proposition 1.8(d).

For (b), we choose $\delta$-preforms $\tilde{\alpha}, \tilde{\beta}$ on an open subset $\tilde{\Omega}$ of $N_{U,\mathbb{R}}$ which represent $\alpha, \beta$, where $\Omega = \mathrm{trop}_U(V) = \tilde{\Omega} \cap \mathrm{Trop}(U)$. Since the positivity notions are functorial, it is enough to show that $\alpha \wedge \beta|_\Omega$ is a (weakly/strongly) positive $\delta$-preform. Let $\mathscr{C}$ be a polyhedral complex of definition for $\tilde{\alpha}$ which means

$$\tilde{\alpha} = \sum_{\Delta \in \mathscr{C}^l} \tilde{\alpha}_\Delta \wedge \delta_\Delta$$

for $\tilde{\alpha}_\Delta \in A^{p,p}(\tilde{\Omega} \cap \Delta)$. Since $\alpha$ is strongly positively representable, the superform $\tilde{\alpha}_\Delta$ is strongly positive on $\tilde{\Omega} \cap \Delta$. We may also assume that $\mathscr{C}$ is a polyhedral complex of definition for $\tilde{\beta}$ and that $\mathrm{Trop}(U)$ is a subcomplex of $\mathscr{C}$. Note that the $\delta$-preform $\beta|_\Omega = \tilde{\beta} \wedge \delta_{\mathrm{Trop}(U)}$ has the polyhedral decomposition

$$\beta|_\Omega = \sum_{\Delta' \in \mathscr{C}^{l'}} \beta_{\Delta'} \wedge \delta_{\Delta'} \in D(\Omega)$$

for (weakly/strongly) positive $\beta_\Delta \in A^{p',p'}(\Omega \cap \Delta)$. By the formula (2.12.3) in [GK14] for the polyhedral representation of the product of $\delta$-preforms and using $\alpha \wedge \beta|_\Omega = \tilde{\alpha} \wedge (\tilde{\beta} \wedge \delta_{\mathrm{Trop}(U)})$, we have

$$\alpha \wedge \beta|_\Omega = \sum_{\tau \in \mathscr{C}^{l+l'}} \sum_{\Delta, \Delta'} [N : N_\Delta + N_{\Delta'}] \cdot \tilde{\alpha}_\Delta \wedge \beta_{\Delta'} \wedge \delta_\tau \in D(\Omega),$$

where $\Delta, \Delta'$ range over all pairs in $\mathscr{C}^l \times \mathscr{C}^{l'}$ with $\tau = \Delta \cap \Delta'$ and with $\Delta \cap (\Delta' + \varepsilon v) \neq \emptyset$ for a fixed generic vector $v \in N_{U,\mathbb{R}}$ and for $\varepsilon > 0$ sufficiently small. Proposition 1.2 shows now that $\alpha \wedge \beta|_\Omega$ is a (weakly/strongly) positive $\delta$-preform. This proves (b).

We can prove (c) in the same way as (b) if we observe Proposition 1.2 (e). If $\alpha$ is a smooth form given on $(V,\varphi_U)$ by a superform $\alpha_U$ on $\Omega$, then

$$\alpha \wedge \beta|_\Omega = \sum_{\Delta' \in \mathscr{C}} \alpha_U \wedge \beta_{\Delta'} \wedge \delta_{\Delta'} \in D(\Omega)$$

and the last claim follows again from Proposition 1.2. $\square$

**Definition 2.10.** A $\delta$-current $T \in E^{p,p}(W)$ is called *(weakly/strongly) positive* if $T$ is symmetric and if $\langle T, \beta\rangle \geq 0$ for all (strongly/weakly) positive $\delta$-forms $\beta \in B_c^{n-p,n-p}(W)$.

Recall from [GK14, Proposition 6.6] that we have a natural map $P^{p,q}(W) \to E^{p,q}(W)$, $\alpha \mapsto [\alpha]_E$ determined by

$$\langle [\alpha]_E, \beta\rangle = \int_W \alpha \wedge \beta$$

for all $\beta \in B_c^{n-p,n-q}(W)$, inducing the map $A^{p,q}(W) \to D^{p,q}(W)$, $\alpha \mapsto [\alpha]_D$.

**Corollary 2.11.** *Let $\alpha \in P^{p,p}(W)$ be a (weakly/strongly) positively representable generalized $\delta$-form. Then $[\alpha]_E$ is a (weakly/strongly) positive $\delta$-current.*

*Proof.* We have to show $\int_W \alpha \wedge \beta \geq 0$ for every (strongly/weakly) positive $\beta \in B_c^{n-p,n-p}(W)$. This follows from Proposition 2.9(c). □

**Remark 2.12.** A symmetric current $T \in D^{p,p}(W)$ is called *(weakly/strongly) positive* if $\langle T, \alpha\rangle \geq 0$ for all (strongly/weakly) positive smooth forms $\alpha \in A^{n-p,n-p}(W)$ with compact support (see [CLD12, §5.3]). Since every smooth form is a $\delta$-form, Remark 2.8 shows that any (weakly/strongly) positive $\delta$-current induces a (weakly/strongly) positive current.

**Proposition 2.13.** *Let $\beta \in P^{p,p}(W)$. Then $\beta$ is a (weakly/strongly) positive generalized $\delta$-form if and only if for every morphism $f : X' \to X$ of varieties the current $[f^*(\beta)]_D \in D^{p,p}(f^{-1}(W))$ is (weakly/strongly) positive.*

*Proof.* We may assume that $\beta$ has codimension $l$ and let $\alpha \in A_c^{n-p,n-p}(W)$. By [GK14, Proposition 5.7], there is a very affine open subset $U \subseteq X$ and an open subset $\Omega$ of $\mathrm{Trop}(U)$ such that $V := \mathrm{trop}_U^{-1}(\Omega)$ contains the support of $\alpha \wedge \beta \in P^{n,n}(W)$ and such that $\alpha$ (resp. $\beta$) is given on $V$ by $\alpha_U \in A^{n-p,n-p}(\Omega)$ (resp. $\beta_U \in P^{p,p}(V, \varphi_U)$). Then $\alpha_U \wedge \beta_U$ has compact support in $\Omega$ [GK14, Proposition 4.21]. Since $\beta_U$ has codimension $l$, there is a polyhedral complex $\mathscr{C}$ of definition for $\beta_U|_\Omega$ with polyhedral representation

$$\beta_U|_\Omega = \sum_{\Delta \in \mathscr{C}^l} \beta_\Delta \wedge \delta_\Delta$$

for superforms $\beta_\Delta \in A^{p-l,p-l}(\Omega \cap \Delta)$. By [GK14, Definition 5.8], we have

$$\langle [\beta]_D, \alpha\rangle = \int_W \beta \wedge \alpha = \int_{|\mathrm{Trop}(U)|} \beta_U \wedge \alpha_U = \sum_{\Delta \in \mathscr{C}^l} \int_\Delta \alpha_\Delta \wedge \beta_\Delta. \tag{2.13.1}$$

Suppose that $\beta$ is a (weakly/strongly) positive generalized $\delta$-form and that $\alpha \in A_c^{n-p,n-p}(W)$ is a (strongly/weakly) positive smooth form. It follows from Proposition 2.9 that $\beta \wedge \alpha$ is a positive generalized $\delta$-form of type $(n, n)$. By Proposition 2.7, the superform $\alpha_\Delta \wedge \beta_\Delta$ of type $(n-l, n-l)$ is positive on $\Delta \cap \Omega$. We conclude that the integral in (2.13.1) is non-negative and hence $[\beta]_D$ is a positive current on $W$. If $f : X' \to X$ is a morphism of varieties, then $f^*(\beta)$ is a (weakly/strongly) positive generalized $\delta$-form on $f^{-1}(W)$ and hence $[f^*(\beta)]_D$ is a positive current on $f^{-1}(W)$.

Conversely, assume that $[f^*(\beta)]_D$ is a positive current on $f^{-1}(W)$ for all morphisms $f : X' \to X$. Using this functoriality, it is enough to show that $\beta_U|_\Omega$ is a (weakly/strongly) positive $\delta$-preform on $\Omega := \mathrm{trop}_U(V)$ for a tropical chart $(V, \varphi_U)$ where $\beta$ is given by $\beta_U \in P^{p,p}(V, \varphi_U)$. By assumption, the integral in (2.13.1) is non-negative for $\alpha$ given by a (strongly/weakly) positive superform $\alpha_U \in A_c^{n-p,n-p}(\Omega)$. It follows from Proposition 1.2(e) that $\beta_\Delta$ is (weakly/strongly) positive in $A^{n-l,n-l}(\Omega \cap \Delta)$ for every $\Delta \in \mathscr{C}^l$. This proves that $\beta_U|_\Omega$ is a (weakly/strongly) positive $\delta$-preform. □

**Remark 2.14.** The same argument shows for a smooth form $\beta \in A^{p,p}(W)$ that $\beta$ is (weakly/strongly) positive if and only if the associated current $[\beta]_D$ is (weakly/strongly) positive on $W$, and furthermore this is equivalent that the $\delta$-current $[\beta]_E$ is (weakly/strongly) positive on $W$.

**Lemma 2.15.** *Let $f : X' \to X$ be a generically finite dominant morphism of varieties over $K$. Then a generalized $\delta$-form $\beta$ on an open subset $W$ of $X^{\mathrm{an}}$ is (weakly/ strongly) positive if and only if $f^*(\beta)$ is (weakly/strongly) positive on $f^{-1}(W)$.*

*Proof.* This follows from Lemma 2.4. □

**Lemma 2.16.** *Let $W$ be an open subset of $X^{\mathrm{an}}$ and let $\alpha \in P^{p,p}(W)$ such that $\alpha$ and $-\alpha$ are weakly positive. Then $\alpha = 0$.*

*Proof.* We choose a tropical chart $(V, \varphi_U)$ of $X^{\mathrm{an}}$ with $V \subseteq W$ such that $\alpha$ is induced by $\alpha_U$ in $P^{p,p}(V, \varphi_U)$. We get from Proposition 2.7 that $\alpha_U$ and $-\alpha_U$ are positive in the sense of 2.2. We conclude from Remark 2.5 that $\alpha_U = 0$. It follows that $\alpha$ vanishes as well. □

## 3. Plurisubharmonic functions and metrics

Let $X$ be an $n$-dimensional variety over $K$. We recall first some definitions from [CLD12].

**Definition 3.1.** A continuous real function $f$ on an open subset $W$ of $X^{\mathrm{an}}$ is called *plurisubharmonic* or *psh* if $d'd''[f]_D$ is a positive current in $D^{1,1}(W)$. This means that $d'd''[f]_D$ has to be non-negative on positive forms in $A_c^{n-1,n-1}(W)$.

**Remark 3.2.** There is an elementary way to describe when a smooth function $f$ is psh. Locally, there is a tropical chart $(V, \varphi_U)$ and a smooth function $\phi$ on $\Omega = \mathrm{trop}_U(V)$ with $f = \phi \circ \mathrm{trop}$. Then it follows from [CLD12, Lemma 5.5.3] that $f$ is psh on $V$ if and only if the restriction of $\phi$ to $\Delta$ is convex for any polyhedron $\Delta \subseteq \Omega$.

Next we show that Remark 3.2 doesn't hold without the smoothness assumption on $f$ (see §8 for a discussion of piecewise smooth functions).

**Example 3.3.** Let us consider the line $x_1 + x_2 = 1$ in $\mathbb{G}_m^2$. Then $\mathrm{Trop}(X)$ is the union of the three half lines $\{u \in \mathbb{R}_+^2 \mid u_1 = 0\}$, $\{u \in \mathbb{R}_+^2 \mid u_2 = 0\}$ and $\{u \in \mathbb{R}_-^2 \mid u_1 = u_2\}$, all equipped with weight 1. Let $\phi$ be the conic function determined by $\phi(1,0) = a$, $\phi(0,1) = b$ and $\phi(-1,-1) = c$. As above, we set $f := \phi \circ \mathrm{trop}$. Then the restriction of $\phi$ to any polyhedron $\Delta \subseteq \mathrm{Trop}(X)$ is linear and hence convex. However, if $\eta$ is a non-negative compactly supported smooth function on $\mathrm{Trop}(X)$ with $\eta(0,0)=1$, then we have

$$\langle d'd''[f]_E, \mathrm{trop}^*(\eta)\rangle = \langle[\phi], d'd''\eta\rangle = (a+b+c)\eta(0,0) = a+b+c$$

which gives a counterexample to $[f]$ psh if and only if $a+b+c<0$.

**3.4.** We give some variants of defining psh functions. We always consider a continuous function $f$ on an open subset $W$ of $X^{\mathrm{an}}$.

