

SKELETONS AND TROPICALIZATIONS

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ABSTRACT. Let K be a complete, algebraically closed non-archimedean field with ring of integers K° and let X be a K -variety. We associate to the data of a strictly semistable K° -model \mathcal{X} of X plus a suitable horizontal divisor H a skeleton $S(\mathcal{X}, H)$ in the analytification of X . This generalizes Berkovich's original construction by admitting unbounded faces in the directions of the components of H . It also generalizes constructions by Tyomkin and Baker–Payne–Rabinoff from curves to higher dimensions. Every such skeleton has an integral polyhedral structure. We show that the valuation of a non-zero rational function is piecewise linear on $S(\mathcal{X}, H)$. For such functions we define slopes along codimension one faces and prove a slope formula expressing a balancing condition on the skeleton. Moreover, we obtain a multiplicity formula for skeletons and tropicalizations in the spirit of a well-known result by Sturmfels–Tevelev. We show a faithful tropicalization result saying roughly that every skeleton can be seen in a suitable tropicalization. We also prove a general result about existence and uniqueness of a continuous section to the tropicalization map on the locus of tropical multiplicity one.

1. INTRODUCTION

1.1. Throughout this paper, K denotes an algebraically closed field which is complete with respect to a non-trivial, non-archimedean valuation $v : K \rightarrow \mathbf{R} \cup \{\infty\}$. The corresponding valuation ring is denoted by K° and the value group by $\Gamma := v(K^\times) \subset \mathbf{R}$.

1.2. Tropicalizations. Let X be a K -variety, i.e. an integral, separated K -scheme of finite type. Suppose that $\varphi : X \rightarrow T = \text{Spec}(K[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ is a closed immersion of X into a multiplicative torus. To φ we associate the *tropicalization* $\text{Trop}(X)$ of X . As a set, $\text{Trop}(X) = \text{trop} \circ \varphi(X^{\text{an}})$, where $\text{trop} : T^{\text{an}} \rightarrow \mathbf{R}^n$ is the valuation map given by

$$\text{trop}(p) = (-\log |x_1(p)|, \dots, -\log |x_n(p)|),$$

and $(\)^{\text{an}}$ denotes analytification in the sense of Berkovich. By the Bieri–Groves theorem and work of Speyer–Sturmfels, the tropicalization $\text{Trop}(X)$ can be enriched with the structure of a balanced, weighted, integral polyhedral complex of pure dimension $d = \dim(X)$ (see 2.3 for details). Tropicalizations have proven to be interesting objects to study: on the one hand they are combinatorial in nature, and as such are amenable to explicit calculations; on the other hand, they are rich enough objects to be used as a tool to study the original variety X . An excellent introduction to the subject can be found in the book of Maclagan and Sturmfels [MS15].

As $\text{Trop}(X)$ depends on the embedding φ , for our purposes we will sometimes call $\text{Trop}(X)$ an *embedded* or *parameterized tropicalization* of X .

1.3. Skeletons. Now suppose that X is a proper, smooth K -variety with a strictly semistable K° -model \mathcal{X} . This is a proper, flat scheme over K° with generic fibre X such that the special fibre is a simple normal crossing divisor (see Definition 3.1). Berkovich introduces the skeleton $S(\mathcal{X})$ of \mathcal{X} as a closed subset of X^{an} in [Ber99]. He shows that $S(\mathcal{X})$ is a piecewise linear space of dimension bounded by $\dim(X)$ which is covered by canonical simplices reflecting the stratification of the special fibre \mathcal{X}_s . In particular, the vertices are in bijective correspondence with the irreducible components of \mathcal{X}_s . The skeleton is in a canonical way a proper strong deformation retraction of X^{an} . For details and generalizations to the analytic setting and to pluristable models, we refer to [Ber99, Ber04].

The piecewise linear structure carried by $S(\mathcal{X})$ is strongly analogous to that of a tropicalization $\text{Trop}(X)$. For the purposes of the introduction, we will regard $S(\mathcal{X})$ as an *intrinsic tropicalization* of the variety X .

1.4. The case of curves. In the special case of a smooth projective curve X the skeleton $S(\mathcal{X})$ is a metric graph whose underlying graph is the incidence graph of the special fibre \mathcal{X}_s . In this case, both the skeleton and tropicalizations are metrized graphs. However, the skeleton is bounded while $\text{Trop}(X)$ is unbounded, which makes direct comparisons between the two awkward. To remedy this, Tyomkin [Tyo12] introduces the skeleton $S(\mathcal{X}, H)$ of a marked curve (X, H) , by adding a ray in the direction of every marked point to $S(\mathcal{X})$. With this tool, Tyomkin obtains an algebraic proof and a generalization of Mikhalkin’s correspondence theorem. The latter is the key in Mikhalkin’s pioneering work on Gromov–Witten invariants on the plane. Mikhalkin’s original proof in [Mik05] is based on complex analytic and symplectic techniques which are fundamentally different from the non-archimedean techniques used in our paper.

The program of comparing skeletons and tropicalizations was launched in a systematic way by Baker, Payne and Rabinoff in [BPR11, BPR13] who work more generally over an algebraically closed field K with a non-trivial, non-archimedean, complete, real valuation. (Tyomkin considers a complete discretely valued ground field; one recovers his construction from the one in [BPR11, BPR13] after base extension.) For a projective smooth curve X with a set H of marked points reducing to distinct smooth points in the special fibre of a given semistable model, the skeleton $S(\mathcal{X}, H)$ is realized in [BPR13] as a metrized unbounded graph which is a subset of X^{an} and is the target of a canonical retraction map $\tau : X^{\text{an}} \setminus H \rightarrow S(\mathcal{X}, H)$. If X is embedded in a toric variety with dense torus T such that $X \setminus H \subset X \cap T$, then comparison theorems relating the skeleton $S(\mathcal{X}, H)$ and the tropicalization $\text{Trop}(X \cap T)$ are proved in [BPR11].

1.5. Goals. The overall goal of this paper is a careful study of the relationship between intrinsic and parameterized tropicalizations of a variety X . We generalize a substantial part of the results in [BPR11, BPR13] to higher dimensions. We hope that they will be useful for correspondence theorems in higher dimensions and for applications to arithmetic geometry as for example for the development of a non-archimedean Arakelov theory.

Let us now describe our main results.

1.6. Skeletons for strictly semistable pairs. Suppose that X is a proper, smooth K -variety. A *strictly semistable pair* roughly consists of a strictly semistable K° -model \mathcal{X} of X along with a Cartier divisor H on X such that H plus the special fibre \mathcal{X}_s of \mathcal{X} is a simple normal crossings divisor. See §3 for a precise definition. To such a pair we associate a *skeleton* $S(\mathcal{X}, H)$ of X . The skeleton is a closed subset of the analytification of $U := X \setminus \text{Supp}(H)$ and its dimension is bounded above by $d = \dim(X)$. It turns out that $S(\mathcal{X}, H)$ is a piecewise linear space whose combinatorics reflect the stratification of \mathcal{X}_s associated to $D = H + \mathcal{X}_s$. This means more precisely that for every stratum $S \subset \mathcal{X}_s$ arising from a finite intersection of components of H and \mathcal{X}_s , there is associated a canonical integral Γ -affine polyhedron Δ_S in X^{an} and such polyhedra form an atlas for the skeleton $S(\mathcal{X}, H)$. Note that S is obtained from a stratum T of \mathcal{X}_s by intersecting with horizontal components H_{i_1}, \dots, H_{i_p} of H . We get Δ_S by expanding Δ_T in linearly independent directions corresponding to H_{i_1}, \dots, H_{i_p} and hence we have $\Delta_S \cong \Delta_T \times \mathbf{R}_+^p$. In other words, the skeleton of a strictly semistable pair generalizes the skeleton of a strictly semistable model in Berkovich’s sense by allowing unbounded faces.

We refer to §3–§5 for a detailed study of these skeletons. In particular, we describe the closure $\widehat{S}(\mathcal{X}, H)$ of $S(\mathcal{X}, H)$ in X^{an} which we call the *compactified skeleton*. The main result from these sections is the following.

Theorem 4.13. *Let (\mathcal{X}, H) be a strictly semistable pair and let X be the generic fibre of \mathcal{X} . Then there is a canonical retraction map τ from $X^{\text{an}} \setminus H^{\text{an}}$ onto the skeleton $S(\mathcal{X}, H)$ which extends to a proper strong deformation retraction $\widehat{\tau}$ from X^{an} onto the compactified skeleton $\widehat{S}(\mathcal{X}, H)$.*

We formulate and prove this theorem in the setting of Raynaud’s admissible formal schemes over K° . The proof follows closely Berkovich’s proof of the corresponding fact for $S(\mathcal{X})$ in [Ber99] taking the unbounded part of our building blocks for the skeleton into account.

If X is a curve, the integral Γ -affine structure on the canonical polyhedra in $S(\mathcal{X}, H)$ amounts to a metric structure on the edges and rays of $S(\mathcal{X}, H)$. In this case, the edge lengths are contained in Γ and are induced by the (logarithmic) modulus of various associated generalized open annuli in X^{an} .

1.7. Slope formula. In his thesis [Thu05], Thuillier develops a non-archimedean potential theory on curves and proves an analogue of the Poincaré–Lelong equation. In [BPR13, Theorem 5.15], an interpretation of the Poincaré–Lelong equation in terms of slopes on the skeleton is given. We generalize this slope formula to higher dimensions.

For a non-zero rational function f on X , we show in Proposition 5.6 that the skeleton $S(\mathcal{X}, H)$ can be covered by finitely many integral Γ -affine polyhedra Δ such that the restriction F of $-\log|f|$ to each Δ is integral Γ -affine. The latter means that $F|_{\Delta}$ is an affine function whose linear part is given by a row vector with \mathbf{Z} coefficients and that the constant term is in the value group Γ . Suppose now that $\operatorname{div}(f)$ is supported on the boundary divisor H . In this case F is integral Γ -affine on each canonical polyhedron Δ_S by Proposition 5.2. For a canonical polyhedron Δ_S of dimension $d := \dim(X)$, we define the slope $\operatorname{slope}(F; \Delta_T, \Delta_S)$ of F at Δ_S along a codimension 1 face Δ_T . If $d = 1$, this amounts to the naïve outgoing slope along the edge or ray Δ_S emanating from the point Δ_T , relative to its metric. In higher dimensions however, it is not clear in which direction in Δ_S one should measure the slope of F . We define a canonical direction using some intersection numbers on the special fibre \mathcal{X}_s (see Definition 6.7). With this in hand, we define the divisor of F as the formal sum

$$\widehat{\operatorname{div}}(F) := \sum_{\Delta_T} \sum_{\Delta_S \succ \Delta_T} \operatorname{slope}(F; \Delta_T, \Delta_S) \Delta_T,$$

where Δ_T ranges over all $d - 1$ -dimensional canonical polyhedra of $S(\mathcal{X}, H)$ and where Δ_S ranges over all d -dimensional canonical polyhedra containing Δ_T . Then we show the following slope formula for $S(\mathcal{X}, H)$:

Theorem 6.9. *Let $f \in K(X)^\times$ be a rational function such that $\operatorname{supp}(\operatorname{div}(f)) \subset \operatorname{supp}(H)_\eta$ and let $F = -\log|f|_{|S(\mathcal{X}, H)}$. Then F is continuous and integral Γ -affine on each canonical polyhedron of $S(\mathcal{X}, H)$, and we have*

$$\widehat{\operatorname{div}}(F) = 0.$$

This is a kind of balancing condition on F which is a direct analogue of the balancing condition for tropical varieties. The proof is based on the refined intersection theory of cycles with Cartier divisors on admissible formal schemes over K° given in [Gub98, Gub03]. As the reader might be unfamiliar with this intersection theory in non-noetherian situations, which we use at several places in our paper, we recall it in Appendix A. In the end, the slope formula follows from the basic fact that the degree of a principal divisor intersected with the curve given by the stratum closure of T has degree 0.

From Theorem 6.9 we deduce a slope formula for the bounded skeleton $S(\mathcal{X})$ (see Theorem 6.12). This formula is inspired by and generalizes work of Cartwright [Car13] on tropical complexes, as well as the slope formula for curves as formulated in [BPR13, Theorem 5.15].

A different higher-dimensional generalization of Thuillier’s Poincaré–Lelong formula is given by Chambert–Loir and Ducros in [CD12, Theorem 4.6.5]. It is formulated in terms of differential forms and currents on Berkovich spaces using tropical charts and hence it is not directly related to our skeletal approach. The work of Chambert–Loir and Ducros does not rely on a skeletal theory, and therefore applies to essentially arbitrary analytic spaces; in contrast, our skeletal version is quite explicit and is amenable to calculations (see below).

1.7.1. A two-dimensional example. In §7, we illustrate skeletons and the slope formula in a non-trivial two-dimensional example which is obtained from the abelian variety $A = E^2$ for a Tate elliptic curve E . We choose a regular triangulation of the canonical skeleton of A leading to a K° -model \mathcal{A} of A by Mumford’s construction. Blowing up the closure of the origin 0 of A in \mathcal{A} , we obtain a strictly semistable pair (\mathcal{X}, H) , where H has five components given by the exceptional divisor and the strict transforms of the diagonal, the anti-diagonal, $E \times \{0\}$ and $\{0\} \times E$. Then we illustrate the slope formula for a certain rational function on the generic fibre X with support in the boundary divisor H . It is interesting to compare intersection numbers on \mathcal{X}_s with the combinatorics of the skeleton of (\mathcal{X}, H) .

1.8. Sturmfels–Tevelev multiplicity formula. Let $\varphi : U \rightarrow U'$ be a dominant generically finite morphism of varieties over K . We suppose that U' (resp. U) is a closed subvariety of a multiplicative torus T' (resp. T) and that φ is the restriction of a homomorphism $T \rightarrow T'$. The original

Sturmfels–Tevelev multiplicity formula relates the tropical multiplicities of $\text{Trop}(U)$ and $\text{Trop}(U')$. It is proved in [ST08, Theorem 1.1] for fields with a trivial valuation and in [BPR11, Corollary 8.4] in general. The Sturmfels–Tevelev multiplicity formula is widely used in tropical geometry. For example, it is the basis for integration of differential forms on Berkovich spaces in [CD12] and it is important for implicitization results (see [ST08, §5]).

In Section 8, we develop a similar formula relating the skeleton $S(\mathcal{X}, H)$ of a strictly semistable pair (\mathcal{X}, H) as above to the tropical variety $\text{Trop}(U')$ in the situation when $\varphi : U := X \setminus H \rightarrow U'$ is a dominant generically finite morphism to a closed subvariety U' of a multiplicative torus T with cocharacter group N . We prove in Proposition 8.2 that the map $\text{trop} \circ \varphi : U^{\text{an}} \rightarrow N_{\mathbf{R}}$ factors through the retraction $\tau : U^{\text{an}} \rightarrow S(\mathcal{X}, H)$ and that its restriction to $S(\mathcal{X}, H)$ induces a piecewise linear map $\varphi_{\text{aff}} : S(\mathcal{X}, H) \rightarrow N_{\mathbf{R}}$ with image $\text{Trop}(U')$. Moreover, the restriction of φ_{aff} to any canonical polyhedron Δ_S of $S(\mathcal{X}, H)$ is an integral Γ -affine map, i.e. it is obtained from a linear map defined over \mathbf{Z} and a translation by a Γ -rational vector in $N_{\mathbf{R}}$.

We consider a *regular* point ω of $\text{Trop}(U')$, which means that ω has an integral Γ -affine polyhedron Δ as a neighbourhood in $\text{Trop}(U')$ such that the tropical multiplicity $m_{\text{Trop}}(\Delta)$ of Δ in $\text{Trop}(U')$ is well-defined. See 2.3 for the definition. We assume that ω is not contained in a polyhedron $\varphi_{\text{aff}}(\Delta_S)$ of dimension $< d$ for any canonical polyhedron Δ_S of $S(\mathcal{X}, H)$. Note that such points are dense in $\text{Trop}(U')$. If Δ_S is any canonical polyhedron of $S(\mathcal{X}, H)$ with $\omega \in \varphi_{\text{aff}}(\Delta_S)$, then our assumptions imply that the linear part of φ_{aff} induces an injective map $N_{\Delta_S} \rightarrow N_{\Delta}$ between the underlying lattices of the corresponding polyhedra. Since $\dim(\Delta_S) = d$, the cokernel is finite and hence we get a lattice index which we denote by $[N_{\Delta} : N_{\Delta_S}]$. We prove the following variant of the Sturmfels–Tevelev multiplicity formula.

Theorem 8.4. *Under the hypotheses above, we have*

$$[U : U'] m_{\text{Trop}}(\Delta) = \sum_{\Delta_S} [N_{\Delta} : N_{\Delta_S}],$$

where the sum ranges over all canonical polyhedra Δ_S of the skeleton $S(\mathcal{X}, H)$ with $\text{relint}(\Delta_S) \cap \varphi_{\text{aff}}^{-1}(\omega) \neq \emptyset$.

It follows that $\text{Trop}(U')$ as a weighted polyhedral complex is essentially determined by $S(\mathcal{X}, H)$ and φ_{aff} . The proof relies on similar techniques from non-archimedean analytic geometry as the proof of the torus case given in [BPR11, Corollary 8.4].

A special case of Theorem 8.4 is proved for a smooth curve embedded as a closed subscheme of a torus in [BPR11, Corollary 6.9]. This formula relates the tropical multiplicity of an edge e in the tropicalization of the curve to the amount that the tropicalization map “stretches” the edges of the skeleton mapping to e . Cueto [Cue12, Theorem 2.5] also proves a version of Theorem 8.4 for a closed subvariety of a torus over a trivalently valued field in characteristic 0 with a compactification whose boundary has simple normal crossings.

1.9. Faithful tropicalization. The results outlined above allow one to compute tropicalizations in terms of skeletons; those outlined below show that in certain situations, one can do the reverse. The following faithful tropicalization result roughly says that a given skeleton can be “seen” in a suitable tropicalization.

Theorem 9.5. *Let (\mathcal{X}, H) be a strictly semistable pair with generic fibre X . Then there exists a dense open subset U of X and a morphism $\varphi : U \rightarrow T = \mathbf{G}_{m,K}^n$ such that the restriction φ_{aff} of $\text{trop} \circ \varphi$ to $S(\mathcal{X}, H)$ is a homeomorphism onto its image in \mathbf{R}^n and is unimodular on every polyhedron of $S(\mathcal{X}, H)$.*

Note that U may be a proper subset of $X \setminus H$ and hence φ_{aff} will not necessarily be affine on canonical polyhedra. The unimodularity condition roughly means that φ_{aff} preserves the piecewise integral Γ -affine structure of the skeleton. More formally, φ_{aff} is *unimodular* provided that $S(\mathcal{X}, H)$ has a finite covering by integral Γ -affine polyhedra Δ which are contained in canonical polyhedra such that φ_{aff} restricts to an integral Γ -affine bijective map $\Delta \rightarrow \Delta'$ onto an integral Γ -affine polyhedron Δ' of $N_{\mathbf{R}} = \mathbf{R}^n$ with $[N_{\Delta'} : N_{\Delta}] = 1$. This is a local condition. In the proof, one first uses local equations for strata on \mathcal{X}_s to produce a φ such that φ_{aff} is unimodular but not necessarily globally injective. When \mathcal{X} is quasiprojective, one can separate generic points of strata

on \mathcal{X}_s using finitely many rational functions; these functions in addition to φ give an injective unimodular map. In general one reduces to the quasiprojective case using Chow's lemma.

If X is a curve, Theorem 9.5 says that there exists a morphism $\varphi : X \setminus H' \rightarrow \mathbf{G}_{m,K}^n$ for a finite set of closed points $H' \supset H$ such that φ_{aff} is a homeomorphism and a local isometry from $S(\mathcal{X}, H)$ onto its image in \mathbf{R}^n , with respect to the lattice length on the target. The unimodularity condition in this case translates into the local isometry condition. Such a result is proven in [BPR11, Theorem 6.22].

1.10. Section of Tropicalization. One consequence of the Sturmfels–Tevelev multiplicity formula is that if $\varphi : U \rightarrow U'$ is a birational morphism and if a polyhedron Δ of dimension $d = \dim(X)$ in $S(\mathcal{X}, H)$ maps to a d -dimensional polyhedron $\varphi_{\text{aff}}(\Delta)$ with tropical multiplicity one, then Δ is the only maximal polyhedron mapping to $\varphi_{\text{aff}}(\Delta)$, and the restriction of φ_{aff} to Δ is unimodular. From this it follows that φ_{aff} has a continuous partial section defined on $\varphi_{\text{aff}}(\Delta)$ which is also an integral Γ -affine map.

Motivated by this observation, we prove the following general result on sections of tropicalization maps, which makes no reference to semistable models or to skeletons. Let U be an (irreducible) very affine variety together with a closed immersion $\varphi : U \hookrightarrow T \cong \mathbf{G}_{m,K}^n$, and let $Z \subset \text{Trop}(U)$ be a subset such that every point of Z has tropical multiplicity one. Set $\text{trop}_\varphi = \text{trop} \circ \varphi^{\text{an}} : U^{\text{an}} \rightarrow N_{\mathbf{R}}$.

Theorem 10.6. *For every $\omega \in Z$, the affinoid space $\text{trop}_\varphi^{-1}(\omega)$ has a unique Shilov boundary point $s(\omega)$, and $\omega \mapsto s(\omega)$ defines a continuous partial section $s : Z \rightarrow U^{\text{an}}$ of the tropicalization map $\text{trop}_\varphi : U^{\text{an}} \rightarrow \text{Trop}(U)$ on the subset Z . Moreover, if Z is contained in the closure of its interior in $\text{Trop}(U)$, then s is the unique continuous section of trop_φ defined on Z .*

An affinoid space has a unique Shilov boundary point if and only if the supremum seminorm on its affinoid algebra is multiplicative, in which case the Shilov boundary point is equal to the supremum seminorm. In order to show that the resulting section s is continuous, we reduce to the case of a torus using a toric Noether normalization argument, i.e. by choosing a homomorphism $\alpha : T \rightarrow \mathbf{G}_m^d$ such that $\alpha \circ \varphi$ is finite.

In the case of curves, such a result is proven in [BPR11, Theorem 6.24]. As a higher-dimensional example, the case of the Grassmannian $\text{Gr}(2, n)$ of planes in n -space is studied in [CHW14]. The Plücker embedding of $\text{Gr}(2, n)$ into projective space gives rise to a tropical Grassmannian $\mathcal{T}\text{Gr}(2, n)$ in tropical projective space, which is an example of an extended tropicalization in the sense of [Pay09]. Then [CHW14, Theorem 1.1] states that the tropicalization map $\text{Gr}(2, n)^{\text{an}} \rightarrow \mathcal{T}\text{Gr}(2, n)$ has a continuous section. Incidentally, the construction of the section implies an algebraic result on the structure of the boundary components of the Grassmannian [CHW14, Lemma 5.3]. Note that Theorem 10.6 does not imply the continuity of the section on the whole tropical Grassmannian $\mathcal{T}\text{Gr}(2, n)$. Draisma and Postingshel [DP14] reprove this result with different techniques and use torus actions to obtain sections of the tropicalization map in other explicit situations.

When there is a strictly semistable pair (\mathcal{X}, H) such that $U = X \setminus \text{supp}(H)$, where X is the generic fibre of \mathcal{X} , we show that the image of the section s is contained in $S(\mathcal{X}, H)$. It follows that $s(Z)$ maps homeomorphically onto Z under φ_{aff} , and that φ_{aff} is unimodular on $s(Z)$ in a suitable sense: see Proposition 10.8. In other words, in this case one “sees” the skeleton in the tropicalization.

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2. PRELIMINARIES

2.1. Notation and conventions. An inclusion $A \subset B$ of sets allows the case $A = B$. The complement of A in B is denoted by $B \setminus A$. The sets \mathbf{N} and \mathbf{R}_+ include 0.

If R is a ring with 1, then the group of multiplicative units is denoted by R^\times .

Throughout the paper, K denotes an algebraically closed field endowed with a non-trivial, non-archimedean, complete absolute value $|\cdot|$. Then $v := -\log|\cdot|$ is the corresponding valuation on K with valuation ring $K^\circ := \{\alpha \in K \mid |\alpha| \leq 1\}$, residue field \tilde{K} and value group $\Gamma := v(K^\times)$. The maximal ideal $\{\alpha \in K \mid |\alpha| < 1\}$ is denoted by $K^{\circ\circ}$. The corresponding point in $\text{Spec}(K^\circ)$ is called the special point s . We have $\text{Spec}(K^\circ) = \{\eta, s\}$, where the generic point η corresponds to the trivial ideal $\{0\}$.

By an *analytic space* we mean a K -analytic space in the sense of Berkovich [Ber93, §1.2]. All analytic spaces which occur in this paper are good and hence we may also use the more restricted definition in [Ber90]. The analytification functor from finite-type K -schemes to analytic spaces is denoted $(\)^{\text{an}}$. We distinguish between *affinoid algebras* and *strictly affinoid algebras* as in [Ber90] where they are called K -affinoid algebras and strictly K -affinoid algebras. Note that this is in contrast to [BPR11] and to the literature in rigid geometry as in [BGR84], where affinoid means strictly affinoid. The Berkovich spectrum of an affinoid algebra \mathcal{A} is denoted $\mathcal{M}(\mathcal{A})$. Let $Y = \mathcal{M}(\mathcal{A})$ be an affinoid space and let $\mathcal{A}^\circ \subset \mathcal{A}$ be the subring of power-bounded elements. If \mathcal{A} is strictly affinoid, then the *canonical model* of Y is the K° -formal scheme $\text{Spf}(\mathcal{A}^\circ)$; this is an affine admissible formal scheme when \mathcal{A} is reduced by [BPR11, Theorem 3.17].

A *variety* is an irreducible, reduced, and separated scheme of finite type over the base. A *very affine variety* over a field is a variety which is isomorphic to a closed subvariety of a multiplicative torus.

If X is a scheme over a ring R and R' is an R -algebra, the extension of scalars is denoted $X_{R'} = X \otimes_R R'$. Similarly, if X is a scheme over a base scheme S and $S' \rightarrow S$ is a morphism, the base change is denoted $X_{S'} = X \times_S S'$.

For a scheme \mathcal{X} over K° , the fibre over η is called *the generic fibre* and is denoted by \mathcal{X}_η , and the fibre over s is called *the special fibre* and is denoted by \mathcal{X}_s . Usually we assume that \mathcal{X} is flat. Note that flatness for a K° -variety \mathcal{X} is equivalent to $\mathcal{X}_\eta \neq \emptyset$. In this situation we call \mathcal{X} an *algebraic K° -model* of the generic fibre \mathcal{X}_η . For a Cartier divisor D on a variety \mathcal{X} , there is an associated Weil divisor $\text{cyc}(D)$ and an intersection theory with cycles on \mathcal{X} . As the variety \mathcal{X} need not be noetherian, this intersection theory is not standard and we recall it in Appendix A. Here, we want to emphasize that $\text{div}(f)$ denotes the Cartier divisor associated to a non-zero rational function f on \mathcal{X} and $\text{cyc}(f)$ is the associated Weil divisor. If \mathcal{X} is proper over K° , then we have a reduction map $\text{red} : \mathcal{X}_\eta \rightarrow \mathcal{X}_s$.