(a) $f$ is *$\delta$-psh* if $d'd''[f]_E$ is a positive $\delta$-current in $E^{1,1}(W)$ as defined in 2.10 using the $\delta$-current $[f]_E$ from [GK14, Proposition 6.16].
(b) $f$ is *functorial psh* if $f\circ\varphi$ is psh on $\varphi^{-1}(W)$ for all morphisms $\varphi : X' \to X$.
(c) $f$ is *functorial $\delta$-psh* if $f\circ\varphi$ is $\delta$-psh on $\varphi^{-1}(W)$ for all morphisms $\varphi : X' \to X$.

Clearly (c) implies (a) and (b). Furthermore either (a) or (b) implies that the function $f$ is psh.

**3.5.** In the following, $L$ is a line bundle on $X$. Let $\| \ \|$ be a continuous metric on $L^{\mathrm{an}}$ over an open subset $W$ of $X^{\mathrm{an}}$. Following [CLD12, 6.3.1] the metric $\| \ \|$ is called *psh* if $-\log\|s\|$ is a psh-function on $U^{\mathrm{an}}\cap W$ for any frame $s$ of $L$ over any open subset $U$. Note that this is equivalent to say that $[c_1(L,\| \ \|)]_D$ is a positive current on $W$. Similarly as above we say that

(a) $\| \ \|$ is *$\delta$-psh* if $-\log\|s\|$ is a $\delta$-psh-function on $U^{\mathrm{an}}\cap W$ for any frame $s$ of $L$ over any open subset $U$ of $X$.
(b) $\| \ \|$ is *functorial psh* if $\varphi^*\| \ \|$ is psh on $(\varphi^{\mathrm{an}})^{-1}(W)$ for all morphisms $\varphi : X' \to X$.
(c) $\| \ \|$ is *functorial $\delta$-psh* if $\varphi^*\| \ \|$ is $\delta$-psh on $(\varphi^{\mathrm{an}})^{-1}(W)$ for all morphisms $\varphi : X' \to X$.

Clearly (c) implies (a) and (b). Furthermore either (a) or (b) implies that the metric $\| \ \|$ is psh. Note also that $\| \ \|$ is $\delta$-psh if and only if the first Chern $\delta$-current $[c_1(L,\| \ \|)]_E$ (defined in [GK14, 7.7]) is a positive $\delta$-current on $W$.

**Proposition 3.6.** *Let $\varphi : X' \to X$ be a surjective proper morphism of $n$-dimensional varieties over $K$ and let $f$ be a continuous function on an open subset $W$ of $X^{\mathrm{an}}$. Then we have the projection formula*

$$\varphi_*[\varphi^* f]_E = \deg(\varphi)[f]_E,$$

*where $[\varphi^* f]_E$ is the $\delta$-current on $\varphi^{-1}(W)$ induced by $f\circ\varphi$.*

*Proof.* Let $\alpha \in B_c^{n,n}(W)$ and let $\mu_\alpha$ be the associated Radon measure on $W$ (see [GK14, Corollary 6.15]). It follows from the projection formula in [GK14, Proposition 5.9] that

$$\deg(\varphi)\int_W g\alpha = \int_{\varphi^{-1}(W)} \varphi^*(g)\varphi^*(\alpha) \tag{3.6.1}$$

for all smooth functions $g$ on $W$. Since the smooth functions with compact support in $W$ are dense in the space of continuous functions with compact support in $W$ equipped with the supremum norm [CLD12, Proposition 3.3.5], we conclude from (3.6.1) that $\deg(\varphi)\mu_\alpha = \varphi_*(\mu_{\varphi^*\alpha})$ as an identity of Radon measures. In particular, we get

$$\deg(\varphi)\langle [f]_E, \alpha\rangle = \deg(\varphi)\int_W f\, d\mu_\alpha = \int_W f\, d\varphi_*(\mu_{\varphi^*\alpha}) = \langle [\varphi^* f]_E, \varphi^*\alpha\rangle$$

proving the claim. □

In the following, we consider a continuous metric $\| \ \|$ on $L^{\mathrm{an}}$ over the open subset $W$ of $X^{\mathrm{an}}$ as before.

**Corollary 3.7.** *Let $\varphi : X' \to X$ be a surjective proper morphism of $n$-dimensional varieties over $K$. Then we have*

$$\varphi_*([c_1(\varphi^*(L)|_{\varphi^{-1}(W)}, \varphi^*\| \ \|)]_E) = \deg(\varphi)[c_1(L|_W, \| \ \|)]_E$$

*as an identity of $\delta$-currents on $W$.*

*Proof.* For a $\delta$-metric (i.e. $c_1(L, \| \ \|)$ is a $\delta$-form), this identity follows directly from the projection formula in [GK14, Proposition 5.9]. It is clear that $L$ has a $\delta$-metric $\| \ \|_0$ as we may choose a smooth metric [CLD12, Proposition 6.2.6] or a formal metric of a compactification of $X$ (see Section 5) using [GK14, Remark 9.16]. Using that the claim holds for $\| \ \|_0$ and linearity, it remains to prove the corollary in the special case $L = O_X$ with a metric induced by a continuous function $f$ on $W$, i.e. $f = -\log \|1\|$. For $\alpha \in B_c^{n,n}(W)$, we have

$$\langle \varphi_*([c_1(\varphi^*(L)|_{\varphi^{-1}(W)}, \varphi^*\| \ \|)]_E), \alpha\rangle = \langle d'd''[\varphi^* f]_E, \varphi^*\alpha\rangle = \langle \varphi_*[\varphi^* f]_E, d'd''\alpha\rangle$$

and

$$\langle [c_1(L|_W, \| \ \|)]_E, \alpha\rangle = \langle d'd''[f]_E, \alpha\rangle = \langle [f]_E, d'd''\alpha\rangle.$$

We conclude that the claim follows from Proposition 3.6. □

**Corollary 3.8.** *Let $\varphi : X' \to X$ be a surjective proper morphism of $n$-dimensional varieties over $K$ and let $\| \ \|$ be a continuous metric on $L^{\mathrm{an}}$ over the open subset $W$ of $X^{\mathrm{an}}$. If $\varphi^*\| \ \|$ is psh (resp. $\delta$-psh, resp. functorial psh, resp. functorial $\delta$-psh) over $\varphi^{-1}(W)$, then $\| \ \|$ is psh (resp. $\delta$-psh, resp. functorial psh, resp. functorial $\delta$-psh) over $W$.*

*Proof.* If $\varphi^*\| \ \|$ is $\delta$-psh over $\varphi^{-1}(W)$, then $\| \ \|$ is $\delta$-psh by Corollary 3.7. Indeed, the proper push-forward of a positive $\delta$-current is positive since positivity of $\delta$-forms is closed under pull-back. All these facts for $\delta$-currents yield immediately the corresponding facts for currents and so the same argument works for psh. Using a suitable cartesian diagram, the remaining two claims involving functoriality follow easily. □

## 4. Lifting varieties

Let $(F, | \ |)$ be a field with a non-archimedean absolute value. Let $F^\circ$, $F^{\circ\circ}$, and $\tilde{F} = F^\circ/F^{\circ\circ}$ denote the valuation ring, its maximal ideal, and the corresponding residue class field.

The following theorem enables us to lift closed subsets from the special fibre of a $F^\circ$-model to the generic fibre. Amaury Thuillier has told the authors that he has found a similar argument.

**Theorem 4.1.** *Let $\mathscr{X}$ denote a flat scheme of finite type over* $\operatorname{Spec} F^\circ$ *with generic fibre $X = \mathscr{X}_\eta$ and special fibre $\mathscr{X}_s$. Let $V$ be an irreducible closed subset of $\mathscr{X}_s$. Then there exists an integral closed subscheme $Y$ of $X$ such that $V$ is an irreducible component of the special fibre $(\overline{Y})_s$ of the schematic closure $\overline{Y}$ of $Y$ in $\mathscr{X}$.*

*Proof.* We may assume without loss of generality that the absolute value on $F$ is non-trivial and that $\mathscr{X} = \operatorname{Spec} A$ is an affine scheme. We consider $V$ as an integral closed subscheme of $\mathscr{X}_s$ . Let $r$ denote its dimension. We choose a closed embedding

$$\mathscr{X} = \operatorname{Spec} A \hookrightarrow \mathbb{A}^N_{F^\circ}.$$

As in the proof of Noether normalization, we can choose a generic projection $\mathbb{A}^N_{\tilde{F}} \to \mathbb{A}^r_{\tilde{F}}$ such that the induced morphism

$$\psi : V \hookrightarrow \mathscr{X}_s \to \mathbb{A}^r_{\tilde{F}}$$

is finite and surjective. The morphism $\mathscr{X}_s \to \mathbb{A}^r_{\tilde{F}}$ clearly lifts to a morphism

$$\varphi : \mathscr{X} \longrightarrow \mathbb{A}^r_{F^\circ}.$$

Let $x_1, \ldots, x_r$ denote the canonical coordinates on $\mathbb{A}^r_{F^\circ}$. We equip the function field $L = F(x_1, \ldots, x_r)$ of $\mathbb{A}^r_{F^\circ}$ with the Gauss norm. Base change to the valuation ring $L^\circ$ of $L$ yields a cartesian diagram

$$\begin{tikzcd} \mathscr{X}' \arrow[r] \arrow[d, "p"] & \operatorname{Spec} L^\circ \arrow[d] \\ \mathscr{X} \arrow[r, "\varphi"] & \mathbb{A}^r_{F^\circ} \end{tikzcd} .$$

and corresponding cartesian diagrams for the generic and the special fibre

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \operatorname{Spec} L \arrow[d] \\ X \arrow[r] & \mathbb{A}^r_F \end{tikzcd} \qquad \begin{tikzcd} (\mathscr{X}')_s \arrow[r] \arrow[d] & \operatorname{Spec}(\tilde{L}) \arrow[d] \\ \mathscr{X}_s \arrow[r] & \mathbb{A}^r_{\tilde{F}}. \end{tikzcd}$$

The residue class field $\tilde{L} = \tilde{F}(x_1, \ldots, x_r)$ of $L$ is canonically isomorphic to the residue class field $\kappa(\eta)$ of the generic point $\eta$ of $\mathbb{A}^r_{\tilde{F}}$. Base change yields the canonical diagram

$$\begin{tikzcd} V' = \psi^{-1}(\{\eta\}) \arrow[r] \arrow[d] & (\mathscr{X}')_s \arrow[r] \arrow[d] & \operatorname{Spec}(\tilde{L}) \arrow[d, "\eta"'] \\ V \arrow[r] & \mathscr{X}_s \arrow[r] & \mathbb{A}^r_{\tilde{F}} \end{tikzcd}$$

which shows that $V'$ is finite over $\tilde{L}$. The generic point $\eta_V$ of $V$ maps to $\eta$ and determines the generic point $\eta_{V'}$ of the fibre $\psi^{-1}(\{\eta\})$. Then $\eta_{V'}$ is a closed point on $(\mathscr{X}')_s$ as $V'$ is finite over $\tilde{L}$. We have $\mathscr{X}' = \operatorname{Spec} A'$ for $A' = A \otimes_{F^\circ[x_1,\ldots,x_r]} L^\circ$. Any finitely generated ideal in a valuation ring $R$ is principal and hence an $R$-module is flat if and only if it is torsion free. It follows that $L^\circ$ is a flat $F^\circ[x_1,\dots,x_r]$-module and hence $A'$ is a flat $A$-module. Since $A$ is a flat $F^\circ$-algebra, we deduce that $A'$ is a flat $F^\circ$-module. Then $A'$ is also torsion free as an $L^\circ$-module and hence flat over $L^\circ$. We choose a non-zero $\rho \in L^{\circ\circ}$ and define

$$\widehat{\mathscr{X}'} = \mathrm{Spf}\ \varprojlim_k A'/\rho^k A'.$$

as the $\rho$-adic completion of $\mathscr{X}'$. Let $\hat{L}$ denote the completion of $L$. Then $\widehat{\mathscr{X}'}$ is an admissible formal $\hat{L}^\circ$-scheme in the sense of [Bos14, Sect. 7.4] as shown in [Ull95, §6.1]. The generic fibre $(X')^\circ$ of $\widehat{\mathscr{X}'}$ is the affinoid Berkovich analytic space associated with the strict affinoid $\hat{L}$-algebra

$$\mathscr{A} = \hat{L} \otimes_{\hat{L}^\circ} \varprojlim_k A'/\rho^k A'.$$

It coincides with the affinoid domain in $(X')^{\mathrm{an}}$ given by all points in $(X')^{\mathrm{an}}$ whose reduction in $(\widehat{\mathscr{X}'})_s = (\mathscr{X}')_s$ is defined [Gub13b, §4]. The reduction map

$$\pi\ : (X')^\circ = \mathscr{M}(\mathscr{A}) \longrightarrow (\widehat{\mathscr{X}'})_s = \mathscr{X}'_s$$

is surjective [Ber90, Prop. 2.4.4] and anti-continuous in the sense that inverse images of closed subsets are open [Ber90, Cor. 2.4.2]. The closed point $\eta_{V'}$ in $(\mathscr{X}')_s$ yields the open subset $\pi^{-1}(\{\eta_{V'}\})$ of $\mathscr{M}(\mathscr{A})$. We also get that $\pi^{-1}(\{\eta_{V'}\})$ is contained in the relative interior $\mathrm{Int}\,(\mathscr{M}(\mathscr{A})/\mathscr{M}(L))$ by [CLD12, Lemme 6.5.1]. From [Ber90, Prop. 3.1.3(ii) and Theorem 3.4.1] we get

$$\mathrm{Int}\,(\mathscr{M}(\mathscr{A})/(X')^{\mathrm{an}}) = \mathrm{Int}\,(\mathscr{M}(\mathscr{A})/\mathscr{M}(L)).$$

The set on the lefthand side coincides by [Ber90, Prop. 3.1.3(i)] with the topological interior of $(X')^\circ$ in $(X')^{\mathrm{an}}$. It follows that $\pi^{-1}(\{\eta_{V'}\})$ is open in $(X')^{\mathrm{an}}$. Each closed point $\xi'$ of $X'$ determines a closed point in $(X')^{\mathrm{an}}$ (which we denote again by $\xi'$) and these points form a dense subset of $(X')^{\mathrm{an}}$ by [Gub13b, 2.6]. Hence there exists a closed point $\xi' \in X'$ such that $\pi\,(\xi') = \eta_{V'}$. We get $\eta_{V'} \in \overline{\{\xi'\}}$ where the (Zariski) closure is taken in $X'$ [Gub13b, 4.8].