Similarly, for an admissible formal scheme \mathfrak{X} over K° , we let \mathfrak{X}_s denote its special fibre and \mathfrak{X}_η its generic fibre. We refer to [BPR11, Section 3.5] for an expository treatment of admissible formal schemes. The generic fibre is an analytic space and the special fibre is a \tilde{K} -scheme. We call \mathfrak{X} a *formal K° -model* of \mathfrak{X}_η . There is a canonical reduction map $\text{red} : \mathfrak{X}_\eta \rightarrow \mathfrak{X}_s$. If \mathfrak{X} is a flat K° -scheme of finite type then its completion $\mathfrak{X} = \widehat{\mathcal{X}}$ with respect to any nonzero element of $K^{\circ\circ}$ is an admissible formal scheme, and $\mathfrak{X}_s = \mathfrak{X}_s$. If \mathcal{X} is proper then $\mathfrak{X}_\eta = \mathcal{X}_\eta^{\text{an}}$. We also use the notation $\text{cyc}(\)$, $\text{div}(\)$ for Cartier and Weil divisors on admissible formal schemes.

We will generally denote schemes over K° using calligraphic letters $\mathcal{X}, \mathcal{Y}, \dots$ and admissible formal schemes over K° using German letters $\mathfrak{X}, \mathfrak{Y}, \dots$. We will use Roman letters X, Y, \dots for both schemes and analytic spaces over K .

The Tate algebra of restricted (i.e. convergent) power series in indeterminates x_1, \dots, x_n with coefficients in K (resp. K°) is denoted $K\langle x_1, \dots, x_n \rangle$ (resp. $K^\circ\langle x_1, \dots, x_n \rangle$). The closed unit ball, viewed as an analytic space, is denoted by \mathbf{B} . It is the Berkovich spectrum $\mathcal{M}(K\langle x \rangle)$. The formal ball $\text{Spf}(K^\circ\langle x \rangle)$ will be denoted by \mathfrak{B} .

If $M \cong \mathbf{Z}^n$ is a finitely generated free abelian group and G is a non-trivial additive subgroup of \mathbf{R} then we set $M_G := M \otimes_{\mathbf{Z}} G$, regarded as a subgroup of the vector space $M_{\mathbf{R}}$.

2.2. Integral Γ -affine polyhedra. Let $M \cong \mathbf{Z}^n$ be a finitely generated free abelian group and let $N = \text{Hom}(M, \mathbf{Z})$. Let $\langle \cdot, \cdot \rangle : M \times N_{\mathbf{R}} \rightarrow \mathbf{R}$ denote the canonical pairing. An *integral Γ -affine polyhedron* in $N_{\mathbf{R}}$ is a subset of $N_{\mathbf{R}}$ of the form

$$\Delta = \{v \in N_{\mathbf{R}} \mid \langle u_i, v \rangle + \gamma_i \geq 0 \text{ for all } i = 1, \dots, r\}$$

for some $u_1, \dots, u_r \in M$ and $\gamma_1, \dots, \gamma_r \in \Gamma$. Any face of an integral Γ -affine polyhedron Δ is again integral Γ -affine. The relative interior of Δ is denoted $\text{relint}(\Delta)$. A bounded polyhedron is called a *polytope*. An *integral Γ -affine polyhedral complex* in $N_{\mathbf{R}}$ is a polyhedral complex whose faces are integral Γ -affine. The *dimension* of an integral Γ -affine polyhedral complex Σ is $\dim(\Sigma) := \max\{\dim(\Delta) \mid \Delta \in \Sigma\}$. We say that Σ has *pure dimension* d provided that every maximal polyhedron of Σ (with respect to inclusion) has dimension d .

An *integral Γ -affine function* on $N_{\mathbf{R}}$ is a function of the form

$$v \mapsto \langle u, v \rangle + \gamma : N_{\mathbf{R}} \rightarrow \mathbf{R}$$

for some $u \in M$ and $\gamma \in \Gamma$. More generally, let M' be a second finitely generated free abelian group and let $N' = \text{Hom}(M', \mathbf{Z})$. An *integral Γ -affine map* from $N_{\mathbf{R}}$ to $N'_{\mathbf{R}}$ is a function of the form $F = \varphi^* + v$, where $\varphi : M' \rightarrow M$ is a homomorphism, $\varphi^* : N_{\mathbf{R}} \rightarrow N'_{\mathbf{R}}$ is the dual homomorphism extended to $N_{\mathbf{R}}$, and $v \in N'_{\mathbf{R}}$. If $N' = M' = \mathbf{Z}^m$ and $F = (F_1, \dots, F_m) : N_{\mathbf{R}} \rightarrow \mathbf{R}^m$ is a function, then F is integral Γ -affine if and only if each coordinate $F_i : N_{\mathbf{R}} \rightarrow \mathbf{R}$ is integral Γ -affine.

An *integral Γ -affine map* from an integral Γ -affine polyhedron $\Delta \subset N_{\mathbf{R}}$ to $N'_{\mathbf{R}}$ is by definition the restriction to Δ of an integral Γ -affine map $N_{\mathbf{R}} \rightarrow N'_{\mathbf{R}}$. If $\Delta' \subset N'_{\mathbf{R}}$ is an integral Γ -affine polyhedron then a function $F : \Delta \rightarrow \Delta'$ is *integral Γ -affine* if the composition $\Delta \rightarrow \Delta' \hookrightarrow N'_{\mathbf{R}}$ is integral Γ -affine.

Let $\Delta \subset N_{\mathbf{R}}$ be an integral Γ -affine polyhedron. Let $(N_{\mathbf{R}})_{\Delta}$ be the linear span of $\Delta - v$ for any $v \in \Delta$ and let $N_{\Delta} = N \cap (N_{\mathbf{R}})_{\Delta}$. This is a saturated subgroup of N . If $F : \Delta \rightarrow N'_{\mathbf{R}}$ is an integral Γ -affine map as above then the image of F is an integral Γ -affine polyhedron Δ' in $N'_{\mathbf{R}}$. Extending F to $N_{\mathbf{R}}$, by definition the linear part of F takes N into N' , hence induces a homomorphism $F_{\Delta} : N_{\Delta} \rightarrow N'_{\Delta'}$, which is independent of the extension of F to $N_{\mathbf{R}}$. We define the *lattice index* of F to be

$$[N'_{\Delta'} : N_{\Delta}] := [N'_{\Delta'} : F_{\Delta}(N_{\Delta})].$$

We say that F is *unimodular* provided that it satisfies any of the following equivalent conditions:

- (1) F is injective and its lattice index is 1.
- (2) Every integral Γ -affine function $f : \Delta \rightarrow \mathbf{R}$ is of the form $f' \circ F$ for an integral Γ -affine function $f' : \Delta' \rightarrow \mathbf{R}$.
- (3) F is injective and the image of N_{Δ} under the linear part of F is saturated in N' .
- (4) F is injective and the inverse function $\Delta' \rightarrow \Delta$ is integral Γ -affine.

2.3. Tropicalization. Let $T \cong \mathbf{G}_{m,K}^n$ be an algebraic K -torus, let $M \cong \mathbf{Z}^n$ be the character lattice of T , and let $N = \text{Hom}(M, \mathbf{Z})$ be its cocharacter lattice. For $u \in M$ the corresponding character of T is denoted χ^u . The *tropicalization map* is the continuous, proper surjection $\text{trop} : T^{\text{an}} \rightarrow N_{\mathbf{R}}$ defined by

$$\langle \text{trop}(p), u \rangle = -\log |\chi^u(p)|,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between M and $N_{\mathbf{R}}$. Choose a basis u_1, \dots, u_n for M , let $x_i = \chi^{u_i}$, and identify $N_{\mathbf{R}}$ with \mathbf{R}^n using the dual basis. Then $K[M] = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and we have

$$\text{trop}(p) = (-\log |x_1(p)|, \dots, -\log |x_n(p)|).$$

If $\varphi : U \hookrightarrow T$ is a closed subscheme, we put $\text{trop}_{\varphi} = \text{trop} \circ \varphi^{\text{an}}$, and we define the *tropicalization* of U to be the subset

$$\text{Trop}(U) := \text{trop}_{\varphi}(U^{\text{an}}) \subset N_{\mathbf{R}}.$$

The tropicalization map restricts to a continuous, proper surjection $\text{trop}_{\varphi} : U^{\text{an}} \rightarrow \text{Trop}(U)$. By the Bieri–Groves theorem [Gub13b, Theorem 3.3], $\text{Trop}(U)$ is the support of an integral Γ -affine polyhedral complex Σ in $N_{\mathbf{R}}$. If U is a variety of dimension d then Σ has pure dimension d .

Via the closed embedding φ we have $U \cong K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/\mathfrak{a}$ for some ideal \mathfrak{a} in the Laurent polynomial ring. Fix $\omega = (\omega_1, \dots, \omega_n) \in \mathbf{R}^n$, and put $r_i = \exp(-\omega_i) \in \mathbf{R}$. We write $r = (r_1, \dots, r_n)$, $x = (x_1, \dots, x_n)$ and use multi-index notation where convenient. In particular, we put $K\langle r^{-1}x, rx^{-1} \rangle = K\langle r_1^{-1}x_1, \dots, r_n^{-1}x_n, r_1x_1^{-1}, \dots, r_nx_n^{-1} \rangle$. The poly-annulus $\mathcal{M}(K\langle r^{-1}x, rx^{-1} \rangle) = \text{trop}^{-1}(\omega)$ is an affinoid subdomain of T^{an} , which is strict if and only if $\omega_1, \dots, \omega_n \in \Gamma = v(K^{\times})$. The Banach norm on $K\langle r^{-1}x, rx^{-1} \rangle$ is denoted by $\|\sum_I a_I x^I\|_r = \max_I |a_I| r^I$. Assume that $\omega \in \text{Trop}(U)$, and put

$$A_{\omega} = K\langle r^{-1}x, rx^{-1} \rangle / \mathfrak{a} K\langle r^{-1}x, rx^{-1} \rangle.$$

Then $U_\omega := \text{trop}_\varphi^{-1}(\omega)$ can be identified with the affinoid subdomain $\mathcal{M}(A_\omega)$ of U^{an} .

Choose an algebraically closed, complete valued extension field L/K whose value group Γ_L is large enough so that $\omega \in N_{\Gamma_L}$. Choose $t \in T(L)$ such that $\text{trop}(t) = \omega$. The *initial degeneration* $\text{in}_\omega(U)$ of U at ω is the special fibre of the schematic closure of $t^{-1}U_L$ in the L° -torus $T_{L^\circ} = \text{Spec}(L^\circ[M])$. This means that $\text{in}_\omega(U)$ is a closed subscheme of $T_{\tilde{L}} = \text{Spec}(\tilde{L}[M])$. See [Gub13b, §5] for a discussion of initial degenerations and the dependence on L and t .

Let g be a non-zero element of $L\langle r^{-1}x, rx^{-1} \rangle$. Since each $r_i = \exp(-\omega_i)$ is contained in the value group of L , we have $\|g\|_r = |c|$ for $c \in L$. As $|t_i| = r_i$ for all i , the Laurent series $c^{-1}g(tx)$ is an element of $L\langle x, x^{-1} \rangle^\circ$. Its image under the reduction map

$$L\langle x, x^{-1} \rangle^\circ \longrightarrow \tilde{L}[x, x^{-1}]$$

is the *initial form* of g , which we denote by $\text{in}_\omega(g)$.

The *initial ideal* of $\mathfrak{a} \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is defined as the ideal generated by all initial forms of polynomials in \mathfrak{a} :

$$\text{in}_\omega(\mathfrak{a}) = (\text{in}_\omega(g) \mid g \in \mathfrak{a}) \subset \tilde{L}[x, x^{-1}].$$

The closed subscheme of $T_{\tilde{L}} = \text{Spec}(\tilde{L}[M])$ given by the initial ideal $\text{in}_\omega(\mathfrak{a})$ is the initial degeneration $\text{in}_\omega(U)$. The initial degeneration is well-defined up to translation by elements of $T_{\tilde{L}}(\tilde{L})$, and up to the choice of the field L . See [Gub13b, Definition 10.6]. If $\omega \in N_\Gamma$ we will always take $L = K$.

Let $\omega \in \text{Trop}(U)$. The *tropical multiplicity* $m_{\text{Trop}}(\omega)$ of the point ω is the number of irreducible components of $\text{in}_\omega(U)$, counted with multiplicity. This quantity is independent of all choices involved in the definition of $\text{in}_\omega(U)$. Suppose now that U is a variety of dimension d . Let Σ be an integral Γ -affine polyhedral complex with support $\text{Trop}(U)$ and let $\Delta \in \Sigma$ be a maximal (i.e. d -dimensional) polyhedron. If Σ is sufficiently fine then for $\omega \in \text{relint}(\Delta) \cap N_\Gamma$ the initial degeneration $\text{in}_\omega(U)$ is isomorphic to $Y \times \mathbb{G}_{m, \tilde{K}}^d$ for a dimension-zero \tilde{K} -scheme Y . Different choices of $\omega \in \text{relint}(\Delta) \cap N_\Gamma$ give rise to isomorphic initial degenerations. By [MS15, Section 3.3] the multiplicity $m_{\text{Trop}}(\omega)$ is constant on the relative interior of Δ , hence we call it the *tropical multiplicity* $m_{\text{Trop}}(\Delta)$ of Δ . Equivalently $m_{\text{Trop}}(\Delta)$ is the length of the dimension-zero scheme Y : this is explained in [BPR11, Theorem 4.29].

More generally, if $\varphi : U \rightarrow T$ is any morphism then we define $\text{Trop}(U)$ to be the tropicalization of the schematic image U' of U . Initial degenerations and tropical multiplicities are all defined with respect to U' .

3. STRICTLY SEMISTABLE PAIRS

In this section, we give a variant of de Jong's notion of a strictly semistable pair (\mathcal{X}, D) in the case of a base $\text{Spec}(K^\circ)$ for the valuation ring K° of our valued field K . Roughly speaking, a strictly semistable pair (\mathcal{X}, D) over K° consists of a strictly semistable proper scheme \mathcal{X} over K° and a divisor D on \mathcal{X} with strictly normal crossings which includes the special fibre. It is convenient to only include the horizontal part of the divisor as part of the data, as it is Cartier whereas the vertical part may not be.