Let $p : \mathscr{X}' \to \mathscr{X}$ denote the base change morphism. Let $Y$ be the Zariski-closure $\overline{\{p(\xi')\}}$ of $p(\xi')$ in $X$. It follows from our construction that $p(\eta_{V'}) = \eta_V$. Let $\overline{Y}$ denote the Zariski-closure of $p(\xi')$ in $\mathscr{X}$. From $\eta_{V'} \in \overline{\{\xi'\}}$ we get $\eta_V \in \overline{Y}$ and hence $V \subseteq \overline{Y}$. The flatness of $\overline{Y}$ over $\mathrm{Spec}\,F^\circ$ yields $\dim Y = \dim(\overline{Y})_s$. It remains to show $\dim Y \leq \dim V$. We get $\mathrm{trdeg}\,\big(\kappa(\xi')/L\big) = 0$ as $\xi'$ is a closed point of $X'$. Hence

$$\begin{aligned}
\dim Y &= \mathrm{trdeg}\,\big(\kappa(p(\xi'))/F\big)\\
&\leq \mathrm{trdeg}\,\big(\kappa(\xi')/F\big)\\
&= \mathrm{trdeg}\,\big(\kappa(\xi')/L\big) + \mathrm{trdeg}\,(L/F)\\
&= \dim V
\end{aligned}$$

yields our claim. □

## 5. Formal metrics

Recall that $K$ denotes always an algebraically closed field endowed with a complete non-trivial non-archimedean absolute value $|\ |$. In this section, we gather various properties of formal metrics on line bundles of strictly $K$-analytic spaces. Such metrics play an important role in Arakelov geometry as we can use the underlying model for intersection theory (see [Gub98]).

**5.1.** Let $X$ be a compact reduced strictly $K$-analytic space in the sense of [Ber93] (resp. a proper algebraic variety). A *formal model* (resp. *algebraic model*) of $X$ over $K^\circ$ is an admissible formal scheme (resp. a flat proper integral scheme) $\mathscr{X}$ over $K^\circ$ with a fixed isomorphism from the generic fibre $\mathscr{X}_\eta$ onto $X$. We recall that an admissible formal scheme $\mathscr{X}$ is a flat formal scheme over $K^\circ$ which is locally of topologically finite type. We will always identify $\mathscr{X}_\eta$ with $X$ along the fixed isomorphism.

**Remark 5.2.** By [Ber93, Theorem 1.6.1], the category of compact strictly $K$-analytic spaces is equivalent to the category of quasicompact and quasiseparated rigid $K$-analytic varieties. This allows us to use *Raynaud's theorem*, proved by Bosch and Lütkebohmert in [BL93a, Theorem 4.1], which shows that the category of quasicompact admissible formal schemes over $K^\circ$ localized in the class of admissible formal blowing ups is equivalent to the category of quasicompact and quasiseparated rigid $K$-analytic varieties. Note that this holds more generally with quasiparacompact replacing quasicompact [Bos14, Theorem 8.4.3], but we don't need to work in such generality.

In the algebraic setting, *Nagata's compactification theorem* replaces Raynaud's theorem from above. It shows that for an algebraic variety $X$ over $K$ with a line bundle $L$, there is a line bundle $\mathscr{L}$ on a flat proper variety $\mathscr{X}$ over $K^\circ$ such that $X$ is an open dense subset of $\mathscr{X}_\eta$ and such that $L = \mathscr{L}|_X$. This was proved by Nagata in the noetherian case (see [Nag62, Nag63] and also in [Voj07, Theorem 5.7]) and generalizes to varieties over valuation rings by noetherian approximation. In particular, it is no restriction of generality working with proper schemes over $K$ in the algebraic case.

**5.3.** We first recall that a line bundle on a strictly $K$-analytic space is a locally free sheaf of rank 1 on the G-topology. If the $K$-analytic space is good, then it is equivalent to have a locally free sheaf on the Berkovich topology [Ber93, Proposition 1.3.4]. In this paper, we consider exclusively the G-topology on strictly $K$-analytic spaces induced by the strictly affinoid subdomains. This G-topology allows us to use results from rigid geometry (see [Ber93, §1.6]).

Let $L$ be a line bundle on the compact reduced strictly $K$-analytic space (resp. proper algebraic variety) $X$. A *formal model* (resp. *algebraic model*) of $(X, L)$ over $K^\circ$ is a pair $(\mathscr{X}, \mathscr{L})$ where $\mathscr{X}$ is a formal model (resp. algebraic model) of $X$ over $K^\circ$ and where $\mathscr{L}$ is a line bundle on $\mathscr{X}$ with a fixed isomorphism $\mathscr{L}|_X \cong L$ which we will use again for identification.

A formal (resp. algebraic) $K^\circ$-model $(\mathscr{X}, \mathscr{L})$ of $(X, L)$ gives rise to an *associated formal metric* (resp. *associated algebraic metric*) $\|\ \|_{\mathscr{L}}$ on $L$ in the following way: For $x \in X$, let us choose a trivialization $\mathscr{U}$ of $\mathscr{L}$ in the reduction $\pi(x)$ of $x$. The induced isomorphism $\mathscr{L}(\mathscr{U}) \cong \mathscr{O}(\mathscr{U})$ allows to identify a local section $s$ with a regular function $\gamma$ and then we define

$$\|s(x)\|_{\mathscr{L}} = |\gamma(x)|. \tag{5.3.1}$$

This is independent of the choice of the trivialization as a change involves multiplication with an invertible function in $\mathscr{O}_{\mathscr{X},\pi(x)}$.

**Definition 5.4.** A metric $\| \ \|$ on $L$ is called a *formal metric* (resp. *algebraic metric*) if there is a formal (resp. algebraic) $K^\circ$-model $(\mathscr{X}, \mathscr{L})$ of $(X, L)$ such that $\| \ \| = \| \ \|_{\mathscr{L}}$. More generally, a $\mathbb{Q}$*-formal (resp.* $\mathbb{Q}$*-algebraic) metric* on $L$ is a metric $\| \ \|$ on $L$ such that there is a non-zero $n \in \mathbb{N}$ with $\| \ \|^{\otimes n}$ a formal (resp. algebraic) metric.

**Proposition 5.5.** *Let $L$ be a line bundle on the compact reduced strictly $K$-analytic space $X$. Then the following properties hold:*

(a) *$\mathbb{Q}$-formal metrics on $L$ are continuous on $X$.*
(b) *$\mathbb{Q}$-formal metrics on $L$ are dense in the space of continuous metrics on $L$ with respect to uniform convergence.*
(c) *$L$ has a formal metric.*
(d) *The isometry classes of formally (resp. $\mathbb{Q}$-formally) metrized line bundles over $X$ form an abelian group.*
(e) *The pull-back of a formal (resp. a $\mathbb{Q}$-formal) metric on $L$ with respect to a morphism $f : X' \to X$ of compact reduced strictly analytic spaces is a formal (resp. a $\mathbb{Q}$-formal) metric on $f^*(L)$.*
(f) *The maximum and the minimum of two formal metrics on $L$ are again formal metrics on $L$.*

*Proof.* Continuity in (a) means that $\|s\|$ is continuous for any local section $s$. This follows easily from (5.3.1).

For (b), we note that the quotient of two metrics on $L$ gives rise to a metric on $O_X$ and evaluation at the constant section 1 gives rise to a continuous function $f$ on $X$. Then we use the maximum norm of $|\log(f)|$ to measure the distance of the two metrics and claim (b) follows from [Gub98, Theorem 7.12].

Property (c) is shown in [Gub98, Lemma 7.6] and (d) follows easily from (5.3.1).

To prove (e), let $\| \ \|_{\mathscr{L}}$ be the formal metric on $L$ associated to the $K^\circ$-model $(\mathscr{X}, \mathscr{L})$ of $(X, L)$. We use Raynaud's theorem from 5.2 which shows the existence of a $K^\circ$-model $\mathscr{X}'$ of $X'$ and of a morphism $\varphi : \mathscr{X}' \to \mathscr{X}$ with generic fibre $f : X' \to X$. Then (e) follows from

$$\varphi^*(\| \ \|_{\mathscr{L}}) = \| \ \|_{\varphi^*(\mathscr{L})}. \tag{5.5.1}$$

Finally, (f) is proven in [Gub98, Lemma 7.8]. $\square$

**Remark 5.6.** For a proper variety $X$ over $K$, we have similar properties as in Proposition 5.5 formulated on $X^{\mathrm{an}}$. Most of them can be proved in the same way replacing Raynaud's theorem by Nagata's compactification theorem. However, they also can be deduced from the fact that algebraic and formal metrics are the same on a line bundle over a proper variety [GK14, Proposition 8.13].

**Remark 5.7.** Let $\mathscr{X}$ be a formal $K^\circ$-model of the compact reduced strictly $K$-analytic space $X$. Then there is a canonical $K^\circ$-model $\mathscr{X}'$ of $X$ over $\mathscr{X}$ and a canonical morphism $\iota : \mathscr{X}' \to \mathscr{X}$ extending the identity such that $\mathscr{X}'$ has reduced special fibre. It is obtained by covering $\mathscr{X}$ by formal affine open subschemes $\mathscr{U} = \mathrm{Spf}(A)$, noting that $\mathscr{A} := A \otimes_{K^\circ} K$ is a reduced strictly affinoid algebra and then gluing the admissible formal affine schemes $\mathrm{Spf}(\mathscr{A}^\circ)$ over $K^\circ$. It is a standard fact that $\iota$ induces a finite surjective morphism between the special fibres (see [Gub98, 1.10, Proposition 1.11]).

**Lemma 5.8.** *Let $\| \ \|$ be a formal metric on the line bundle $L$ of the compact reduced strictly $K$-analytic space $X$. Then there is a model $(\mathscr{X}, \mathscr{L})$ of $(X, L)$ over $K^\circ$ with $\mathscr{X}_s$ reduced and with $\| \ \| = \| \ \|_{\mathscr{L}}$. Moreover, for such a model $\mathscr{X}$, the sheaf $\mathscr{L}$ is canonically isomorphic to the sheaf*

$$\mathscr{U} \mapsto \{s \in L(U) \mid \|s(x)\| \leq 1\},$$

*where $\mathscr{U}$ ranges over all open subsets of $\mathscr{X}$ and where $U$ is the generic fibre of $\mathscr{U}$.*

*Proof.* The first claim follows from Proposition 5.5(c) and Remark 5.7. The second claim follows from [Gub98, Proposition 7.5]. □

**5.9.** Let $X$ be a compact strictly $K$-analytic space which is not necessarily reduced and let $L$ be a line bundle on $X$. Then it is better to work with piecewise linear (resp. piecewise $\mathbb{Q}$-linear) metrics on $L$ (see [Gub98, §7] for details). Here, a metric $\| \ \|$ on $L$ is called *piecewise linear* if there is a G-covering of $X$ which has frames of norm identically one, and *piecewise $\mathbb{Q}$-linear* if there is a non-zero $n \in \mathbb{N}$ such that $\| \ \|^{\otimes n}$ is a piecewise linear metric of $L^{\otimes n}$. The properties of Proposition 5.5 hold also for piecewise linear (resp. piecewise $\mathbb{Q}$-linear) metrics. If $X$ is reduced, then a piecewise linear (resp. piecewise $\mathbb{Q}$-linear) metric is the same as a formal (resp. $\mathbb{Q}$-formal) metric. In general, $X$ and the analytic space $X_{\mathrm{red}}$ with the induced reduced structure have the same G-topology (see [BGR84, p. 389]). We conclude that pull-back gives a bijective correspondence between piecewise linear (resp. piecewise $\mathbb{Q}$-linear) metrics on $L$ and formal (resp. $\mathbb{Q}$-formal) metrics on $L|_{X_{\mathrm{red}}}$.