Definition 3.1. A *strictly semistable pair* (\mathcal{X}, H) consists of an irreducible proper flat scheme \mathcal{X} over K° and a sum $H = H_1 + \dots + H_S$ of distinguished effective Cartier divisors H_i on \mathcal{X} such that \mathcal{X} is covered by open subsets \mathcal{U} which admit an étale morphism

$$(3.1.1) \quad \psi : \mathcal{U} \longrightarrow \mathcal{S} := \text{Spec}(K^\circ[x_0, \dots, x_d]/\langle x_0 \cdots x_r - \pi \rangle)$$

for $r \leq d$ and $\pi \in K^\times$ with $|\pi| < 1$. We assume that each H_i has irreducible support and that the restriction of H_i to \mathcal{U} is either trivial or defined by $\psi^*(x_j)$ for some $j \in \{r+1, \dots, d\}$.

3.2. The underlying scheme \mathcal{X} is called a *strictly semistable scheme* over K° . Note that such a scheme is a variety over K° in the sense of 2.1. Indeed, reducedness follows from [EGAIV₄, Proposition 17.5.7]. Note that \mathcal{X} is not noetherian, but the underlying topological space is noetherian and of topological dimension $d+1$. This follows from the fact that the generic fibre and the special fibre are both noetherian.

Similarly, de Jong [dJ96] defined strictly semistable schemes and strictly semistable pairs over a complete discrete valuation ring R , where π is a uniformizer. Suppose that R is a subring of our algebraically closed field K and that the discrete valuation of R extends to our given complete valuation v . By [dJ96, 2.16], the base change of a strictly semistable scheme over R (in the sense of de Jong) to the valuation ring K° is a strictly semistable scheme as above. By [dJ96, 6.4], the same applies for strictly semistable pairs if we neglect the vertical components of the divisor.

Notation 3.3. We fix the following notation for a strictly semistable pair (\mathcal{X}, H) . The generic fibre of \mathcal{X} is denoted by X . Let \mathcal{H}_i be the closed subscheme of \mathcal{X} locally cut out by a defining equation for H_i , and let $\mathcal{H} = \bigcup_{i=1}^S \mathcal{H}_i$. Let V_1, \dots, V_R be the irreducible components of the special fibre \mathcal{X}_s . Define $D_i = V_i$ for $i \leq R$ and $D_i = \mathcal{H}_{i-R}$ for $R < i \leq N := R + S$, and set $D = \sum_{i=1}^N D_i$. This is a Weil divisor on \mathcal{X} . We call the \mathcal{H}_i (resp. V_j) the *horizontal components* (resp. *vertical components*) of D .

3.4. Let (\mathcal{X}, H) be a strictly semistable pair. It is clear that the generic fibre of each \mathcal{H}_i is smooth. Since H_i is Cartier on \mathcal{X} , the special fibre $(\mathcal{H}_i)_s$ is the support of a Cartier divisor on \mathcal{X}_s , so it has pure codimension one in \mathcal{X}_s . Each V_j is a smooth \tilde{K} -scheme because, locally, we have $\psi^*(x_k) = 0$ on $V_j \cap \mathcal{U}_s$ for some k , so $V_j \cap \mathcal{U}_s$ is étale over an affine space. We may regard V_j as a Weil divisor on \mathcal{X}_s , but it is not necessarily the support of a Cartier divisor on \mathcal{X} : see Proposition 4.17.

The generic fibre X of \mathcal{X} is smooth as the generic fibre of the scheme \mathcal{S} of (3.1.1) is smooth. It is clear that $d = \dim(X)$, but r, s may depend on the choice of \mathcal{U} . In this generality, $v(\pi)$ may also depend on the choice of \mathcal{U} ; when X is a curve, this reflects the fact that the edges of the skeleton of \mathcal{X} may have different lengths. This is related to the fact that the V_j may not be Cartier; again see Proposition 4.17.

Example 3.5. Let (\mathcal{X}, H) be a strictly semistable pair of relative dimension one. Then its generic fibre $X = \mathcal{X}_\eta$ is a smooth, proper, connected K -curve, and its special fibre \mathcal{X}_s has smooth irreducible components and at worst nodal singularities. The Cartier divisor H amounts to a finite collection of points in $X(K)$ which reduce to distinct smooth points of $\mathcal{X}_s(\tilde{K})$.

Remark 3.6. Let (\mathcal{X}, H) be a strictly semistable pair and let \mathcal{H}_k be a horizontal component of D . Then $(\mathcal{H}_k, H|_{\mathcal{H}_k})$ is again a strictly semistable pair, where $H|_{\mathcal{H}_k} = \sum_{j \neq k} H_j|_{\mathcal{H}_k}$. This is immediate from Definition 3.1. Note however that $H_j|_{\mathcal{H}_k}$ does not necessarily have irreducible support; it must be broken up into a sum of irreducible Cartier divisors.

It is also useful to have a notion of a strictly semistable pair in the category of admissible formal schemes. Definition 3.1 carries over verbatim. The definition is constructed so as not to allow the horizontal components to intersect themselves in the generic fibre — this condition is not local in the analytic topology.

Definition 3.7. A *formal strictly semistable pair* (\mathfrak{X}, H) consists of a connected quasi-compact admissible formal K° -scheme \mathfrak{X} and a sum $H = H_1 + \dots + H_S$ of distinguished effective Cartier divisors on \mathfrak{X} such that \mathfrak{X} is covered by formal open subsets \mathfrak{U} which admit an étale morphism

$$(3.7.1) \quad \psi : \mathfrak{U} \longrightarrow \mathrm{Spf} \left(K^\circ \langle x_0, \dots, x_d \rangle / \langle x_0 \dots x_r - \pi \rangle \right)$$

for $r \leq d$ and $\pi \in K^\times$ with $|\pi| < 1$. We assume that each $H_i|_{\mathfrak{X}_\eta}$ has irreducible support and that $H_i|_{\mathfrak{U}}$ is defined by $\psi^*(x_j)$ for some $j > r$ unless it is trivial.

3.8. We use notation analogous to 3.3 for formal strictly semistable pairs. That is, we let $X = \mathfrak{X}_\eta$ be the generic fibre of \mathfrak{X} and \mathfrak{X}_s its special fibre. We define \mathfrak{H}_i as the admissible formal closed subscheme of \mathfrak{X} locally cut out by a defining equation for H_i . Its generic fibre $(\mathfrak{H}_i)_\eta$ is an irreducible Weil divisor on X and its special fibre is a Weil divisor on \mathfrak{X}_s . We also set $\mathfrak{H} = \bigcup_{i=1}^S \mathfrak{H}_i$. Let V_1, \dots, V_R be the irreducible components of \mathfrak{X}_s , and define $D_i = V_i$ for $i \leq R$ and $D_i = \mathfrak{H}_{i-R}$ for $R < i \leq N := R + S$. Set $D = \sum_{i=1}^N D_i$. We may regard D as a Weil divisor on \mathfrak{X} in the sense of [Gub98, §3], with horizontal components \mathfrak{H}_i and vertical components V_j . (In *loc. cit.* the horizontal divisors live on the generic fibre X , but for our purposes it is convenient to remember the special fibre of the \mathfrak{H}_i .)

The remarks made in 3.4 apply to formal strictly semistable pairs as well.

3.9. Let \mathfrak{X} be a *strictly semistable formal scheme* over K° which means that $(\mathfrak{X}, 0)$ is a formal strictly semistable pair. We consider a formal open subset \mathfrak{U} of \mathfrak{X} . Then \mathfrak{U} is formal affine if and only if \mathfrak{U}_s is an affine open subscheme of \mathfrak{X}_s . Indeed, the special fibre of \mathfrak{X}_s is reduced and hence we may use [Gub98, Proposition 1.11] to deduce the claim from a theorem of Bosch [Bos77, Theorem 3.1] about formal analytic varieties. Note that in this case, $\text{red}^{-1}(\mathfrak{U}_s)$ is an affinoid domain in \mathfrak{X}_η .

Example 3.10. Let $r, s, d \in \mathbb{N}$ with $r + s \leq d$. In the case $r > 0$, we choose $\pi \in K^\times$ with $|\pi| < 1$. If $r = 0$, then we always take $\pi := 1$ to avoid ambiguities. Let \mathfrak{B} be the formal ball of radius 1 and let $\mathfrak{U}_{\Delta(r, \pi)}$ be the canonical model of the polytopal domain [Gub13b, §6] in the hyperplane $x_0 \cdots x_r = \pi$ of $\mathbf{B}^{r+1} := \mathfrak{B}_\eta^{r+1}$ associated to the simplex

$$(3.10.1) \quad \Delta(r, \pi) := \{\mathbf{v} \in \mathbf{R}_+^{r+1} \mid v_0 + \cdots + v_r = v(\pi)\}.$$

We consider the strictly semistable formal scheme $\mathfrak{S} := \mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{B}^{d-r}$. Note that

$$\mathfrak{S} \cong \text{Spf}(K^\circ \langle x_0, \dots, x_d \rangle / \langle x_0 \cdots x_r - \pi \rangle).$$

Then $(\mathfrak{S}, H(s))$ is a formal strictly semistable pair, where $H(s)$ is the principal Cartier divisor defined by $x_{r+1} \cdots x_{r+s}$. We call $(\mathfrak{S}, H(s))$ a *standard pair*. The isomorphism class of the formal scheme \mathfrak{S} of a standard pair is uniquely determined by $(r, d, v(\pi))$.

Remark 3.11. By the definitions, any formal strictly semistable pair (\mathfrak{X}, H) is covered by formal open subsets \mathfrak{U} with an étale morphism

$$(3.11.1) \quad \psi : \mathfrak{U} \longrightarrow \mathfrak{S} = \text{Spf}(K^\circ \langle x_0, \dots, x_d \rangle / \langle x_0 \cdots x_r - \pi \rangle)$$

to the formal scheme \mathfrak{S} of a standard pair $(\mathfrak{S}, H(s))$ such that $H|_{\mathfrak{U}} = \psi^*(H(s))$. We have only to note that the morphism in (3.7.1) is not changed by our choice $\pi = 1$ in case of $r = 0$.

3.12. For a proper, flat K° -scheme we let $\widehat{\mathcal{X}}$ be its completion with respect to a non-zero element of K° . Let $\mathcal{U} \subset \mathcal{X}$ be an open subset which admits an étale morphism $\psi : \mathcal{U} \rightarrow \mathcal{S}$ as in (3.11.1). Taking completions, we have an étale morphism in the category of admissible formal K° -schemes

$$\psi : \widehat{\mathcal{U}} \longrightarrow \mathfrak{S} := \text{Spf}(K^\circ \langle x_0, \dots, x_d \rangle / \langle x_0 \cdots x_r - \pi \rangle).$$

Since \mathcal{X} is proper, the analytification of \mathcal{X}_η is naturally identified with the analytic generic fibre of $\widehat{\mathcal{X}}$. A Cartier divisor H on \mathcal{X} naturally induces a Cartier divisor \widehat{H} on $\widehat{\mathcal{X}}$, and the analytification of an irreducible closed subscheme of \mathcal{X}_η is an irreducible Zariski-closed subspace of $\mathcal{X}_\eta^{\text{an}}$ [Con99, Theorem 2.3.1]. Hence we have shown:

Proposition 3.13. *If (\mathcal{X}, H) is a strictly semistable pair then $(\widehat{\mathcal{X}}, \widehat{H})$ is a formal strictly semistable pair.*

Remark 3.14. The converse to Proposition 3.13 is also true: if \mathcal{X} is an irreducible proper flat K° -scheme and H is an effective Cartier divisor on \mathcal{X} such that $(\widehat{\mathcal{X}}, \widehat{H})$ is a formal strictly semistable pair then (\mathcal{X}, H) is a strictly semistable pair. We will not need this fact.

3.15. The Weil divisor $D = \sum_{i=1}^N D_i = \sum_{i=1}^R V_i + \sum_{j=1}^S \mathcal{H}_j$ of a strictly semistable pair (\mathcal{X}, H) has a stratification, where a *stratum* S is given as an irreducible component of $\bigcap_{i \in I} D_i \setminus \bigcup_{i \notin I} D_i$ for any $I \subset \{1, \dots, N\}$. This is a special case of a more general definition of strata given in [Ber99], §2. It is easy to see that the closure \overline{S} of a stratum S is a strata subset, i.e. a disjoint union of strata. Note also that two strata are either disjoint or one of them is contained in the closure of the other. The set of strata of D is partially ordered by $S \leq T$ if and only if $\overline{S} \subseteq \overline{T}$.

If S is contained in the generic fibre, then it follows from Remark 3.6 that $(\overline{S}, H|_{\overline{S}})$ is a strictly semistable pair. We let $\text{str}(\mathcal{X}_s, H)$ denote the set of vertical strata, i.e. the strata contained in the special fibre \mathcal{X}_s . Note that the closure of a vertical stratum is smooth over \tilde{K} .

3.15.1. The Weil divisor $D = \sum_{i=1}^R V_i + \sum_{j=1}^S \mathfrak{H}_j$ of a formal strictly semistable pair (\mathfrak{X}, H) has a stratification defined in the same way, where we consider \mathfrak{H}_j as a disjoint union of its generic and special fibres. If we handle the horizontal strata with some care, then everything from above applies. We denote the set of vertical strata by $\text{str}(\mathfrak{X}_s, H)$. Let us explain how we treat a horizontal stratum S of D . The closure of S in the generic fibre \mathfrak{X}_η is a closed analytic subvariety Y of \mathfrak{X}_η .

The closure of Y in \mathfrak{X} , defined similarly to the scheme-theoretic closure in algebraic geometry, is an admissible formal scheme \mathfrak{Y} over K° which is a closed formal subscheme of \mathfrak{X} and which has generic fibre Y (see [Gub98, Proposition 3.3]). Setting $\overline{S} := \mathfrak{Y}$, we can show as above that $(\overline{S}, H|_{\overline{S}})$ is a formal strictly semistable pair. For the definition of the partial ordering \leq , we always view \overline{S} as the strata subset given by the disjoint union of \mathfrak{Y}_η and \mathfrak{Y}_s .

4. THE SKELETON OF A STRICTLY SEMISTABLE PAIR

In this section, (\mathfrak{X}, H) denotes a formal strictly semistable pair. As always we use the associated notation introduced in 3.8. We will define the skeleton $S(\mathfrak{X}, H)$ which generalizes the skeleton for strictly semistable formal schemes over K° introduced by Berkovich in [Ber99] and [Ber04]. The horizontal divisor H is the new ingredient here. The skeleton $S(\mathfrak{X}, H)$ is well-known in the case of curves (see e.g. [Tyo12] and [BPR13]). In particular, we will obtain the skeleton of an algebraic strictly semistable pair (\mathcal{X}, H) by using the formal completion $(\widehat{\mathcal{X}}, \widehat{H})$ as in 3.12 and setting $S(\mathcal{X}, H) := S(\widehat{\mathcal{X}}, \widehat{H})$.

We will define $S(\mathfrak{X}, H)$ in such a way that the vertical strata $S \in \text{str}(\mathfrak{X}_s, H)$ are in one-to-one correspondence with the open faces of the skeleton. We will do so first on the standard pairs $(\mathfrak{S}, H(s))$ from 3.10 and then on \mathfrak{X} using the étale morphisms $\psi : \mathfrak{U} \rightarrow \mathfrak{S}$ of (3.11.1). For this, it is easier if there is a unique stratum S on \mathfrak{U}_s which maps to the minimal vertical stratum of $(\mathfrak{S}, H(s))$; in this case, the part of the skeleton of (\mathfrak{X}, H) contained in \mathfrak{U}_η will map homeomorphically onto the skeleton of $(\mathfrak{S}, H(s))$. The following proposition states that such \mathfrak{U} exist for each stratum $S \in \text{str}(\mathfrak{X}_s, H)$. Recall from 3.8 that D is the Weil divisor on \mathfrak{X} given as the sum of the Weil divisor induced by H and the Weil divisor induced by the special fibre of \mathfrak{X} .

Proposition 4.1. *Let (\mathfrak{X}, H) be a formal strictly semistable pair. Any open covering of \mathfrak{X}_s admits a refinement $\{\mathfrak{U}_s\}$ by affine open subsets \mathfrak{U}_s satisfying the following properties:*

- The formal open subscheme \mathfrak{U} of \mathfrak{X} with underlying set \mathfrak{U}_s admits an étale morphism $\psi : \mathfrak{U} \rightarrow \mathfrak{S} = \mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{B}^{d-r}$ as in (3.11.1) such that $(\mathfrak{U}, H|_{\mathfrak{U}})$ is the pull-back of the standard pair $(\mathfrak{S}, H(s))$ for some $s \in \{0, \dots, d-r\}$.*
- There is a distinguished vertical stratum S of D associated to \mathfrak{U}_s such that for any stratum T of D , we have $S \subset \overline{T}$ if and only if $\mathfrak{U}_s \cap \overline{T} \neq \emptyset$.*
- The distinguished stratum S from (b) is given on \mathfrak{U}_s by $\psi^{-1}(x_0 = \dots = x_{r+s} = 0)$ in terms of the étale morphism ψ in (a).*
- Every vertical stratum of D is the distinguished stratum of a suitable \mathfrak{U}_s .*

The arguments are similar as in the proof of Proposition 5.2 in [Gub10]. We leave the details to the reader.

When choosing a covering of \mathfrak{X} as in Definition 3.7, we will always pass to a refinement whose special fibre $\{\mathfrak{U}_s\}$ satisfies Proposition 4.1. The formal open subschemes \mathfrak{U} induced on the open subsets \mathfrak{U}_s are called the *building blocks* of the formal strictly semistable pair (\mathfrak{X}, H) . By 3.9, building blocks are formal affine open subschemes of \mathfrak{X} . For any building block \mathfrak{U} , the étale morphism ψ from (a) induces a bijective correspondence between $\text{str}(\mathfrak{U}_s, H|_{\mathfrak{U}})$ and the vertical strata of the standard pair $(\mathfrak{S}, H(s))$ of 3.10.

4.2. The skeleton of a standard pair. We start by defining the skeleton of the standard pair $(\mathfrak{S}, H(s))$ of 3.10, where $\mathfrak{S} = \mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{B}^{d-r} = \text{Spf}(K^\circ \langle x_0, \dots, x_d \rangle / \langle x_0 \cdots x_r - \pi \rangle)$. For $\varepsilon \in |K^\times|$ with $0 < \varepsilon < 1$, consider the affinoid annulus $U_\varepsilon := \{x \in \mathbf{B}^1 \mid |x| \geq \varepsilon\}$. The canonical model \mathfrak{U}_ε of U_ε is an admissible formal affine scheme over K° which is strictly semistable; indeed, we have $\mathfrak{U}_\varepsilon \cong \text{Spf}(K^\circ \langle y_0, y_1 \rangle / \langle y_0 y_1 - a_\varepsilon \rangle)$, where $a_\varepsilon \in K^\times$ is any element with $|a_\varepsilon| = \varepsilon$. We conclude that $\mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{U}_\varepsilon^s \times \mathfrak{B}^{d-r-s}$ is a strictly polystable formal scheme and we define its skeleton $S(\mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{U}_\varepsilon^s \times \mathfrak{B}^{d-r-s})$ as in [Ber99, §5], or in [Ber04, §4]. This is a closed subset of the generic fibre $U_{\Delta(r, \pi)} \times U_\varepsilon^s \times \mathbf{B}^{d-r-s}$ which is homeomorphic to $\Delta(r, \pi) \times [0, -\log \varepsilon]^s$. It has the following explicit description. First note that projection onto x_1, \dots, x_d induces an isomorphism of $U_{\Delta(r, \pi)} \times U_\varepsilon^s \times \mathbf{B}^{d-r-s}$ onto the affinoid domain

$$U = \{p \in \mathbf{B}^d \mid -\log |x_1(p)| - \dots - \log |x_r(p)| \leq v(\pi), \\ -\log |x_j(p)| \leq -\log \varepsilon \text{ for } j = r+1, \dots, r+s\}$$

in \mathbf{B}^d , and similarly that projection onto x_1, \dots, x_{r+s} maps $\Delta(r, \pi) \times [0, -\log \varepsilon]^s$ homeomorphically onto

$$S = \{(v_1, \dots, v_{r+s}) \in \mathbf{R}_+^{r+s} \mid v_1 + \dots + v_r \leq v(\pi), v_j \leq -\log \varepsilon \text{ for } j = r+1, \dots, r+s\}.$$

For $\mathbf{v} = (v_1, \dots, v_{r+s}) \in S$ we define a bounded multiplicative norm $\|\cdot\|_{\mathbf{v}} \in U$ by the formula

$$\left\| \sum_{\mathbf{m}} a_{\mathbf{m}} x^{\mathbf{m}} \right\|_{\mathbf{v}} = \max_{\mathbf{m}} |a_{\mathbf{m}}| \exp(-m_1 v_1 - \dots - m_{r+s} v_{r+s}),$$

where $\mathbf{m} = (m_1, \dots, m_d)$ ranges over $\mathbf{Z}^{r+s} \times \mathbf{N}^{d-r-s}$. This gives a continuous inclusion $S \hookrightarrow U$, hence an inclusion $\Delta(r, \pi) \times [0, -\log \varepsilon]^s \hookrightarrow U_{\Delta(r, \pi)} \times U_{\varepsilon}^s \times \mathbf{B}^{d-r-s}$. The image is the skeleton $S(\mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{U}_{\varepsilon}^s \times \mathfrak{B}^{d-r-s})$. Note that

$$p \mapsto (-\log |x_0(p)|, \dots, -\log |x_{r+s}(p)|) : S(\mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{U}_{\varepsilon}^s \times \mathfrak{B}^{d-r-s}) \longrightarrow \Delta(r, \pi) \times [0, -\log \varepsilon]^s$$

is a continuous inverse to the above inclusion.

We define $S(\mathfrak{S}, H(s)) \subset \mathfrak{S}_{\eta} = U_{\Delta(r, \pi)} \times \mathbf{B}^{d-r}$ as the union of the skeletons $S(\mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{U}_{\varepsilon}^s \times \mathfrak{B}^{d-r-s})$ as $\varepsilon \rightarrow 0$. We see by the above description that $S(\mathfrak{S}, H(s))$ is a closed subset of

$$U_{\Delta(r, \pi)} \times (\mathbf{B} \setminus \{0\})^s \times \mathbf{B}^{d-r-s} = \bigcup_{\varepsilon \rightarrow 0} U_{\Delta(r, \pi)} \times U_{\varepsilon}^s \times \mathbf{B}^{d-r-s}$$

which is homeomorphic to $\Delta(r, \pi) \times \mathbf{R}_+^s$. The homeomorphism is given by the restriction of the map

$$\text{Val} : U_{\Delta(r, \pi)} \times (\mathbf{B} \setminus \{0\})^s \times \mathbf{B}^{d-r-s} \longrightarrow \Delta(r, \pi) \times \mathbf{R}_+^s, \quad p \mapsto (-\log |x_0(p)|, \dots, -\log |x_{r+s}(p)|)$$

to $S(\mathfrak{S}, H(s))$. Composing Val with the inverse homeomorphism $\Delta(r, \pi) \times \mathbf{R}_+^s \xrightarrow{\sim} S(\mathfrak{S}, H(s))$ yields a map

$$\tau : U_{\Delta(r, \pi)} \times (\mathbf{B} \setminus \{0\})^s \times \mathbf{B}^{d-r-s} \longrightarrow S(\mathfrak{S}, H(s))$$

which is a proper strong deformation retraction. The latter follows from [Ber99, Theorem 5.2]. By construction, we have that $\text{Val} = \text{Val} \circ \tau$, i.e. that Val factors through the retraction to the skeleton.

4.3. The skeleton of a building block. Next we consider a formal affine open building block $\mathfrak{U} = \text{Spf}(A)$ as in Proposition 4.1 and let

$$\psi : \mathfrak{U} \longrightarrow \mathfrak{S} = \mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{B}^{d-r} = \text{Spf}(K^{\circ} \langle x_0, \dots, x_d \rangle / \langle x_0 \cdots x_r - \pi \rangle)$$

be the étale map from (a). Recall that \mathfrak{H}_{η} is the support of $H|_{\mathfrak{X}_{\eta}}$. Define

$$\text{Val} := \text{Val} \circ \psi : U \setminus \mathfrak{H}_{\eta} \longrightarrow \Delta(r, \pi) \times \mathbf{R}_+^s,$$

where $U = \mathfrak{U}_{\eta}$. Then we have

$$\text{Val}(p) = (-\log |\psi^*(x_0)(p)|, \dots, -\log |\psi^*(x_{r+s})(p)|).$$

We define the skeleton $S(\mathfrak{U}, H|_{\mathfrak{U}})$ as the preimage of $S(\mathfrak{S}, H(s))$ under ψ . This is a closed subset of $U \setminus \mathfrak{H}_{\eta}$. Using Berkovich's results in [Ber99, §5] and the ε -approximation argument from 4.2, one can show that:

- (1) ψ induces a homeomorphism of $S(\mathfrak{U}, H|_{\mathfrak{U}})$ onto $S(\mathfrak{S}, H(s))$, so Val restricts to a homeomorphism $S(\mathfrak{U}, H|_{\mathfrak{U}}) \xrightarrow{\sim} \Delta(r, \pi) \times \mathbf{R}_+^s$;
- (2) composing Val with the inverse homeomorphism $\Delta(r, \pi) \times \mathbf{R}_+^s \xrightarrow{\sim} S(\mathfrak{U}, H|_{\mathfrak{U}})$ yields a proper strong deformation retraction $\tau : U \setminus \mathfrak{H}_{\eta} \longrightarrow S(\mathfrak{U}, H|_{\mathfrak{U}})$;
- (3) $S(\mathfrak{U}, H|_{\mathfrak{U}})$ is intrinsic to (\mathfrak{U}, H) and does not depend on the choice of ψ .

We wish to emphasize that the above map Val factors through the retraction to the skeleton. Moreover, Val is essentially intrinsic to $(\mathfrak{U}, H|_{\mathfrak{U}})$ and is independent of ψ up to reordering the coordinates, as we now prove.

Lemma 4.4. *In the notation above, suppose that \mathfrak{U}_s is not irreducible. Let V be an irreducible component of \mathfrak{U}_s . There exists $i \leq r$ such that the zero set of the restriction of $f := \psi^*(x_i) \in A$ to \mathfrak{U}_s is equal to V . Moreover, if $f' \in A$ is any other function whose restriction to \mathfrak{U}_s has zero set contained in V and whose restriction to the generic fibre is a unit, then $f' = u f^n$ for some $u \in A^{\times}$ and $n \in \mathbf{N}$. We have $\text{cyc}(f) = v(\pi) V$.*

Proof. This follows from [Gub07, Proposition 2.11(c)]. See Appendix A for the definition of the Weil divisor $\text{cyc}(f)$ associated to the Cartier divisor $\text{div}(f)$. \blacksquare

Note that \mathfrak{U}_s is not irreducible if and only if $r > 0$. We claim that $\text{Val} : U \setminus \mathfrak{H}_\eta \rightarrow \Delta(r, \pi) \times \mathbf{R}_+^s$ is intrinsic to (\mathfrak{U}, H) up to reordering the coordinates. If $r > 0$, then Lemma 4.4 shows that the functions $\psi^*(x_i)$ for $i = 0, \dots, r$ are intrinsic to (\mathfrak{U}, H) up to units on \mathfrak{U} . Since $H = \psi^*H(s)$ is a Cartier divisor on \mathfrak{U} with distinguished components H_i , the functions ψ^*x_i for $i = r+1, \dots, r+s$ are also intrinsic up to units on \mathfrak{U} . This proves that $\text{Val} : U \setminus \mathfrak{H}_\eta \rightarrow \Delta(r, \pi) \times \mathbf{R}_+^s$ is well-defined up to reordering the coordinates. If $r = 0$ then $\Delta(0, \pi) = \{0\} \subset \mathbf{R}$ by our choice $\pi := 1$ in the definition of a standard pair (see Example 3.10), hence there is no additional ambiguity in the definition of Val .