**Proposition 5.10.** *Let $X$ be a compact strictly $K$-analytic space with a line bundle $L$. Then the definition of a piecewise linear metric is* G*-local.*

*Proof.* We have to show that if there is a G-covering $(V_i)_{i \in I}$ of $X$ such that the restriction of the metric $\| \ \|$ on $L$ to $V_i$ is piecewise linear for all $i \in I$, then $\| \ \|$ is a piecewise linear metric. Passing to a refinement, we may assume that every $V_i$ has a frame of norm identically 1 and hence $\| \ \|$ is piecewise linear. □

We have the following extension result which allows us often to work globally.

**Proposition 5.11.** *Let $L$ be a line bundle on the compact strictly $K$-analytic space $X$ and let $V$ be a compact analytic domain in $X$. Let $\| \ \|_V$ be a piecewise linear metric on $L|_V$. Then there is a piecewise linear metric $\| \ \|$ on $L$ which extends $\| \ \|_V$.*

*Proof.* By the final remark in 5.9, we may assume that $X$ is reduced and we may prove the claim for formal metrics. Then there is a formal model $(\mathscr{V}, \mathscr{L}_{\mathscr{V}})$ of $(V, L)$. By Proposition 5.5(c), there is a formal model $(\mathscr{X}, \mathscr{L})$ of $(X, L)$. Modifiying $\mathscr{X}$ by an admissible blowing up and replacing $\mathscr{V}$ by a dominating formal model of $V$, Raynaud's theorem gives a morphism $\mathscr{V} \to \mathscr{X}$ extending the G-open immersion $V \to X$. By [BL93b, Corollary 5.4], we may even assume that $\mathscr{V} \to \mathscr{X}$ is an open immersion. By Lemma 5.8, we may assume that $\mathscr{X}_s$ and hence $\mathscr{V}_s$ are reduced.

We compare the sheaves $\mathscr{L}_{\mathscr{V}}$ and $\mathscr{L}$ using the generic fibre $L$ as a reference, i.e. for any formal open subset $\mathscr{U}$ of $\mathscr{V}$ (resp. $\mathscr{X}$) with generic fibre $U$, we view $\mathscr{L}_{\mathscr{V}}(\mathscr{U})$ (resp. $\mathscr{L}(\mathscr{U})$) as a subset of $L(U)$. Using compactness of $V$ and replacing $\| \ \|_{\mathscr{L}}$ by a suitable small multiple, we may assume that $\| \ \|_{\mathscr{L}} \leq \| \ \|_V$ on $L|_V$. By Lemma 5.8, we deduce that $\mathscr{L}_{\mathscr{V}}$ is a coherent submodule of $\mathscr{L}$. Similarly as in the proof of [BL93a, Lemma 5.7], we can extend $\mathscr{L}_{\mathscr{V}}$ to a coherent submodule $\mathscr{N}$ of $\mathscr{L}$. Since $\mathscr{L}|_{\mathscr{V}}$ is coherent and $V$ is compact, there is a sufficiently small $\pi \in K^\circ \setminus \{0\}$ with $\pi\mathscr{L}|_{\mathscr{V}}$ a submodule of $\mathscr{L}_{\mathscr{V}}$. Then the generic fibre of the coherent submodule $\mathscr{M} := \mathscr{N} + \pi\mathscr{L}$ of $\mathscr{L}$ is $L$ and $\mathscr{M}$ agrees with $\mathscr{L}_{\mathscr{V}}$ on $\mathscr{V}$. Using the flattening techniques from [BL93b, Theorem 4.1, Proposition 4.2], there is an admissible formal blowing up $\mathscr{X}'$ of $\mathscr{X}$ with center outside $\mathscr{V}$ such that the strict transform $\mathscr{M}'$ of $\mathscr{M}$ is flat over $\mathscr{X}'$. We conclude that $\mathscr{M}'$ is a line bundle on $\mathscr{X}'$ which agrees with $\mathscr{L}_{\mathscr{V}}$ over $\mathscr{V}$. Since $\mathscr{M}'|_X = L$, the formal metric $\| \ \|$ associated to $\mathscr{M}'$ does the job. □

The following result shows that local analytic considerations for formal metrics on an algebraic variety can be always done with algebraic metrics.

**Corollary 5.12.** *Let $L$ be a line bundle on a proper variety $X$ over $K$. Suppose that $\| \ \|$ is a formal metric on $L^{\mathrm{an}}|_V$ for a compact strictly $K$-analytic domain $V$ of $X^{\mathrm{an}}$. Then there is an algebraic metric $\| \ \|'$ on $L$ which agrees with $\| \ \|$ over $V$.*

*Proof.* By Proposition 5.11, we can extend $\| \ \|$ to a formal metric $\| \ \|'$ on $L$. By Remark 5.6, this is an algebraic metric. □

## 6. Semipositive piecewise linear metrics

In this section, we start with a line bundle $L$ on a strictly $K$-analytic space $X$ over $K$ endowed with a piecewise linear metric $\| \ \|$. We will introduce semipositive piecewise linear metrics from a point of view which is local on $X$. Assuming that $X$ is the analytification of a proper algebraic variety, we have seen in the previous section that piecewise linear, formal and algebraic metrics are the same. In this case, we will show that a formal metric is semipositive in all points of $X$ if and only if an associated model is vertically nef which is Zhang's definition used in arithmetic intersection theory. Then we show that semipositivity for formal metrics agrees with various other positivity notions introduced before.

**6.1.** First, we generalize the definitions from Section 5 to our setting. Let $L$ be a line bundle on the strictly $K$-analytic space $X$ over $K$ which means that $L$ is a locally free sheaf of rank 1 on the G-topology of $X$. We say that a metric $\| \ \|$

on $L$ is *piecewise linear* if there is G-open covering which has frames of norm identically one. It is easy to see that piecewise linearity is closed with respect to the following operations: tensor product of metrics, passing to the dual metric and pull-back of metrics.

We have seen in 5.9 that on a reduced compact strictly $K$-analytic space, a metric is piecewise linear if and only if it is formal. This will be used for the following local definition of semipositivity which was suggested to us by Tony Yue Yu.

**Definition 6.2.** Let $\| \ \|$ be a piecewise linear metric on the line bundle $L$ of the strictly $K$-analytic space $X$. If $X$ is reduced, then $\| \ \|$ is called *semipositive in $x \in X$* if $x$ has a neighbourhood $V$ in $X$ with the following properties:

(i) $V$ is a compact strictly $K$-analytic domain;
(ii) $(V, L^{\mathrm{an}}|_V)$ has a formal $K^\circ$-model $(\mathscr{V}, \mathscr{L})$ with $\| \ \|_{\mathscr{L}} = \| \ \|$;
(iii) If $Y$ is a closed curve in $\mathscr{V}_s$ with $Y$ proper over $\tilde{K}$, then $\deg_{\mathscr{L}}(Y) \geq 0$.

If $X$ is not necessarily reduced, then $\| \ \|$ is called *semipositive in $x$* if the induced metric on $L|_{X_{\mathrm{red}}}$ is semipositive in the above sense. We call $\| \ \|$ *semipositive* on an open subset $W$ of $X$ if $\| \ \|$ is semipositive in all points of $W$.

**Lemma 6.3.** *Suppose that $X$ is reduced and that $\| \ \|$ is semipositive in $x \in X$. Let $W \subseteq V$ be compact strictly $K$-analytic domains in $X$ such that $W$ is a neighbourhood of $x$. Suppose that $(\mathscr{V}, \mathscr{L})$ (resp. $(\mathscr{W}, \mathscr{M})$) is a formal $K^\circ$-model of $(V, L^{\mathrm{an}}|_V)$ (resp. $(W, L^{\mathrm{an}}|_W)$). If $(\mathscr{V}, \mathscr{L})$ satisfies (ii) and (iii) in Definition 6.2 and if $(\mathscr{W}, \mathscr{M})$ satisfies (ii), then $(\mathscr{W}, \mathscr{M})$ also satisfies (iii).*

*Proof.* We first note that we may always replace the formal $K^\circ$-models $\mathscr{V}$ (resp. $\mathscr{W}$) by dominating formal $K^\circ$-models $\mathscr{V}'$ of $V$ (resp. $\mathscr{W}'$ of $W$) as property (iii) is equivalent under such a change. This follows from the fact that any curve of $\mathscr{V}_s$ is dominated by a curve $\mathscr{V}_s'$ with respect to the proper morphism $\mathscr{V}_s' \to \mathscr{V}_s$ [Tem00, Corollary 4.4] and from projection formula. In this way, Raynaud's theorem and [BL93b, Corollary 5.4] show that we may assume that the G-open immersion $W \to V$ extends to an open immersion $\mathscr{W} \to \mathscr{V}$. By Lemma 5.8, we may assume that $\mathscr{V}_s$ and hence $\mathscr{W}_s$ are reduced, and then $\mathscr{L}|_{\mathscr{W}} \cong \mathscr{M}$ again by Lemma 5.8. Therefore property (iii) for $(\mathscr{V}, \mathscr{L})$ implies the same property for $(\mathscr{W}, \mathscr{M})$. □

**Proposition 6.4.** *Let $\| \ \|$ be a piecewise linear metric on the line bundle $L$ on the strictly $K$-analytic space $X$ and let $x \in X$.*

(a) *The set of points in $X$ where $\| \ \|$ is semipositive is open.*
(b) *The trivial metric on $O_X$ is semipositive on $X$.*
(c) *The tensor product of two piecewise linear metrics which are semipositive in $x$ is again semipositive in $x$.*
(d) *Let $f : X' \to X$ be a morphism of strictly $K$-analytic spaces and let $x' \in X'$ with $x = f(x')$. If $\| \ \|$ is semipositive in $x$, then $\| \ \|' := f^*\| \ \|$ is semipositive in $x'$.*

*Proof.* We may assume that the $X, X'$ are reduced. Properties (a) and (b) are obvious from Definition 6.2. Lemma 6.3 and linearity of the degree of a proper curve with respect to the divisor shows (c).

For (d), we choose $V$, $\mathscr{V}$ and $\mathscr{L}$ as in Definition 6.2. Then there is a compact G-open neighbourhood $W$ of $x'$ in $(X')^{\mathrm{an}}$ with $f(W) \subseteq V$. By Raynaud's theorem, the morphism $f : W \to V$ extends to a morphism $\varphi : \mathscr{W} \to \mathscr{V}$ of formal $K^\circ$-models. Let $Y$ be a closed curve in $\mathscr{W}_s$ which is proper over $\tilde{K}$. Then the restriction of $\varphi$ to $Y$ is proper. By (5.5.1), we have $\| \|' = \| \|_{\varphi^*\mathscr{L}}$, and (d) follows from projection formula. □

In the following, we always assume that $L$ is a line bundle on the proper algebraic variety $X$ over $K$.

**Proposition 6.5.** *We assume that $\| \|$ is the formal metric associated to the formal $K^\circ$-model $(\mathscr{X}, \mathscr{L})$ of $(X, L)$ and we denote by $\pi : X^{\mathrm{an}} \to \mathscr{X}_s$ the reduction map. Then $\| \|$ is semipositive on the open subset $W$ of $X^{\mathrm{an}}$ if and only if $\deg_{\mathscr{L}}(Y) \geq 0$ for any closed curve $Y$ in $\mathscr{X}_s$ with $Y \subseteq \pi(W)$.*

*Proof.* We note first that on the right hand side of the equivalence we may always pass to a formal $K^\circ$-model $\mathscr{X}'$ dominating $\mathscr{X}$ using that any curve in $\mathscr{X}_s$ is dominated by a curve in $\mathscr{X}'_s$ with respect to the proper morphism $\mathscr{X}'_s \to \mathscr{X}_s$ and using the projection formula.