4.5. We make one final remark about the skeleton $S(\mathfrak{U}, H|_{\mathfrak{U}})$. Using [Ber99, Theorem 5.2(iv)], one sees that for every $x \in U \setminus \mathfrak{H}_\eta$, $\text{red}(\tau(x))$ is equal to the generic point of the stratum of D containing $\text{red}(x)$. This implies that if $\mathfrak{U}' \subset \mathfrak{U}$ is a formal open subset intersecting the minimal stratum and if U' is the generic fibre of \mathfrak{U}' , then $S(\mathfrak{U}', H|_{\mathfrak{U}'})$ is equal to $S(\mathfrak{U}, H|_{\mathfrak{U}})$. In particular, $S(\mathfrak{U}, H|_{\mathfrak{U}}) \subset U'$. When \mathfrak{U} is a building block of a formal strictly semistable pair (\mathfrak{X}, H) , this proves that the skeleton $S(\mathfrak{U}, H|_{\mathfrak{U}}) \subset X$ depends only on the minimal stratum and not the choice of building block.

4.6. The skeleton of a formal strictly semistable pair. Let (\mathfrak{X}, H) be a formal strictly semistable pair. Recall that $X = \mathfrak{X}_\eta$ in our standard notation 3.8. In 4.5 we associated to every building block \mathfrak{U} of (\mathfrak{X}, H) the skeleton $S(\mathfrak{U}, H|_{\mathfrak{U}}) \subset X$, and we showed that this skeleton depends only on the distinguished stratum S of \mathfrak{U} and not on the choice of \mathfrak{U} . For clarity we decorate the map Val associated to this stratum with the subscript S , i.e. we have $\text{Val}_S : \mathfrak{U}_\eta \setminus \mathfrak{H}_\eta \rightarrow \Delta(r, \pi) \times \mathbf{R}_+^s$. We call $\Delta_S := S(\mathfrak{U}, H|_{\mathfrak{U}}) \subset X$ the *canonical polyhedron* of S . The name is justified by the fact that Val_S restricts to an identification $\Delta_S \xrightarrow{\sim} \Delta(r, \pi) \times \mathbf{R}_+^s \subset \mathbf{R}_+^{r+s+1}$ which is canonical up to reordering the coordinates; the range is a polyhedron with a single maximal bounded face $\Delta(r, \pi)$. We call $\text{Val}_S^{-1}(\Delta(r, \pi)) \cap \Delta_S$ the *finite part* of Δ_S and we call r the *dimension of the finite part* of Δ_S . It is equal to the number of irreducible components of \mathfrak{X}_s containing S minus 1. We call $v(\pi)$ the *length* of Δ_S . Note that $\dim(\Delta_S)$ is equal to the codimension of S in \mathfrak{X}_s . The canonical polyhedra satisfy the following properties:

- (a) For a vertical stratum S of D , the map $T \mapsto \Delta_T$ gives a bijective order reversing correspondence between vertical strata T of D with $S \subset \overline{T}$ and closed faces of Δ_S .
- (b) For vertical strata R, S of D , $\Delta_R \cap \Delta_S$ is the union of all Δ_T with T ranging over all vertical strata of D such that $\overline{T} \supset R \cup S$.

We define the *skeleton* of (\mathfrak{X}, H) to be

$$S(\mathfrak{X}, H) := \bigcup_{S \in \text{str}(\mathfrak{X}_s, H)} \Delta_S.$$

This is a closed subset of $X \setminus \mathfrak{H}_\eta$ which depends only on the formal strictly semistable pair (\mathfrak{X}, H) . The above incidence relations endow $S(\mathfrak{X}, H)$ with the canonical structure of a piecewise linear space whose charts are integral Γ -affine polyhedra and whose transition functions are integral Γ -affine maps. More precisely, if $\Delta_S \cong \Delta(r, \pi) \times \mathbf{R}_+^s \subset \mathbf{R}_+^{r+s+1}$ and $\Delta_{S'} \cong \Delta(r', \pi') \times \mathbf{R}_+^{s'} \subset \mathbf{R}_+^{r'+s'+1}$ are canonical polyhedra associated to strata S, S' with $S' \subset \overline{S}$, we have the following description: If $r > 0$, then after potentially reordering the coordinates $v_0, \dots, v_{r'+s'}$, the polyhedron Δ_S is the intersection of $\Delta_{S'}$ with the linear subspace $\{v_{r+1} = \dots = v_{r'} = 0, v_{r'+s+1} = \dots = v_{r'+s'} = 0\}$. In particular, we have $v(\pi) = v(\pi')$; as we noted after Lemma 4.4, $v(\pi)$ is intrinsic to S and S' in this case. The case $r = 0$ is trivial as Δ_S is a vertex of $\Delta_{S'}$.

As a consequence, we see that if two canonical polyhedra $\Delta_S \cong \Delta(r, \pi) \times \mathbf{R}_+^s$ and $\Delta_{S'} \cong \Delta(r', \pi') \times \mathbf{R}_+^{s'}$ with $r, r' \geq 1$ are connected by a chain of canonical polyhedra Δ_T whose finite part has positive dimension, then $v(\pi) = v(\pi')$. In this case we say that $\Delta_S, \Delta_{S'}$ are *connected by finite faces of positive dimension*. The skeleton $S(\mathfrak{X}, H)$ canonically decomposes into components which are connected by finite faces of positive dimension.

4.6.1. When $H = 0$ the skeleton $S(\mathfrak{X}, 0)$ coincides with the skeleton $S(\mathfrak{X})$ of the strictly semistable formal scheme \mathfrak{X} in the sense of Berkovich [Ber99]. We will use the notations $S(\mathfrak{X})$ and $S(\mathfrak{X}, 0)$ interchangeably.

Remark 4.7. Every point x of the skeleton is Abhyankar: that is,

$$\text{rank}_{\mathbf{Z}}(|\mathcal{H}(x)^\times|/|K^\times|) + \text{tr. deg}(\widetilde{\mathcal{H}}(x)/\widetilde{K}) = \dim(X).$$

Indeed, any point of the skeleton can be interpreted as a monomial valuation with respect to some system of local coordinates, and it is easy to see that any monomial valuation is Abhyankar. Moreover, x induces a valuation on the function field of X . As a consequence, the skeleton $S(\mathfrak{X}, H)$ is contained in the complement of every closed analytic subvariety $Y \neq X$ (see [Ber90, Proposition 9.1.3]), hence is a “birational” feature of X .

Remark 4.8. A *face* of the skeleton $S(\mathfrak{X}, H)$ is the same as a canonical polyhedron. The relative interior of a canonical polyhedron is called an *open face*. We use the partial ordering $\Delta_T \preceq \Delta_S$ for canonical polyhedra meaning Δ_T is a face of Δ_S . If additionally $\Delta_T \neq \Delta_S$, then we write $\Delta_T \prec \Delta_S$. We use this partial ordering also for open faces by applying it to the closures.

4.9. Retraction to the skeleton. The retractions onto the skeletons of the building blocks $\tau : \mathfrak{U}_\eta \setminus \mathfrak{H}_\eta \rightarrow S(\mathfrak{U}, H|_{\mathfrak{U}})$ glue to give a proper strong deformation retraction

$$(4.9.1) \quad \tau : X \setminus \mathfrak{H}_\eta \longrightarrow S(\mathfrak{X}, H).$$

Since τ is continuous and surjective, $S(\mathfrak{X}, H)$ is connected. We can use the retraction map to give a different description of the correspondence between $\text{str}(\mathfrak{X}_s, H)$ and the faces of $S(\mathfrak{X}, H)$. In the following *orbit-face correspondence*, we use the restriction $\text{red}_{H^c} : X \setminus \mathfrak{H}_\eta \rightarrow \mathfrak{X}_s$ of the reduction map $\text{red} : X \rightarrow \mathfrak{X}_s$. Recall also the partial orders \preceq defined on the open faces of $S(\mathfrak{X}, H)$ and \leq defined on the strata of D (see 3.15, Remark 4.8).

Proposition 4.10. *There is a bijective order-reversing correspondence between open faces σ of the skeleton $S(\mathfrak{X}, H)$ and vertical strata S of D , given by*

$$(4.10.1) \quad S = \text{red}_{H^c}(\tau^{-1}(\sigma)), \quad \sigma = \tau(\text{red}_{H^c}^{-1}(Y)),$$

where Y is any non-empty subset of S . We have $\dim(S) + \dim(\sigma) = \dim(X)$.

Proof. This is completely analogous to Proposition 5.7 in [Gub10]. ■

Corollary 4.11. *There is a bijective correspondence between open faces σ of $S(\mathfrak{X}, H)$ and the generic points ζ_S of vertical strata S of D , given by*

$$(4.11.1) \quad \{\zeta_S\} = \text{red}(\Omega), \quad \sigma = \text{red}^{-1}(\zeta_S) \cap S(\mathfrak{X}, H),$$

where Ω is any non-empty subset of σ .

Proof. As remarked in 4.5, for any $p \in X \setminus \mathfrak{H}_\eta$, $\text{red}(\tau(p))$ is equal to the generic point of the stratum containing $\text{red}(p)$. For $p \in \text{relint}(\Delta_S)$ we then have $\text{red}(p) = \zeta_S$ by Proposition 4.10. ■

Remark 4.12. We can compactify the skeleton $S(\mathfrak{X}, H)$ of a formal strictly semistable pair (\mathfrak{X}, H) by taking its closure in X . We get a compact subset $\widehat{S}(\mathfrak{X}, H)$ of X whose boundary has the following interpretation: We note that for $k = 1, \dots, S$, we have canonical formal strictly semistable pairs $(\mathfrak{H}_k, H|_{\mathfrak{H}_k})$ (see Remark 3.6) and hence we get corresponding skeletons $S(\mathfrak{H}_k, H|_{\mathfrak{H}_k})$. Proceeding inductively, we get formal strictly semistable pairs $(\overline{T}, H|_{\overline{T}})$ for every horizontal stratum T of D . Then it follows from the construction that $\widehat{S}(\mathfrak{X}, H)$ contains the disjoint union of $S(\mathfrak{X}, H)$ and of all these skeletons $S(\overline{T}, H|_{\overline{T}})$. The retraction map τ extends to a map $\widehat{\tau} : X \rightarrow \widehat{S}(\mathfrak{X}, H)$ such that the restriction of $\widehat{\tau}$ to any horizontal stratum T is the canonical retraction map to $S(\overline{T}, H|_{\overline{T}})$. It will follow from Theorem 4.13 that $\widehat{\tau}$ is a continuous map. We conclude that $\widehat{\tau}(X)$ is compact which means that $\widehat{S}(\mathfrak{X}, H)$ is indeed the closure of S and that $\widehat{\tau}$ is surjective.

Theorem 4.13. *The map $\widehat{\tau} : X \rightarrow \widehat{S}(\mathfrak{X}, H)$ is a proper strong deformation retraction.*

Proof. By [Ber99, Theorem 5.2] and the ε -approximation from 4.2, we have a map $\Phi : X \times [0, 1] \rightarrow X$. The restriction of Φ to $X \setminus \mathfrak{H}_\eta \times [0, 1]$ gives the homotopy leading to the proper strong deformation retraction $\tau : X \setminus \mathfrak{H}_\eta \rightarrow S(\mathfrak{X}, H)$ from 4.9. Moreover, for every horizontal stratum T of D as in Remark 4.12, the restriction of Φ to T is the homotopy giving the proper strong deformation retraction of T to $S(\overline{T}, H|_{\overline{T}})$. It remains to show Φ is continuous (properness is then

obvious from compactness of X). This can be done completely similar as in Berkovich's proof of [Ber99, Theorem 5.2] along the following lines:

Let $\mathfrak{G}_m = \mathrm{Spf} K^\circ \langle \mathbf{Z} \rangle$ be the formal affine torus of rank 1 over K° . Recall that we have considered standard pairs $(\mathfrak{S} := \mathfrak{U}_{\Delta(r,\pi)} \times \mathfrak{B}^{d-r}, H(s))$ in 3.10. We first check the claim for the slightly restricted standard pair $(\mathfrak{S}' := \mathfrak{U}_{\Delta(r,\pi)} \times \mathfrak{G}_m^s \times \mathfrak{B}^{d-r-s}, H(s)|_{\mathfrak{S}'})$. Note that any \mathfrak{S} is a finite open union of such \mathfrak{S}' and we may use them as well for building blocks in the construction of the skeleton which fits better to Berkovich's setting.