Suppose that $\| \|$ is semipositive on $W$ and let $Y$ be a closed curve in $\mathscr{X}_s$ with $Y \subseteq \pi(W)$. There is $x \in W$ with $\pi(x)$ equal to the generic point of $Y$. Since $\| \|$ is semipositive in $x$, there is a neighbourhood $V$ of $x$ in $X^{\mathrm{an}}$ and a formal $K^\circ$-model $(\mathscr{V}, \mathscr{L})$ of $(V, L^{\mathrm{an}}|_V)$ with properties (i)–(iii) from Definition 6.2. Using Raynaud's theorem and [BL93b, Corollary 5.4], we may assume that $\mathscr{V}$ is an open subset of $\mathscr{X}$. Since $x$ is contained in an open neighbourhood of $X^{\mathrm{an}}$ which is contained in $V$ and since $X^{\mathrm{an}}$ is boundaryless [Ber90, Theorem 3.4.1], we deduce that $x$ is not a boundary point of $V$ and hence the closure of $\pi(x)$ in $\mathscr{V}_s$ is proper (see [CLD12, Lemme 6.5.1]). We conclude that this closure is $Y$ and hence $\deg_{\mathscr{L}}(Y) \geq 0$ by (iii).

Conversely, assume that $\deg_{\mathscr{L}}(Y) \geq 0$ for any closed curve $Y$ in $\mathscr{X}_s$ with $Y \subseteq \pi(W)$. For $x \in W$, we choose a neighbourhood $V$ of $x$ in $W$ such that $V$ is a compact strictly $K$-analytic domain. By Raynaud's theorem and [BL93b, Corollary 5.4], we may assume that $V$ has a $K^\circ$-model $\mathscr{V}$ which is a formal open subset of $\mathscr{X}$. Then (iii) in Definition 6.2 follows from our assumption on the degree of curves since $\mathscr{V}_s \subseteq \pi(W)$. This proves semipositivity of $\| \|$ in $x$. □

**Remark 6.6.** It follows that any formal metric $\| \|$ is semipositive in all $K$-rational points of $X$. Indeed, using the notation from Proposition 6.5, we note that that the reduction $\pi(x)$ is a closed point of the special fibre $\mathscr{X}_s$. By anticontinuity of the reduction map $\pi$, we get an open neighbourhood $W := \pi^{-1}(\pi(x))$ of $x$ in $X^{\mathrm{an}}$ for which no closed curve of $\mathscr{X}_s$ is contained in $\pi(W)$. Then Proposition 6.5 proves semipositivity of $\| \|$ on $W$.

**6.7.** Let $(L, \| \|)$ be a formally metrized line bundle on the $n$-dimensional proper variety $X$ over $K$ and let $W$ be an open subset of $X^{\mathrm{an}}$. Then the $\delta$-form $c_1(L, \| \|)^n$ of type $(n, n)$ induces a unique Radon measure on $W$ extending the current $[c_1(L, \| \|)^n]_D$ (see [GK14, Corollary 6.15]). It is shown in [GK14, Theorem 10.5] that this Monge–Ampère measure on $X^{\mathrm{an}}$ agrees with the corresponding Chambert–Loir measure in arithmetic geometry and hence it is supported in finitely many points. Note that the restriction of this measure to $W$ is the unique Radon measure on $W$ extending the current $[c_1(L|_W, \| \ \|)^n]_D$.

**Lemma 6.8.** *Let $L$ be a line bundle on a proper curve $C$ over $K$ endowed with a formal metric $\| \ \|$ and let $W$ be an open subset of $C^{\mathrm{an}}$. Then $\| \ \|$ is a semipositive formal metric over $W$ if and only if $c_1(L|_W, \| \ \|)$ induces a positive measure on $W$.*

*Proof.* Let $\mathscr{L}$ be a formal model of $L$ over a proper formal model $\mathscr{X}$ of $C$ which induces the given metric and let $\pi : C^{\mathrm{an}} \to \mathscr{X}_s$ be the reduction map. By 5.7, we may assume $\mathscr{X}_s$ reduced. The equality of the Monge–Ampère measure and the Chambert-Loir measure [GK14, Theorem 10.5] gives the formula

$$c_1(L, \| \ \|) = \sum_Y \deg(\mathscr{L}|_Y) \cdot \delta_{\xi_Y} \tag{6.8.1}$$

where $Y$ runs over the irreducible components of the special fibre of $\mathscr{X}$ and $\xi_Y$ denotes the corresponding point in the Berkovich space $C^{\mathrm{an}}$ with reduction equal to the generic point of $Y$. In view of (6.8.1) and using Proposition 6.5, the lemma follows from the claim that $\xi_Y \in W$ if and only if $Y \subseteq \pi(W)$.

To see the equivalence, let $\xi_Y \in W$. Then $\xi_Y$ has a compact strictly $K$-analytic domain $V \subseteq W$ as a neighbourhood with a formal $K^\circ$-model $\mathscr{V}$ of $V$. By Raynaud's theorem, we may assume that the inclusion $V \to C^{\mathrm{an}}$ extends to a morphism $\iota : \mathscr{V} \to \mathscr{X}$. Since $\xi_Y$ is an inner point of $V$, the closure $Y'$ of the reduction of $\xi_Y$ in $\mathscr{V}$ is proper over $\tilde{K}$ [CLD12, Lemme 6.5.1] and hence $\iota(Y')$ is a proper curve over $\tilde{K}$. By functoriality of the reduction map, we have $\pi(\xi_Y) \in \iota(Y') \subseteq \pi(V) \subseteq \pi(W)$. Since $\pi(\xi_Y)$ is dense in $Y$ and $\iota(Y')$ is proper, we get $Y = \iota(Y') \subseteq \pi(W)$.

Conversely, if $Y \subseteq W$, then there is $x \in W$ with $\pi(x)$ equal to the generic point of $Y$. By the characterization of $\xi_Y$, we get $\xi_Y = x \in W$. $\square$

We recall that a line bundle is called semiample if a strictly positive tensorpower is generated by global sections. Note also that a formal metric $\| \ \|$ on $L$ has a canonical first Chern $\delta$-form $c_1(L, \| \ \|) \in B^{1,1}(X^{\mathrm{an}})$ [GK14, Remark 9.16].

**Lemma 6.9.** *Let $V$ be a compact strictly $K$-analytic neighbourhood of $x$ in $X^{\mathrm{an}}$. Suppose that the formal metric $\| \ \|$ on $L|_V$ is induced by a formal model $(\mathscr{V}, \mathscr{L})$ of $(V, L^{\mathrm{an}}|_V)$ and let $Y$ be the closure of the reduction of $x$ in $\mathscr{V}_s$. If the restriction of $\mathscr{L}$ to $Y$ is semiample, then the first Chern $\delta$-form $c_1(L, \| \ \|)$ is positively representable in the sense of 2.6 on an open neighbourhood of $x$.*

*Proof.* We will show that $x$ is contained in a tropical chart $(V, \varphi_U)$ with $V \subseteq W$ such that $c_1(L|_V, \| \ \|)$ is induced by a positively representable element $\alpha$ in $P^{1,1}(V, \varphi_U)$. Recall from 2.2(iii) that this means that $\alpha$ admits a positive representative $\tilde{\alpha}$ which is a positive $\delta$-preform on $N_{U,\mathbb{R}}$. We may always replace $\mathscr{V}$ by a dominating formal $K^\circ$-model of $V$. Then the proof of Proposition 5.11 shows that we may assume that $\mathscr{V}$ is a formal open subset of a formal $K^\circ$-model $\mathscr{X}$ of $X^{\mathrm{an}}$ such that $L$ and $\mathscr{L}$ extend to a line bundle on $\mathscr{X}$. The formal GAGA theorem of Fujiwara–Kato [FK13, Theorem I.10.1.2] shows that $\mathscr{X}$ is dominated by a (proper flat) algebraic $K^\circ$-model of $X$ (see also the proof of [GK14, Proposition 8.13]) and that the pull-back of $\mathscr{L}$ is an algebraic line bundle. So we may assume that $\mathscr{X}$ and $\mathscr{L}$ are both algebraic $K^\circ$-models.

Replacing $\mathscr{L}$ by a strictly positive tensor power, there is a generating set $\{\tilde{s}_1, \ldots, \tilde{s}_n\}$ of global sections of $\mathscr{L}|_Y$. Since $x$ is an inner point of $V$, the closure $Y$ of the reduction $\pi(x)$ of $x$ in $\mathscr{V}_s$ is proper [CLD12, Lemme 6.5.1] and hence $Y$ is also the closure of $\pi(x)$ in $\mathscr{X}_s$. By anticontinuity of the reduction map $\pi : X^{\mathrm{an}} \to \mathscr{X}_s$, the subset $\pi^{-1}(Y)$ of $V$ is an open neighbourhood of $x$ in $X^{\mathrm{an}}$.

We cover $Y$ by finitely many trivializations $(\mathscr{U}_i)_{i=1,\ldots,t}$ of $\mathscr{L}$ with special fibre $(\mathscr{U}_i)_s$ contained in $\mathscr{V}_s$ and intersecting $Y$. We may assume that there are meromorphic (algebraic) sections $s_i$ of $\mathscr{L}$ which restrict to invertible sections on $\mathscr{U}_i$ and agree with $\tilde{s}_i$ on $Y$. In particular, there is an open subset $U$ of $X$ such that $x \in U^{\mathrm{an}}$ and such that every $s_i$ restricts to an invertible section of $L|_U$. Since tropical charts form a basis for the topology on $X^{\mathrm{an}}$, we may assume that $U$ is very affine and that $(W, \varphi_U)$ is a tropical chart with $x \in W \subseteq \pi^{-1}(Y) \subseteq V$. For any $w \in W$, we have $\|s_i(w)\| \leq 1$ for all $i \in \{1, \ldots, t\}$ using that $s_i$ restricts to a global section on $Y$. Moreover, there is an $i$ such that $\pi(w) \in (\mathscr{U}_i)_s$ and hence $\|s_i(w)\| = 1$. For a fixed frame $s$ of $L|_U$, we get

$$\|s(w)\| = \max_i \left| \frac{s}{s_i}(w) \right|.$$

We consider the character lattice $M_U = \mathscr{O}(U)^*/K^*$ of the torus $T$ associated to $U$. By definition, we have $u_i := \frac{s}{s_i} \in M_U = N_U^*$ and hence $c_1(L|_V, \|\ \|)$ is represented by the $\delta$-preform $\tilde{\alpha} := d'd''[\max_i u_i]$ on $N_{U,\mathbb{R}}$. Since $\max_i u_i$ is a convex function, Example 1.5 yields that $\tilde{\alpha}$ is a positive $\delta$-preform on $N_{U,\mathbb{R}}$. $\square$

**Theorem 6.10.** *Let $L$ be a line bundle on an algebraic variety $X$ over $K$ and let $W$ be an open subset of $X^{\mathrm{an}}$. Then the following properties are equivalent for a piecewise linear metric $\|\ \|$ on $L$ over $W$:*

1. *The piecewise linear metric $\|\ \|$ is semipositive on $W$.*
2. *The metric is functorial $\delta$-psh.*
3. *The metric $\|\ \|$ is functorial psh.*
4. *The $\delta$-form $c_1(L|_W, \|\ \|)$ is positive on $W$.*
5. *The restriction of $\|\ \|$ to $W \cap C^{\mathrm{an}}$ is psh for any closed curve $C$ of $X$.*

There is also an equivalent version of Theorem 6.10 in terms of formal metrics. This version was given in Theorem 0.1. To see that they are equivalent, we note that we may assume $X$ proper over $K$ by Nagata's compactification theorem [Nag62]. The properties (1)–(5) are local in the analytic topology and hence we may assume that $\|\ \|$ extends to a formal metric on $L$. This shows the desired equivalence.

*Proof.* The above remark shows that we may assume that $\|\ \|$ extends to a metric on $L$ which we also denote by $\|\ \|$. Let $(\mathscr{X}, \mathscr{L})$ be a formal $K^\circ$-model of $(X, L)$ with $\|\ \| = \|\ \|_{\mathscr{L}}$ over $X^{\mathrm{an}}$.

(1) $\Rightarrow$ (2): Since semipositivity of formal metrics is functorial, it is enough to show that $[c_1(L|_W, \| \ \|)]_E$ is a positive $\delta$-current. By the projection formula [GK14, Proposition 5.9(iii)], we may check that on a generically finite projective covering and so we may assume that $X$ is projective using Chow's lemma.

For any $x \in W$, there is a compact strictly $K$-analytic domain $V$ as a neighbourhood in $W$ such that $(V, L^{\mathrm{an}}|_V)$ has a formal $K^\circ$-model $(\mathscr{V}, \mathscr{M} = \mathscr{L}|_{\mathscr{V}})$. Since $\| \ \|$ is semipositive in $x$, we may choose this model such that $\deg_{\mathscr{M}}(Y) \geq 0$ for all curves $Y \subseteq \mathscr{V}_s$ which are proper over $\tilde{K}$. Let $Z$ be the closure of the reduction of $x$ in $\mathscr{V}_s$. Since $x$ is an inner point of $V$, the variety $Z$ is proper over $\tilde{K}$ [CLD12, Lemme 6.5.1]. By construction, the restriction of $\mathscr{M}$ to $Z$ is nef.

By Lemma 6.3, we may always pass to a dominating formal $K^\circ$-model of $\mathscr{V}$. Using Raynaud's theorem and [BL93b, Corollary 5.4], we may assume that $\mathscr{V}$ is a formal open subset of $\mathscr{X}$. By [Gub03, Proposition 10.5], $\mathscr{X}$ is dominated by the formal completion of a projective flat $K^\circ$-model and hence we may assume that $\mathscr{X}$ is projective. Then the formal GAGA-theorem of Ullrich [Ull95, Theorem 6.8] shows that $\mathscr{L}$ is an algebraic line bundle as well.

We fix a very ample line bundle $\mathscr{H}$ on $\mathscr{X}$ with generic fibre $H$. Let $\| \ \|_H$ be the semipositive algebraic metric on $H$ given by the very ample model $\mathscr{H}$. For every rational $\varepsilon > 0$, the $\mathbb{Q}$-line bundle $L_\varepsilon := L \otimes H^\varepsilon$ has the metric $\| \ \|_\varepsilon := \| \ \| \otimes \| \ \|_H^\varepsilon$ over $W$ given by the model $\mathscr{L}_\varepsilon := \mathscr{L} \otimes \mathscr{H}^\varepsilon$ on $\mathscr{X}$.

The restriction of $\mathscr{L}_\varepsilon$ to $Z$ is ample as it is the tensor product of an ample line bundle with a nef line bundle. By Lemma 6.9, the $\delta$-form $c_1(L, \| \ \|_\varepsilon)$ is positively representable on an open neighbourhood of $x$ in $W$. As $x$ was any point of $W$, we conclude that $c_1(L|_W, \| \ \|)$ is positively representable on $W$. By Corollary 2.11, the associated $\delta$-current $[c_1(L|_W, \| \ \|)]_E$ is positive and hence

$$\langle [c_1(L|_W, \| \ \|)]_E, \beta\rangle + \varepsilon \langle [c_1(H|_W, \| \ \|_H)]_E, \beta\rangle = \langle [c_1(L_\varepsilon|_W, \| \ \|_\varepsilon)]_E, \beta\rangle \geq 0$$

for any positive $\beta \in B_c^{n-1,n-1}(W)$ where $n := \dim(X)$. Using $\varepsilon \to 0$, we deduce $\langle [c_1(L|_W, \| \ \|)]_E, \beta\rangle \geq 0$ and hence $[c_1(L|_W, \| \ \|)]_E$ is positive proving (2).

(2) $\Rightarrow$ (3): Since any positive $\delta$-current is a positive current (see Remark 2.12), this is obvious.

(3) $\Leftrightarrow$ (4): This follows from Proposition 2.13.

(3) $\Rightarrow$ (5): This is obvious.

(5) $\Rightarrow$ (1): By Remark 5.6, we may assume that $\| \ \|$ is given by an algebraic $K^\circ$-model $(\mathscr{X}, \mathscr{L})$ of $(X, L)$. Let $Y$ be a closed curve in $\mathscr{X}_s$ with $Y \subseteq \pi(W)$. Since $\mathscr{X}$ is proper, it is clear that $Y$ is proper over the residue field $\tilde{K}$. By Theorem 4.1, there is a closed curve $C$ in $X$ whose closure in $\mathscr{X}$ has $Y$ as an irreducible component. We look again at the discrete Radon measure which extends the current $[c_1(L|_{C^{\mathrm{an}}}, \| \ \|)]_D$ on $C^{\mathrm{an}}$. Positivity of $[c_1(L|_{W\cap C^{\mathrm{an}}}, \| \ \|)]_D$ yields positivity of the discrete Radon measure. By Lemma 6.8, we deduce that the restriction of $\| \ \|$ to $C^{\mathrm{an}} \cap W$ is semipositive. This means in particular $\deg_{\mathscr{L}}(Y) \geq 0$ proving that $\mathscr{L}$ is vertically nef and (1). $\square$

## 7. Semipositive approximable metrics

We consider a metric $\| \ \|$ on a line bundle $L$ of a proper variety $X$ over $K$.

**7.1.** We say that $\| \ \|$ is *semipositive approximable* if it is the uniform limit of a sequence $(\| \ \|_n)_{n\in\mathbb{N}}$ of semipositive $\mathbb{Q}$-formal metrics on $L^{\mathrm{an}}$.

This class of metrics was introduced by Zhang and includes canonical metrics of dynamical systems. It is clear that every semipositive formal metric is semipositive approximable. We will show below that semipositive and semipositive approximable is the same for formal metrics. Semipositive approximable metrized line bundles are closed under tensor product and pull-back with respect to morphisms of proper varieties. This notion is also stable under base extension of the ground field. Note also that a semipositive approximable metric is continuous on the analytification of $L$.

The following result was proved in the special case of a discretely valued field with residue characteristic 0 by Boucksom, Favre and Jonsson [BFJ12, Remark after Theorem 5.12].

**Proposition 7.2.** *Suppose that $\| \ \|$ is a formal metric. Then $\| \ \|$ is semipositive approximable if and only if it is semipositive.*

*Proof.* We have to show that a semipositive approximable formal metric $\| \ \|$ is semipositive. Let $\| \ \|_n$ be semipositive $\mathbb{Q}$-formal metrics on $L^{\mathrm{an}}$ approximating the formal metric $\| \ \|$ uniformly. By Remark 5.6, there is an algebraic $K^\circ$-model $(\mathscr{X},\mathscr{L})$ of $(X,L)$ with $\| \ \| = \| \ \|_{\mathscr{L}}$. Let $V$ be a closed curve contained in $\mathscr{X}_s$. Then Theorem 4.1 shows that there is a closed curve $Y$ in $X$ such that $V$ is an irreducible component of the special fibre of the closure $\overline{Y}$ in $\mathscr{X}$. The restriction of the metrics $\| \ \|_n$ to $Y$ are semipositive and the Chambert–Loir measures $c_1(L|_Y, \| \ \|_n)$ converge weakly to $c_1(L|_Y, \| \ \|)$. We conclude that $c_1(L|_Y, \| \ \|)$ is a positive discrete measure. By Lemma 6.8, the restriction of $\| \ \|$ to $L|_Y$ is semipositive and hence $\deg_{\mathscr{L}}(V) \geq 0$. This proves semipositivity of the formal metric $\| \ \|$. □

**Proposition 7.3.** *If $\| \ \|$ is semipositive approximable, then the pull-back of $\| \ \|$ to any curve is psh.*

*Proof.* Semipositive approximable is stable under pull-back, so it is enough to show that $\| \ \|$ is psh in the case of a curve $X$. By Corollary 3.8, it is enough to show that the pull-back metric is psh on $X'$ for a proper surjective morphism $X' \to X$ of curves. We conclude that we may assume $X$ projective over $K$. Then the first Chern current $[c_1(L, \| \ \|)]_D$ is induced by the corresponding Chambert–Loir measure $c_1(L, \| \ \|)$. By definition, the latter is the weak limit of positive discrete measures and hence $c_1(L, \| \ \|)$ is also a positive measure. This means that the first Chern current $[c_1(L, \| \ \|)]_D$ is positive proving the claim. □

## 8. Piecewise smooth metrics

We have introduced piecewise smooth metrics on line bundles in [GK14, §8]. They include smooth metrics, formal metrics and canonical metrics. In this section, we relate them to the positivity notions from Section 3.

**8.1.** Let $C = (\mathscr{C}, m)$ be a tropical cycle on $N_{\mathbb{R}}$ for a lattice $N$ of finite rank and let $\Omega$ be an open subset of the support $|\mathscr{C}|$. We say that $\phi : \Omega \to \mathbb{R}$ is a *piecewise smooth* function if there is an integral $\mathbb{R}$-affine polyhedral subdivision $\mathscr{D}$ of $\mathscr{C}$ and smooth functions $\phi_\sigma : \Omega \cap \sigma \to \mathbb{R}$ with $\phi|_{\Omega\cap\sigma} = \phi_\sigma$ for every $\sigma \in \mathscr{D}$.

In a similar way as above, piecewise smooth superforms on $\Omega$ are defined in [GK14, 3.10]. In particular, we get a piecewise smooth superform $d'_{\mathrm{P}}\phi$ (resp. $d''_{\mathrm{P}}\phi$) given by the superform $d'\phi_\sigma$ (resp. $d''\phi_\sigma$) on $\Omega \cap \sigma$ for every $\sigma \in \mathscr{C}$.

We recall from [GK14, 1.10–1.12] that a piecewise smooth function $\phi$ on $|\mathscr{C}|$ induces a tropical cycle $\phi \cdot C$ of codimension 1 in $|\mathscr{C}|$ called the *corner locus of* $\phi$. Its support is the non-differentiability locus of $\phi$ and its smooth weights are defined in terms of the outgoing slopes of $\phi$. The above notions are related by

$$d'd''[\phi] = [d'_{\mathrm{P}}d''_{\mathrm{P}}\phi] + \delta_{\phi\cdot\mathrm{Trop}(U)} \in D^{1,1}(|\mathscr{C}|) \tag{8.1.1}$$

as a consequence of the tropical Poincaré–Lelong formula (see [GK14, Corollary 3.19]), where $[\alpha]$ denotes the supercurrent associated to a piecewise smooth $\alpha$.

**8.2.** Let $L$ be a line bundle on the algebraic variety $X$ over $K$. We recall that a metric $\|\ \|$ on $L$ over an open subset $W$ of $X^{\mathrm{an}}$ is called *piecewise smooth* if for any $x \in W$ there is a tropical chart $(V, \varphi_U)$ with $x \in V \subseteq W$, a frame $s$ of $L$ over $U$ and a piecewise smooth function $\phi : \Omega \to \mathbb{R}$ such that $-\log\|s\||_V = \phi \circ \mathrm{trop}_U|_V$. Here, the open subsets $\Omega := \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$ may be assumed to be convex and we may assume that $\phi$ extends to a piecewise smooth function $\tilde{\phi} : N_{\mathbb{R}} \to \mathbb{R}$. In this case, we will call $(V, \varphi_U, \Omega, s, \phi)$ a *tropical frame* for the piecewise smooth metric $\|\ \|$.

The choice of $\tilde{\phi}$ will be not be important for the following. We need this extension only to make the corner locus $\tilde{\phi} \cdot \mathrm{Trop}(U)$ well-defined as a tropical cycle contained in $\mathrm{Trop}(U)$. The definition of the weights of the corner locus in [GK14, 1.10] shows that the restrictions of the weights to $\Omega$ depend only on $\phi$. We have therefore decided to drop $\tilde{\phi}$ from our notation for tropical frames.

**8.3.** Let $\|\ \|$ be a piecewise smooth metric on $L$ over $W$. We recall from [GK14, 9.5–9.8] that there is a canonical piecewise smooth form $c_1(L, \|\ \|)_{\mathrm{ps}}$ on $W$ and a canonical generalized $\delta$-form $c_1(L, \|\ \|)_{\mathrm{res}} \in P^{1,1}(W)$ of codimension 1 such that for the associated $\delta$-currents, we have

$$[c_1(L|_W, \|\ \|)]_E = [c_1(L|_W, \|\ \|)_{\mathrm{ps}}]_E + [c_1(L|_W, \|\ \|)_{\mathrm{res}}]_E \tag{8.3.1}$$

in a functorial way. If $(V, \varphi_U, \Omega, s, \phi)$ is a tropical frame for $\|\ \|$, then $c_1(L, \|\ \|)_{\mathrm{ps}}$ is given on the tropical chart $(V, \varphi_U)$ by $d'_{\mathrm{P}}d''_{\mathrm{P}}\phi$ and the generalized $\delta$-form $c_1(L, \|\ \|)_{\mathrm{res}}$ is represented on $(V, \varphi_U)$ by the $\delta$-preform $d'd''\delta_{\tilde{\phi}\cdot N_{U,\mathbb{R}}} \in P^{1,1}(N_{U,\mathbb{R}})$.