Let $\mathfrak{G}^{(r)}$ be the kernel of the multiplication map $\mathfrak{G}_m^r \rightarrow \mathfrak{G}_m$. Then $\mathfrak{G}^{(r)}$ acts canonically on $\mathfrak{U}_{\Delta(r,\pi)}$ and hence $\mathfrak{G} := \mathfrak{G}^{(r)} \times \mathfrak{G}_m^{d-r}$ acts canonically on \mathfrak{S}' . It is clear that \mathfrak{G} is a formal affine torus of rank d and the generic fibre G of \mathfrak{G} is a formal affinoid torus of rank d . In Step 2, Berkovich gives a canonical continuous map $[0, 1] \rightarrow G$, mapping $t \in [0, 1)$ to the Shilov boundary point g_t of the closed disk in T with center 1 and radius t . Moreover, $t = 1$ is mapped to the Shilov boundary point of G . By [Ber90, §5.2], the points g_t are peaked and the group action induces well-defined points $g_t * x$ on \mathfrak{S}'_η for every $x \in \mathfrak{S}'_\eta$. In this way we get a continuous homotopy $\mathfrak{S}'_\eta \times [0, 1] \rightarrow \mathfrak{S}'_\eta$, given by $(x, t) \mapsto g_t * x$. Note that the action by g_t leaves any horizontal stratum T invariant and acts there in the same way as the corresponding peaked point g'_t for the formal affinoid torus G' of rank $\dim(T)$ (apply [Ber90, Proposition 5.2.8(ii)] with $X = X' = T$ and φ the projection from G onto G'). By construction, the homotopy agrees with Φ and leads to the strong deformation retraction $\tau : \mathfrak{S}'_\eta \rightarrow S(\mathfrak{S}', H(s)) \cong \Delta(r, \pi) \times \mathbf{R}_+^s$.

The general case is deduced from the above case with the same arguments as in the proof of [Ber99, Theorem 5.2]. The special shape of the building blocks was not used there. \blacksquare

4.14. Fibres of the retraction. Let (\mathfrak{X}, H) be a formal strictly semistable pair and let $S \in \mathrm{str}(\mathfrak{X}_s, H)$ be a zero-dimensional stratum, so $S = \{x\}$ for $x \in \mathfrak{X}_s(\tilde{K})$. Let $\Delta_S \subset S(\mathfrak{X}, H)$ be the corresponding canonical polyhedron and let ω be a Γ -rational point in the relative interior of Δ_S . In the proof of Theorem 8.4 it will be important to understand the analytic subdomain $X_\omega := \tau^{-1}(\omega) \subset X$. We will prove that X_ω is isomorphic to the affinoid torus $T = \mathcal{M}(K \langle x_1^{\pm 1}, \dots, x_d^{\pm 1} \rangle)$, where $d = \dim(X) = \dim(\Delta_S)$.

Let $\mathfrak{U} \subset \mathfrak{X}$ be a building block with distinguished stratum S . Let $\psi : \mathfrak{U} \rightarrow \mathfrak{S} = \mathfrak{U}_{\Delta(r,\pi)} \times \mathfrak{B}^{d-r}$ be an étale morphism as in Proposition 4.1 and let $y = \psi(x) = (0, 0, \dots, 0)$, so $\{y\}$ is the minimal stratum of $(\mathfrak{S}, H(d-r))$. By Proposition 4.10 we have $\mathrm{red}(X_\omega) = \mathrm{red}(\tau^{-1}(\omega)) = \{x\}$, so X_ω is contained in the formal fibre $\mathrm{red}^{-1}(x)$. In particular, $X_\omega \subset \mathfrak{U}_\eta$. Since ψ is étale at $x \in \mathfrak{U}_s(\tilde{K})$, the induced map on formal fibres $\mathrm{red}^{-1}(x) \rightarrow \mathrm{red}^{-1}(y)$ is an isomorphism by [Gub07, Proposition 2.9].

Let $v = \mathrm{Val}_S(\omega) \in \Delta(r, \pi) \times \mathbf{R}_+^{d-r} \subset \mathbf{R}^{d+1}$. The coordinates of v are contained in Γ , so $\mathrm{Val}_S^{-1}(v) \subset U_{\Delta(r,\pi)} \times (\mathbf{B} \setminus \{0\})^{d-r} \subset \mathbf{G}_m^{d+1, \mathrm{an}}$ is non-canonically isomorphic to the affinoid torus T defined above. Since ψ commutes with the retraction maps $\tau : \mathfrak{U}_\eta \setminus \mathfrak{H}_\eta \rightarrow \Delta_S$ and $\tau : U_{\Delta(r,\pi)} \times (\mathbf{B} \setminus \{0\})^{d-r} \rightarrow S(\mathfrak{S}, H(d-r))$, we have

$$X_\omega = \tau^{-1}(\omega) = \psi^{-1}(\tau^{-1}(\psi(\omega))) = \psi^{-1}(\mathrm{Val}_S^{-1}(v)).$$

On the other hand, as above $\tau^{-1}(\psi(\omega))$ is contained in the formal fibre $\mathrm{red}^{-1}(y)$, so as $\mathrm{red}^{-1}(x) \rightarrow \mathrm{red}^{-1}(y)$ is an isomorphism, the same is true for the map $\psi : X_\omega \xrightarrow{\sim} \mathrm{Val}_S^{-1}(v) \cong T$. Hence we have proved:

Proposition 4.15. *Let (\mathfrak{X}, H) be a formal strictly semistable pair; let $S \in \mathrm{str}(\mathfrak{X}_s, H)$ be a zero-dimensional stratum, and let $\omega \in \Delta_S$ be a Γ -rational point contained in the relative interior of Δ_S . Then $X_\omega := \tau^{-1}(\omega)$ is isomorphic to the affinoid torus $T = \mathcal{M}(K \langle x_1^{\pm 1}, \dots, x_d^{\pm 1} \rangle)$, where $d = \dim(X) = \dim(\Delta_S)$.*

Let $Y = \mathcal{M}(A)$ be an affinoid space and let $A^\circ \subset A$ be the subring of power-bounded elements. Recall from 2.1 that the *canonical model* of Y is the K° -formal scheme $\mathrm{Spf}(A^\circ)$; this is an affine admissible formal scheme when A is reduced by [BPR11, Theorem 3.17].

Corollary 4.16. *With the notation in Proposition 4.15, the analytic subdomain X_ω is strictly affinoid, and its canonical model \mathfrak{X}_ω is isomorphic to a multiplicative formal torus over K° of rank d .*

At this point we are able to formulate a precise statement about which vertical components of D are Cartier, and where. See A.5 for the definition of the Weil divisor $\text{cyc}(C)$ associated to the Cartier divisor C .

Proposition 4.17. *Let V be an irreducible component of \mathfrak{X}_s and let v be the vertex of $S(\mathfrak{X}, H)$ associated to the open stratum in V .*

- (1) *If $\mathfrak{X}_s = V$, then any vertical effective Cartier divisor on \mathfrak{X} is equal to $\text{div}(\lambda)$ for some non-zero $\lambda \in K^\circ$.*
- (2) *If $\mathfrak{X}_s \neq V$ and if all canonical polyhedra containing v with positive-dimensional finite part have the same length $v(\pi)$, then there is a unique effective Cartier divisor C on \mathfrak{X} with $\text{cyc}(C) = v(\pi)V$, and any effective Cartier divisor with support contained in V is equal to nC for a unique $n \in \mathbf{N}$.*
- (3) *If v is adjacent to canonical polyhedra $\Delta_{S_1}, \Delta_{S_2}$ with positive-dimensional finite part and whose lengths $v(\pi_1), v(\pi_2)$ are not commensurable, then V is not the support of a Cartier divisor on \mathfrak{X} .*

Proof. Note that (1) follows from the fact that a vertical Cartier divisor on an admissible formal scheme over K° with reduced special fibre is uniquely determined by the associated Weil divisor (see Proposition A.7).

To prove (2), we assume that $\mathfrak{X}_s \neq V$. Since X is connected, continuity and surjectivity of the retraction map τ yield that $S(\mathfrak{X}, H)$ is connected. We conclude from the stratum-face correspondence in Proposition 4.10 that there is at least one canonical polyhedron of $S(\mathfrak{X}, H)$ containing v with positive-dimensional finite part. By assumption, all such canonical polyhedra have the same length $v(\pi)$ for some non-zero $\pi \in K^{\circ\circ}$. Choose a cover of \mathfrak{X} by building blocks \mathfrak{U}_i . If $(\mathfrak{U}_i)_s \cap V = \emptyset$ then a local equation for C on \mathfrak{U}_i is $f_i = 1$. If V is the distinguished stratum of \mathfrak{U}_i then a local equation for C on \mathfrak{U}_i is $f_i = \pi$. Otherwise the special fibre of \mathfrak{U}_i is not irreducible; we choose the function $f_i = f$ of Lemma 4.4 as the local equation for C on \mathfrak{U}_i . As the Weil divisor associated to $\text{div}(f_i)$ is $v(\pi)(V \cap (\mathfrak{U}_i)_s)$ for all i , these indeed define an effective Cartier divisor C . This and uniqueness follows again from Proposition A.7.

To prove the last statement in (2), let C' be an effective Cartier divisor with support contained in V . Then the associated Weil divisor is equal to $v(\lambda)V$ for $\lambda \in K^\circ$. We may apply Lemma 4.4 to a building block \mathfrak{U}_i whose special fibre is not irreducible. The latter is equivalent to the property that the finite part of the canonical polyhedron of the distinguished stratum is positive-dimensional. This proves that $v(\lambda) = nv(\pi)$ for some non-zero $n \in \mathbf{N}$. Then Lemma 4.4 again and part (1) applied to the building blocks with irreducible special fibre prove that $C' = nC$. Since n is the multiplicity of C' in V , it is unique.

In the situation of (3), we may shrink \mathfrak{X} to assume that it is covered by two building blocks $\mathfrak{U}_1, \mathfrak{U}_2$ with distinguished strata S_1 and S_2 . It follows from Lemma 4.4 that any Cartier divisor C_i on \mathfrak{U}_i with support equal to $(\mathfrak{U}_i)_s \cap V$ has $\text{cyc}(C_i) = n_i v(\pi_i)(V \cap (\mathfrak{U}_i)_s)$ for $n_i \in \mathbf{Z} \setminus \{0\}$. Since $n_1 v(\pi_1) \neq n_2 v(\pi_2)$ for $n_1, n_2 \neq 0$ there does not exist a Cartier divisor C on \mathfrak{X} such that $C|_{\mathfrak{U}_i} = C_i$ for $i = 1, 2$. \blacksquare

5. FUNCTORIALITY

In this section, (\mathcal{X}, H) is a strictly semistable pair. As always we use the associated notation introduced in 3.8; in particular, $X := \mathcal{X}_\eta$ is the generic fibre. Let $d := \dim(X)$. Consider a non-zero rational function f on X . This induces a meromorphic function on X^{an} . The goal of this section is to show that the restriction of this meromorphic function to the skeleton $S(\mathcal{X}, H)$ is an everywhere-defined piecewise linear function. If the support of $\text{div}(f)$ is contained in the boundary $\text{supp}(H)_\eta$, then the restriction to any canonical polyhedron of $S(\mathcal{X}, H)$ is integral Γ -affine for the value group $\Gamma = v(K^\times)$. This basic result will be used in later sections. As a further consequence, we will show functoriality of the retraction to the skeleton. We will deduce piecewise linearity of the restriction of $-\log|f|$ to the skeleton $S(\mathcal{X}, H)$ without boundary assumptions. Note that in this generality, the restriction is not necessarily integral Γ -affine on canonical polyhedra as above.

5.1. Let f be a non-zero rational function on X such that the support of $\text{div}(f)$ is contained in the generic fibre of $\text{supp}(H)$. As in 3.3, let $D_1 = V_1, \dots, D_R = V_R$ be the irreducible components of the

special fibre \mathcal{X}_s , let $D_{R+1} = \mathcal{H}_1, \dots, D_{R+S} = \mathcal{H}_S$ be the prime components of the (horizontal) Weil divisor \mathcal{H} on \mathcal{X} associated to H and let $D = D_1 + \dots + D_{R+S}$. We have

$$(5.1.1) \quad \text{cyc}(f) = \sum_{i=1}^{R+S} \text{ord}(f, D_i) D_i = \sum_{i=1}^R \text{ord}(f, V_i) V_i + \sum_{j=1}^S \text{ord}(f, \mathcal{H}_j) \mathcal{H}_j.$$

for the associated Weil divisor $\text{cyc}(f)$ on \mathcal{X} . Here $\text{ord}(f, \mathcal{H}_j) \in \mathbf{Z}$ is the usual order of vanishing of f along $(\mathcal{H}_j)_\eta$ in X and $\text{ord}(f, V_i) = -\log |f(\xi_i)|$, where $\xi_i \in X^{\text{an}}$ is the unique point reducing to the generic point of V_i ; see Appendix A.

The next result generalizes parts (1) and (2) of [BPR13, Theorem 5.15].