**Theorem 8.4.** *Let $L$ be a line bundle on the algebraic variety $X$ over $K$. Let $\|\ \|$ be a piecewise smooth metric on $L$ over an open subset $W$ of $X^{\mathrm{an}}$. Then the metric $\|\ \|$ is plurisubharmonic if and only if for each tropical frame $(V, \varphi_U, \Omega, s, \phi)$ of $\|\ \|$ we have*

(i) *the restriction of $\phi$ to each maximal face of* $\mathrm{Trop}(U) \cap \Omega$ *is a convex function and*
(ii) *the corner locus $\tilde{\phi} \cdot \mathrm{Trop}(U)$ is effective on $\Omega$ .*

*Proof.* Let $n := \dim(X)$ and let $(V, \varphi_U, \Omega, s, \phi)$ be a tropical frame for $\|\ \|$. A positive superform $\alpha_U \in A^{n-1,n-1}(\Omega)$ induces a positive $(n-1, n-1)$-form $\alpha$ on $V$. Note that $\alpha$ has compact support in $V$ if and only if $\alpha_U$ has compact support in $V$ (see [CLD12, Corollaire 3.2.3]). Assuming $\alpha$ with compact support and using

$$\langle [c_1(L|_W, \|\ \|)]_D, \alpha\rangle = \langle d'd''[-\log \|s\|]_D, \alpha\rangle = \langle d'd''[\phi], \alpha_U\rangle, \tag{8.4.1}$$

the tropical Poincaré–Lelong formula (8.1.1) yields

$$\langle [c_1(L|_W, \|\ \|)]_D, \alpha\rangle = \langle [d'_{\mathrm{P}}d''_{\mathrm{P}}\phi], \alpha_U\rangle + \langle \delta_{\tilde{\phi}\cdot \mathrm{Trop}(U)}, \alpha_U\rangle. \tag{8.4.2}$$

We first assume that $\|\ \|$ is plurisubharmonic which means that the first Chern current $[c_1(L|_W, \|\ \|)]_D$ is positive. We choose any positive superform $\alpha_U$ of bidegree $(n-1, n-1)$ with compact support in $\Omega$ and let $\alpha$ be the induced smooth form on $V$. Since $\alpha$ is positive, we deduce that (8.4.1) is non-negative and hence $[d'd''\phi]$ is a positive supercurrent on $\Omega$. We deduce that the piecewise smooth extension $\tilde{\phi}$ induces a positive supercurrent on $\Omega \cap \mathrm{relint}(\Delta)$ for any maximal face $\Delta$ of $\mathrm{Trop}(U)$. It follows from Example 1.5 that $\phi$ is a convex function on $\Omega\cap\Delta$ proving (i). This implies that the supercurrent $[d'_{\mathrm{P}}d''_{\mathrm{P}}\phi]$ is positive. Property (ii) follows then from (8.4.2) and Example 1.4.

Conversely, we assume that (i) and (ii) are always satisfied. Given a positive smooth form $\alpha$ on $X^{\mathrm{an}}$ with compact support in $W$ and of bidegree $(n-1, n-1)$, we have to show that $\langle [c_1(L|_W, \|\ \|)]_D, \alpha\rangle \geq 0$. We cover the support of $\alpha$ by finitely many non-empty tropical frames $(V_i, \varphi_{U_i}, \Omega_i, s_i, \phi_i)$, $i = 1, \ldots, s$, such that $\alpha$ is given on $V_i$ by $\alpha_i \in A^{n-1,n-1}(\Omega_i)$ for $\Omega_i := \mathrm{trop}_{U_i}(V_i)$. Then we consider a non-empty very affine open subset $U \subseteq U_1 \cap \cdots \cap U_s$. Then there is a unique $\alpha_U \in A^{n-1,n-1}(\mathrm{Trop}(U))$ such that $\alpha$ is given on $U^{\mathrm{an}}$ by $\alpha_U$ (as the argument in [Gub13a, Proposition 5.13] shows, see also [GK14, Proposition 5.7]). Note that the support of $\alpha_U$ is not necessarily compact, but it is contained in $\Omega := \bigcup_{i=1}^{s} F_i^{-1}(\Omega)$ for the canonical integral $\mathbb{R}$-affine maps $F_i : N_{U,\mathbb{R}} \to N_{U_i,\mathbb{R}}$. We define $\phi : \Omega \to \mathbb{R}$ on $F_i^{-1}(\Omega)$ by $\phi := \phi_i \circ F_i$ and $V := \mathrm{trop}_U^{-1}(\Omega)$. Then $V$ contains $\mathrm{supp}(\alpha) \cap U^{\mathrm{an}}$. Note that $c_1(L|_W, \|\ \|)_{\mathrm{res}} \wedge \alpha$ and $c_1(L|_W, \|\ \|)_{\mathrm{ps}} \wedge \alpha$ are of type $(n, n)$ and hence their support is contained in $U^{\mathrm{an}}$ (see [GK14, Corollary 5.6, 9.5]). We conclude that both supports are contained in a compact subset $C$ of $V$. By properness of the tropicalization map, we get that $\mathrm{trop}_U(C)$ is a compact subset of $\Omega$. We may shrink $\Omega$ a bit still containing $\mathrm{trop}_U(C)$ such that $\phi$ is the restriction of a piecewise smooth function $\tilde{\phi}$ on $N_{U,\mathbb{R}}$. Then $(V, \varphi_U, \Omega, s, \phi)$ is a tropical frame for $\|\ \|$. Evaluating (8.3.1) at $\alpha$ gives

$$\langle [c_1(L|_W, \|\ \|)]_D, \alpha\rangle = \int_{\mathrm{Trop}(U)} d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \alpha_U + \int_{|\mathrm{Trop}(U)|} \delta_{\tilde{\phi}\cdot\mathrm{Trop}(U)} \wedge \alpha_U. \tag{8.4.3}$$

Since $\alpha$ is positive, the superform $\alpha_U$ is positive (see Proposition 2.7 and Remark 2.8). By Example 1.5, $d'_{\mathrm{P}}d''_{\mathrm{P}}\phi$ restricts to a positive superform on any maximal face of $\mathrm{Trop}(U) \cap \Omega$ where $\phi$ is smooth. Then (i) and Proposition 1.2 show that $d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \alpha_U$ is also positive on any such maximal face of $\mathrm{Trop}(U)\cap\Omega$. It follows that the first integral in (8.4.3) is non-negative. Using (ii) and Example 1.4, we deduce that also the second integral is non-negative and hence (8.4.3) is non-negative. □

**Remark 8.5.** Let $L$ be a line bundle on the algebraic variety $X$. Let $\| \ \|$ be a piecewise smooth metric on $L$ over an open subset $W$ of $X^{\rm an}$.

(a) Our proof of Theorem 8.4 shows as well that $\| \ \|$ is already plurisubharmonic if $W$ admits a basis of tropical frames $(V, \varphi_U, \Omega, s, \phi)$ for $\| \ \|$ such that conditions (i) and (ii) in Theorem 8.4 hold. We may then even replace (i) by the seemingly weaker condition:

(i') *there is a polyhedral complex $\mathscr{C}$ with $|\mathscr{C}| = |{\rm Trop}(U)|$ such that $\phi|_{\Omega\cap\Delta}$ is a smooth convex function for any maximal face $\Delta$ of $\mathscr{C}$.*

(b) Let $(V, \varphi_U, \Omega, s, \phi)$ be a tropical frame of $\| \ \|$. Let $f : X' \to X$ be a morphism of varieties and $(V', \varphi_{U'})$ a tropical chart on $X'$ such that $\Omega' = {\rm trop}_{U'}(V')$ is convex, $f(U') \subseteq U$ and $f^{\rm an}(V') \subseteq V$. Let $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ be the integral $\mathbb{R}$-affine map induced by $f$, let $\phi'$ be the restriction of $\phi \circ F$ to $\Omega'$ and let us consider the frame $s' := f^*(s)|_{U'}$ of $L' := f^*(L)$ over $U'$. Then $(V', \varphi_{U'}, \Omega', s', \phi')$ is a tropical frame for the piecewise smooth metric $f^*\| \ \|$ of $L$ over $f^{-1}(W)$.

Assume that $(V, \varphi_U, \Omega, s, \phi)$ satisfies condition 8.4(i). Then condition 8.4(i) holds for $(V', \varphi_{U'}, \Omega', s', \phi')$ as well.

(c) Part (b) applies to the special case when $(V', \varphi_{U'})$ is also a tropical chart of $X$ with $V' \subseteq V$ and $U' \subseteq U$. Then $f = {\rm id}_X$, $L' = L$ and $s' = s$. We conclude that the tropical frame $(V', \varphi_{U'}, \Omega', s, \varphi')$ satisfies 8.4(i) if the tropical frame $(V, \varphi_U, \Omega, s, \varphi)$ satisfies 8.4(i). Observe that it is not clear whether an analog of the above statement holds for condition (ii) in Theorem 8.4. To achieve that, we have to work below with positive representable $\delta$-preforms.

(d) Let $(V, \varphi_U, \Omega, s, \phi)$ be a tropical frame of $\| \ \|$. Instead of 8.4(ii) we may consider the following stronger condition:

(ii') *there exists a piecewise smooth function $\tilde\phi : N_{U,\mathbb{R}} \to \mathbb{R}$ with $\tilde\phi|_\Omega = \phi$ such that the corner locus $\tilde\phi \cdot N_{U,\mathbb{R}}$ defines a positive $\delta$-preform on $\tilde\Omega$ for some open $\tilde\Omega$ in $N_{U,\mathbb{R}}$ with $\Omega = \tilde\Omega \cap {\rm Trop}(U)$.*

In the setup of part (b) we conclude from Proposition 1.8(c) that the tropical frame $(V', \varphi_{U'}, \Omega', s', \varphi')$ of $f^*\| \ \|$ fulfills conditions (ii') if the tropical frame $(V, \varphi_U, \Omega, s, \varphi)$ satisfies condition (ii').

**Example 8.6.** Assume in Theorem 8.4 that $X$ is a curve. Then ${\rm Trop}(U)$ is a metrized graph, where the length of a primitive vector of an edge is defined as 1, and $\Omega$ is an open subset of ${\rm Trop}(U)$. In this case, condition (i) and (ii) can be summarized by the condition that for any $\omega \in \Omega$, the sum of the outgoing slopes of $\phi$ at $\omega$ (along the finitely many edges emerging from $\omega$) is non-negative.

The following result gives a sufficient condition for a piecewise smooth metric to be plurisubharmonic. It can be checked on a given covering by tropical charts.

**Proposition 8.7.** *Let $L$ be a line bundle on the algebraic variety $X$. Let $\| \ \|$ be a piecewise smooth metric on $L$ over an open subset $W$ of $X^{\rm an}$. Then the metric $\| \ \|$ is functorial $\delta$-psh if $W$ admits a covering by tropical frames $((V_i, \varphi_{U_i}, \Omega_i, s_i, \phi_i))_{i\in I}$ for $\| \ \|$ satisfying the conditions:*

(i') *there is a polyhedral complex $\mathscr{C}$ with $|\mathscr{C}| = |{\rm Trop}(U_i)|$ such that $\phi_i|_{\Delta\cap\Omega}$ is smooth and convex for every maximal face $\Delta$ of $\mathscr{C}$,* (ii') *$\phi_i$ is the restriction of a piecewise smooth function $\tilde{\phi}_i : N_{U,\mathbb{R}} \to \mathbb{R}$ as in Remark 8.5(d) such that $\tilde{\phi}_i \cdot N_{U_i,\mathbb{R}}$ defines a positive $\delta$-preform on $\tilde{\Omega}_i$.*

*In particular, the metric $\| \ \|$ is then plurisubharmonic.*

*Proof.* We have seen in Remark 8.5 that conditions (i') and (ii') are functorial, so it is enough to show that the $\delta$-current $[c_1(L|_W, \| \ \|)]_E$ is positive. Let $n := \dim(X)$ and let $\alpha$ be a positive $\delta$-form of type $(n-1, n-1)$ with compact support in $W$. We have to show that $\langle [c_1(L|_W, \| \ \|)]_E, \alpha\rangle \geq 0$. We proceed similarly as in the second part of the proof of Theorem 8.4. We cover the support of $\alpha$ by finitely many non-empty tropical frames $(V_i, \varphi_{U_i}, \Omega_i, s_i, \phi_i)$, $i = 1, \ldots, s$, satisfying (i), (ii') such that $\alpha$ is given on $V_i$ by $\alpha_i \in P^{n-1,n-1}(V_i, \varphi_{U_i})$. Then there is a unique $\alpha_U \in P^{n-1,n-1}(\mathrm{Trop}(U), \varphi_U)$ such that $\alpha$ is given on $U^{\mathrm{an}}$ by $\alpha_U$ (see [GK14, Proposition 5.7]). We use the same $\Omega$ and $\phi$ as in the proof of Theorem 8.4. We have again $\mathrm{supp}(\alpha) \cap U^{\mathrm{an}} \subseteq V = \mathrm{trop}_U^{-1}(\Omega)$. Evaluating (8.3.1) at $\alpha$ gives

$$\langle [c_1(L|_W, \| \ \|)]_E, \alpha\rangle = \int_{|\mathrm{Trop}(U)|} d'_{\mathrm{P}} d''_{\mathrm{P}} \phi \wedge \alpha_U + \int_{|\mathrm{Trop}(U)|} \delta_{\tilde{\phi}\cdot \mathrm{Trop}(U)} \wedge \alpha_U. \tag{8.7.1}$$

We have seen in Remark 8.5(c),(d) that the tropical frame $(V, \varphi_U, \Omega, s, \phi)$ also satisfies (i') and (ii'). Since $\alpha$ is a positive $\delta$-form, $\alpha_U$ is positive in $P(V, \varphi_U)$ (see Proposition 2.7). By Examples 1.4 and 1.5, $d'_{\mathrm{P}} d''_{\mathrm{P}} \phi$ restricts to a positive superform on $\mathrm{relint}(\Delta) \cap \Omega$ for any maximal face of $\mathscr{C}$. Then Proposition 1.2 and Example 1.4 show that $d'_{\mathrm{P}} d''_{\mathrm{P}} \phi \wedge \alpha_U$ induces a positive polyhedral supercurrent on $\Omega$ and hence the first integral in (8.7.1) is non-negative. Let $\tilde{\Omega}$ be an open subset of $N_{U,\mathbb{R}}$ with $\Omega = \tilde{\Omega} \cap \mathrm{Trop}(U)$. We choose a $\delta$-preform $\tilde{\alpha}_U \in P^{n-1,n-1}(\tilde{\Omega})$ representing $\alpha_U$. By [GK14, Proposition 1.14], we have

$$\int_{|\mathrm{Trop}(U)|} \delta_{\tilde{\phi}\cdot \mathrm{Trop}(U)} \wedge \alpha_U = \int_{N_{U,\mathbb{R}}} \delta_{\tilde{\phi}\cdot N_{U,\mathbb{R}}} \wedge \delta_{\mathrm{Trop}(U)} \wedge \tilde{\alpha}_U.$$

Since $\alpha_U$ is positive, we get immediately that the $\delta$-preform $\delta_{\mathrm{Trop}(U)} \wedge \tilde{\alpha}_U$ on $\tilde{\Omega}$ is positive. Using (ii') and Proposition 1.8, we deduce that the above integral is non-negative and hence (8.7.1) is non-negative. □

Let $L$ be a line bundle on $X$ equipped with a piecewise smooth metric $\| \ \|$ over the open subset $W$ of $X^{\mathrm{an}}$. Recall from [GK14, 9.9–9.11] that $\| \ \|$ is called a *$\delta$-metric* if $c_1(L, \| \ \|)$ is a well-defined $\delta$-form.

**Proposition 8.8.** *The following properties are equivalent for a $\delta$-metric:*

(i) *The $\delta$-form $c_1(L|_W, \| \ \|)$ is positive on $W$.*
(ii) *The metric $\| \ \|$ is functorial psh over $W$.*

*Proof.* The formation of the first Chern $\delta$-form is compatible with pull-back. Hence the equivalence of (i) and (ii) follows directly from Proposition 2.13. □

**Corollary 8.9.** *Let $L$ be a line bundle on a variety $X$ equipped with a smooth metric $\| \ \|$ over the open subset $W$ of $X^{\mathrm{an}}$. Then the following properties are equivalent:*

(i) *The smooth form $c_1(L|_W, \| \ \|)$ is positive on $W$.*
(ii) *The metric $\| \ \|$ is functorial $\delta$-psh over $W$.*

(iii) *The metric $\| \ \|$ is functorial psh over $W$.*
(iv) *The metric $\| \ \|$ is psh over $W$.*

*Proof.* The equivalence of (i) and (iii) is an immediate consequence of Proposition 8.8 if one observes that a smooth form is positive if and only if it is a positive $\delta$-form (see Remark 2.8). The remaining equivalences follow easily from Remark 2.14. □

**Example 8.10.** Let $A$ be an abelian variety over $K$. Let $L$ be a line bundle on $A$ equipped with a canonical metric $\| \ \|$. We investigate $c_1(L, \| \ \|)$. Hence we assume that our metric is induced by a rigidification of $L$ at zero.

(a) If $L$ is ample, then $\| \ \|$ is plurisubharmonic.
(b) If $L$ is algebraically equivalent to zero, then the $\delta$-form $c_1(L, \| \ \|)$ vanishes.

*Proof.* We recall from [GK14, Example 9.17] that the canonical metric $\| \ \|$ is a $\delta$-metric. The argument was based on [GK14, Example 8.15], where it was shown that $\| \ \|$ is locally with respect to the analytic topology equal to the product of a smooth metric with a piecewise linear metric. We recall some details from the proof of this local factorization. We used the Raynaud extension

$$1 \longrightarrow T^{\mathrm{an}} \longrightarrow E \xrightarrow{q} B^{\mathrm{an}} \longrightarrow 0, \tag{8.10.1}$$

where $T$ is a multiplicative torus of rank $r$ and $B$ is an abelian variety of good reduction. There is an analytic quotient map $p : E \to A^{\mathrm{an}}$ which is locally an isomorphism. There is a canonical map $\mathrm{val} : E \to N_{\mathbb{R}}$, where $N$ is the cocharacter lattice of $T$. The cocycles of the line bundle $L$ determine a canonical quadratic function $q_0 : N_{\mathbb{R}} \to \mathbb{R}$ with associated symmetric bilinear form $b : N_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R}$. Then we have $p^*\| \ \| := e^{-q_0 \circ \mathrm{val}} q^*\| \ \|_{\mathscr{H}}$ on $p^*(L^{\mathrm{an}})$. Here, $\mathscr{H}$ is an line bundle on the abelian $K^\circ$-scheme $\mathscr{B}$ with generic fibre $B$ and hence $q^*\| \ \|_{\mathscr{H}}$ is a piecewise linear metric. Moreover, there is a unique metric $\| \ \|_{\mathrm{sm}}$ on $O_E$ with $\|1\|_{\mathrm{sm}} = e^{-q_0 \circ \mathrm{val}}$. Since $\| \ \|_{\mathrm{sm}}$ is a smooth metric and $p$ is a local isomorphism, this proves the desired local factorization.

In case (a), the symmetric bilinear form $b$ is positive definite and hence the first Chern form of $\| \ \|_{\mathrm{sm}}$ is a positive smooth form. Moreover, the line bundle $\mathscr{H}$ is again ample. Both claims are proved in [BL91, Theorem 6.13]. Hence $q^*\| \ \|_{\mathscr{H}}$ is a semipositive piecewise linear metric and $\| \ \|$ is locally the product of a smooth metric with positive first Chern form and a semipositive piecewise linear metric. By Corollary 5.12 and Theorem 0.1, the latter has a positive first Chern $\delta$-form and hence the first Chern $\delta$-form of the canonical metric $\| \ \|$ on $L$ is positive as well.

In case (b), the symmetric bilinear form $b$ is zero (see the comments before [BL91, Theorem 6.8]). Hence $q_0$ is linear and hence $d'd''q_0 = 0$. This means that the first Chern form of $\| \ \|_{\mathrm{sm}}$ is zero. Moreover, the line bundle $\mathscr{H}$ is algebraically equivalent to 0. We conclude that $q^*\| \ \|_{\mathscr{H}}$ is a semipositive piecewise linear metric. Since semipositivity is a local analytic property, we deduce that $\| \ \|$ is locally the product of a smooth metric with zero Chern form and of a semipositive piecewise linear metric. By Corollary 5.12 and Theorem 0.1, we see that $c_1(L, \| \ \|)$ is a positive $\delta$-form. The same argument shows that

$c_1(L^{-1}, \| \ \|) = -c_1(L, \| \ \|)$ is a positive $\delta$-form. We get $c_1(L, \| \ \|) = 0$ from Lemma 2.16. □

**Remark 8.11.** Let $L$ be a line bundle on a proper smooth variety over $K$ which is algebraically equivalent to zero. We have seen in [GK14, Example 8.16] that a canonical metric $\| \ \|_{\rm can}$ on $L$ is a $\delta$-metric as a positive tensor power is piecewise linear. Since the canonical metric is obtained by pull-back from a canonical metric on an odd line bundle on an abelian variety (see [GK14, Example 8.16]), we deduce from Example 8.10 that $c_1(L, \| \ \|_{\rm can}) = 0$.

**Proposition 8.12.** *Let $L$ be a line bundle on a variety $X$ over $K$ and let $U$ be a dense Zariski open subset of $X$. We consider a piecewise smooth metric $\| \ \|$ on $L$ over an open subset $W$ of $X^{\rm an}$. Let $(P)$ be one of the four properties: psh, functorial psh, $\delta$-psh, functorial $\delta$-psh. Then $\| \ \|$ has property $(P)$ if and only if the restriction of $\| \ \|$ to $L^{\rm an}|_{U^{\rm an}\cap W}$ fulfills $(P)$.*

*Proof.* The preimage of a Zariski dense open subset is again a Zariski dense open subset and so the functoriality assertions follow from the corresponding assertions on $X$. The restriction of a psh (resp. $\delta$-psh) metric over $W$ to any open subset of $W$ is obviously psh (resp. $\delta$-psh). Conversely, assume that the restriction of $\| \ \|$ to $L^{\rm an}|_{U^{\rm an}\cap W}$ is psh (resp. $\delta$-psh). Let $n := \dim(X)$ and let $\alpha \in P^{n-1,n-1}(W)$. Using 8.3, we note that $\beta := c_1(L^{\rm an}|_W, \| \ \|)_{\rm res} \wedge \alpha$ and $\omega := c_1(L^{\rm an}|_W, \| \ \|)_{\rm ps} \wedge \alpha$ have both support in $U^{\rm an} \cap W$ by [GK14, Corollary 5.6, 9.5]. Now assume that $\alpha$ is a positive smooth form (resp. a positive $\delta$-form) with compact support in $W$. Then $\beta$ and $\omega$ have both compact support in $U^{\rm an} \cap W$. Using partition of unity [CLD12, Corollaire 3.3.4], there is a smooth function $\phi \geq 0$ with compact support in $W$ such that $\phi$ is identically 1 on the supports of $\beta$ and $\omega$. Since $\phi\alpha$ is positive on $W \cap U^{\rm an}$, we get from 8.3.1

$$\langle [c_1(L^{\rm an}|_W, \| \ \|)], \alpha \rangle = \langle [c_1(L^{\rm an}|_{U^{\rm an}\cap W}, \| \ \|)], \phi\alpha \rangle \geq 0$$

using currents (resp. $\delta$-currents). This proves that $\| \ \|$ is psh (resp. $\delta$-psh). □

Now let $X$ be a toric variety over $K$ with dense open torus $T$ and let $\| \ \|$ be a piecewise smooth toric metric of the line bundle $L$ on $X$. Then a toric section $s$ of $L$ (i.e. a rational section which is invertible over $T$) induces a function $\phi$ on $N_{\mathbb{R}}$ with

$$\phi \circ \mathrm{trop}_T = -\log \|s\| \tag{8.12.1}$$

on $T^{\rm an}$, where $N$ is the cocharacter lattice of $T$. Note that $\phi$ is locally a piecewise smooth function.

**Proposition 8.13.** *For a line bundle $L$ on a toric variety $X$, we assume that $\| \ \|$ is a piecewise smooth toric metric as above. Then the following properties are equivalent.*

(i) *The metric $\| \ \|$ is functorial $\delta$-psh.*
(ii) *The metric $\| \ \|$ is functorial psh.*
(iii) *The metric $\| \ \|$ is psh.*
(iv) *The function $\phi$ from (8.12.1) is convex.*

*Proof.* By Proposition 8.12, we may assume that $X = T$. It is clear that (i) yields (ii) and that (ii) yields (iii). By Theorem 8.4, property (iii) implies (iv). Finally, assume that $\phi$ is convex. It follows from Examples 1.4 and 1.5 that the assumptions from Proposition 8.7 are satisfied and hence (i) holds. □

**Corollary 8.14.** *We assume that* $\| \ \|$ *is a toric formal metric on the line bundle* $L$ *of a toric variety* $X$ *over* $K$*. Then there is a unique piecewise linear function* $\phi : N_{\mathbb{R}} \to \mathbb{R}$ *with* (8.12.1)*. Moreover, the formal metric* $\| \ \|$ *is semipositive if and only if* $\phi$ *is convex.*

*Proof.* It follows from 5.9 that a metric is formal if and only if it is piecewise linear. This proves the first claim easily. For a formal metric, Theorem 0.1 shows that semipositive is equivalent to functorial psh. Now the final claim follows from Proposition 8.13. □

This corollary is important for the characterization of all toric continuous metrics on a proper toric variety over $K$ given in the paper of Burgos–Philippon–Sombra over discretely valued fields [BPS14] and generalized in the forthcoming thesis of Julius Hertel.

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W. Gubler, Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

*E-mail address*: walter.gubler@mathematik.uni-regensburg.de

K. Künnemann, Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

*E-mail address*: klaus.kuennemann@mathematik.uni-regensburg.de