Proposition 5.2. *Let f be a non-zero rational function on X such that the support of $\text{div}(f)$ is contained in the generic fibre of $\mathcal{H} = \text{supp}(H)$. We consider $F = -\log |f|$ as a function $F : U^{\text{an}} := X^{\text{an}} \setminus \mathcal{H}_\eta^{\text{an}} \rightarrow \mathbf{R}$. Then F factors through the retraction map $\tau : U^{\text{an}} \rightarrow S(\mathcal{X}, H)$. Moreover, the restriction of F to $S(\mathcal{X}, H)$ is an integral Γ -affine function on each canonical polyhedron.*

Proof. Let $\mathfrak{X} = \widehat{\mathcal{X}}$ and let $\mathfrak{U} \subset \mathfrak{X}$ be a building block with distinguished stratum S . Choose an étale morphism $\psi : \mathfrak{U} \rightarrow \mathfrak{S} = \mathfrak{U}_{\Delta(r, \pi)} \times \mathfrak{B}^{d-r} = \text{Spf}(K^\circ \langle x_0, \dots, x_d \rangle / \langle x_0 \dots x_r - \pi \rangle)$ as in Proposition 4.1. This means that the restriction of the formal completion of (\mathcal{X}, H) to \mathfrak{U} is equal to the pull-back of the standard pair $(\mathfrak{S}, H(s))$ with respect to ψ for some $s \in \{0, \dots, d-r\}$. Since the support of $\text{div}(f)$ on the generic fibre is contained in the generic fibre of the horizontal divisor, there exist integers n_{r+1}, \dots, n_{r+s} such that $f|_{\mathfrak{U}} / \prod_{i=r+1}^{r+s} \psi^*(x_i)^{n_i}$ is a unit on \mathfrak{U}_η . By [Gub07, Proposition 2.11], we find

$$(5.2.1) \quad f|_{\mathfrak{U}} = \lambda u \psi^*(x_0)^{n_0} \dots \psi^*(x_{r+s})^{n_{r+s}},$$

with $\lambda \in K^\times$, $u \in \mathcal{O}(\mathfrak{U})^\times$ and $n_0, \dots, n_r \in \mathbf{Z}$. It follows that for $p \in \mathfrak{U}_\eta$ we have

$$F(p) = v(\lambda) - \sum_{j=0}^{r+s} n_j \log |\psi^* x_j(p)|.$$

By definition $\text{Val}_S : \mathfrak{U}_\eta \setminus \mathcal{H}_\eta^{\text{an}} \rightarrow \mathbf{R}^{r+s+1}$ is given by

$$\text{Val}_S(p) = (-\log |\psi^* x_0(p)|, \dots, -\log |\psi^* x_{r+s}(p)|),$$

so the restriction of F to $\mathfrak{U}_\eta \setminus \mathcal{H}_\eta^{\text{an}}$ is the composition of Val_S followed by

$$(v_0, \dots, v_{r+s}) \mapsto v(\lambda) + \sum_{j=0}^{r+s} n_j v_j.$$

We conclude that F factors through the retraction τ . We also see that the restriction of F to Δ_S is integral Γ -affine. \blacksquare

Remark 5.3. The function F is completely determined by the Weil divisor $\text{cyc}(f)$ in (5.1.1). From the proof of Proposition 5.2, we deduce the following explicit formula for F in terms of the multiplicities $(\text{ord}(f, D_i))_{i=1, \dots, R+S}$: For $j = 0, \dots, s$, the zero set of $\psi^*(x_j)$ on the building block \mathfrak{U} is equal to V_{i_j} for a unique $i_j \in \{1, \dots, R\}$. For $j = r+1, \dots, r+s$, the coordinate $\psi^*(x_j)$ on \mathfrak{U} is a local equation for a unique \widehat{H}_{i_j} with $i_j \in \{1, \dots, S\}$. By Lemma 4.4 and (5.1.1), we have the relations $\text{ord}(f, V_{i_j}) = v(\lambda) + n_j v(\pi)$ for $j \in \{0, \dots, r\}$ and $\text{ord}(f, \mathcal{H}_{i_j}) = n_j$ for $j \in \{r+1, \dots, r+s\}$. For $p \in \mathfrak{U}_\eta$, we get

$$\begin{aligned} F(p) &= v(\lambda) - \sum_{j=0}^{r+s} n_j \log |\psi^* x_j(p)| \\ &= v(\lambda) - \sum_{j=0}^r \frac{\text{ord}(f, V_{i_j}) - v(\lambda)}{v(\pi)} \log |\psi^* x_j(p)| - \sum_{j=r+1}^s \text{ord}(f, \mathcal{H}_{i_j}) \log |\psi^* x_j(p)| \\ &= -\frac{1}{v(\pi)} \sum_{j=0}^r \text{ord}(f, V_{i_j}) \log |\psi^* x_j(p)| - \sum_{j=r+1}^s \text{ord}(f, \mathcal{H}_{i_j}) \log |\psi^* x_j(p)|, \end{aligned}$$

where the last equality holds because $-\sum_{j=0}^r \log |\psi^* x_j(p)| = v(\pi)$. We conclude that the restriction of F to $\mathfrak{U}_\eta \setminus \mathcal{H}_\eta^{\text{an}}$ is the composition of $\text{Val}_S : \mathfrak{U}_\eta \setminus \mathcal{H}_\eta^{\text{an}} \rightarrow \mathbf{R}^{r+s+1}$ followed by the linear

form

$$(5.3.1) \quad (v_0, \dots, v_{r+s}) \mapsto \sum_{j=0}^r \frac{\text{ord}(f, V_{i_j})}{v(\pi)} v_j + \sum_{j=r+1}^s \text{ord}(f, \mathcal{H}_{i_j}) v_j.$$

This is the desired formula in terms of the multiplicities.

By the stratum–face correspondence in Proposition 4.10, a vertex u of the canonical polyhedron Δ_S corresponds to a vertical component V_i of D . Then we deduce from (5.3.1) that

$$F(u) = \text{ord}(f, V_i).$$

Alternatively, by Corollary 4.11, u is the unique point of $S(\mathcal{X})$ reducing to the generic point of V_i , so $\text{ord}(f, V_i) = -\log |f(u)| = F(u)$ by definition.

Remark 5.4. The above results hold also for a non-zero meromorphic function f on a formal strictly semistable pair (\mathfrak{X}, H) assuming that the generic fibre of the support of $\text{div}(f)$ is contained in the generic fibre of $\text{supp}(H)$. The same proof applies without any change.

Next, we will show **functoriality** of the retraction to the skeleton. This holds for formal strictly semistable pairs as well, but we will restrict our attention to the algebraic case.

Proposition 5.5. *Let $(\mathcal{X}, H), (\mathcal{X}', H')$ be strictly semistable pairs and let $\varphi : \mathcal{X}' \rightarrow \mathcal{X}$ be a morphism with generic fibre $\varphi_\eta : X' \rightarrow X$. We assume that $\varphi_\eta^{-1}(\text{supp}(H)_\eta) \subset \text{supp}(H')_\eta$. Then there is a unique map $\varphi_{\text{aff}} : S(\mathcal{X}', H') \rightarrow S(\mathcal{X}, H)$ with*

$$\varphi_{\text{aff}} \circ \tau' = \tau \circ \varphi_\eta$$

on $X' = \mathcal{X}'_\eta$, where τ (resp. τ') is the retraction to the skeleton $S(\mathcal{X}, H)$ (resp. $S(\mathcal{X}', H')$) from 4.9. The map φ_{aff} is continuous. For any canonical polyhedron $\Delta_{S'}$ of $S(\mathcal{X}', H')$, there is a canonical polyhedron Δ_S of $S(\mathcal{X}, H)$ with $\varphi_{\text{aff}}(\Delta_{S'}) \subset \Delta_S$. Moreover, the induced map $\Delta_{S'} \rightarrow \Delta_S$ is integral Γ -affine.

Proof. Put $\mathfrak{X} = \widehat{\mathcal{X}}$ and consider a covering by building blocks \mathfrak{U}_i as in Proposition 4.1. Now refine the covering $\varphi_s^{-1}(\mathfrak{U}_{i,s})$ of \mathfrak{X}'_s according to Proposition 4.1. In this way we get a covering of \mathfrak{X}' by building blocks $\mathfrak{U}'_{i,k}$ such that each $\mathfrak{U}'_{i,k}$ maps to a building block \mathfrak{U}_i of \mathfrak{X} via the formal completion $\widehat{\varphi}$ of φ . Let \mathfrak{U}' be one of those building blocks of \mathfrak{X}' with associated stratum S' such that $\widehat{\varphi}$ maps \mathfrak{U}' to a building block \mathfrak{U} of \mathfrak{X} with associated stratum S . Choose an étale morphism $\psi : \mathfrak{U} \rightarrow \mathfrak{S} = \mathfrak{U}_{\Delta(r,\pi)} \times \mathfrak{B}^{d-r} = \text{Spf}(K^\circ \langle x_0, \dots, x_d \rangle / \langle x_0 \dots x_r - \pi \rangle)$ as in Proposition 4.1 (a).

Consider the analytic functions $f_i = \varphi_\eta^* \psi_\eta^* x_i$ on \mathfrak{U}'_η for $i = 0, \dots, r+s$. Since $\varphi_\eta^{-1}(\text{supp}(H)_\eta) \subset \text{supp}(H')_\eta$, all f_i are units on $\mathfrak{U}'_\eta \setminus (\mathcal{H}'_\eta)^{\text{an}}$. Choose an étale morphism $\psi' : \mathfrak{U}' \rightarrow \mathfrak{S}' = \mathfrak{U}_{\Delta(r',\pi')} \times \mathfrak{B}^{d-r'} = \text{Spf}(K^\circ \langle x'_0, \dots, x'_{d'} \rangle / \langle x'_0 \dots x'_{r'} - \pi' \rangle)$ as in Proposition 4.1 (a). Now we argue similarly as in the proof of Proposition 5.2. It follows from [Gub07, Proposition 2.11] that

$$(5.5.1) \quad f_i = \lambda u \psi'^*(x'_0)^{n_0} \dots \psi'^*(x'_{r'+s'})^{n_{r'+s'}},$$

with $\lambda \in K^\times$, $u \in \mathcal{O}(\mathfrak{U}')^\times$ and $n_0, \dots, n_{r'+s'} \in \mathbf{Z}$. Hence the map $-\log |f_i|$ factors through an integral Γ -affine map on $\Delta_{S'} = S(\mathfrak{U}', H')|_{\mathfrak{U}'}$.

Note that the map $(\tau \circ \varphi_\eta)|_{\mathfrak{U}'_\eta \setminus (\mathcal{H}'_\eta)^{\text{an}}}$ is given by $(-\log |f_0|, \dots, -\log |f_{r+s}|)$. Hence there exists a unique map $\varphi_{\text{aff}} : \Delta_{S'} \rightarrow \Delta_S$ satisfying $\varphi_{\text{aff}} \circ \tau' = \tau \circ \varphi_\eta$. By construction, it is integral Γ -affine. As these maps fit together on the intersection of building blocks, we get a well-defined map $\varphi_{\text{aff}} : S(\mathcal{X}', H') \rightarrow S(\mathcal{X}, H)$ with the required properties. Uniqueness is obvious from surjectivity of τ' . \blacksquare

Proposition 5.6. *Let (\mathcal{X}, H) be a strictly semistable pair and let f be a non-zero rational function on $X = \mathcal{X}_\eta$. Then $S(\mathcal{X}, H)$ can be covered by finitely many integral Γ -affine polyhedra Δ such that $-\log |f|$ restricts to an integral Γ -affine function on Δ .*

Proof. Locally, \mathcal{X} is given by an open subset \mathcal{U} with an étale morphism

$$(5.6.1) \quad \psi : \mathcal{U} \longrightarrow \mathcal{S} := \text{Spec}(K^\circ[x_0, \dots, x_d] / \langle x_0 \dots x_r - \pi \rangle)$$

and $H|_{\mathcal{U}} = \text{div}(\psi^*(x_{r+1} \dots x_{r+s}))$. Then the formal completion $\mathfrak{U} := \widehat{\mathcal{U}}$ with respect to a non-zero $\pi \in K$ with $|\pi| < 1$ together with the Cartier divisor $H_{\mathfrak{U}}$ induced by $\psi^*(x_{r+1} \dots x_{r+s})$ is the

pull-back of the standard pair $(\mathfrak{S}, H(s))$ as in Example 3.10. We may assume that \mathfrak{U} is a building block with distinguished stratum S . Then we get an étale morphism

$$(5.6.2) \quad \widehat{\psi} : \mathfrak{U} \longrightarrow \mathfrak{S} := \mathrm{Spf}(K^\circ \langle x_0, \dots, x_d \rangle / \langle x_0 \cdots x_r - \pi \rangle)$$

of affine formal schemes. The corresponding face Δ_S of $S(\mathcal{X}, H)$ is mapped homeomorphically onto $S(\mathfrak{S}, H(s)) = \Delta(r, \pi) \times \mathbf{R}_+^s$ by $\widehat{\psi}^{\mathrm{an}}$. In the following, we will use this to identify Δ_S with $\Delta(r, \pi) \times \mathbf{R}_+^s$.

We have to show that Δ_S can be covered by finitely many integral Γ -affine polyhedra Δ such that $(-\log |f|)|_\Delta$ is integral Γ -affine. Let H' be the Cartier divisor of \mathfrak{U} given by $\widehat{\psi}^*(x_{r+1} \cdots x_d)$ and let \mathfrak{H}' be the associated Weil divisor on \mathfrak{U} . Then (\mathfrak{U}, H') is a formal affine strictly semistable pair with skeleton $\Delta(r, \pi) \times \mathbf{R}_+^{d-r}$ containing $\Delta_S = \Delta(r, \pi) \times \mathbf{R}_+^s \times \{0\}$ as a closed face. Moreover, the retraction

$$\tau_d : \mathfrak{U}_\eta \setminus \mathfrak{H}'_\eta \longrightarrow \Delta(r, \pi) \times \mathbf{R}_+^{d-r} \subset \mathbf{R}_+^{r+1} \times \mathbf{R}_+^{d-r} = \mathbf{R}_+^{d+1}$$

composed with the projection $\mathbf{R}_+^{d+1} = \mathbf{R}_+^{r+s+1} \times \mathbf{R}_+^{d-r-s} \rightarrow \mathbf{R}_+^{r+s+1}$ is the restriction of the retraction $\tau : X^{\mathrm{an}} \setminus \mathcal{H}_\eta \rightarrow S(\mathcal{X}, H)$ to $\mathfrak{U}_\eta \setminus \mathfrak{H}'_\eta$.

Let Y be the support of the Cartier divisor $\mathrm{div}(f)$ on X . We claim that $\tau_d(Y \cap \mathfrak{U}_\eta \setminus \mathfrak{H}'_\eta)$ is contained in the support of an integral Γ -affine polyhedral complex in $\Delta(r, \pi) \times \mathbf{R}_+^{d-r}$ of dimension at most $d-1$. This follows from the fact that $\tau_d(Y^{\mathrm{an}} \cap \mathfrak{U}_\eta \setminus \mathfrak{H}'_\eta)$ is contained in the tropicalization of the closure of $\psi_\eta(Y \cap \mathcal{U}_\eta)$ in the torus $\mathrm{Spec}(K[x_0^{\pm 1}, \dots, x_d^{\pm 1}])$. We conclude that there is an integral Γ -affine polyhedral subdivision Σ of $\Delta(r, \pi) \times \mathbf{R}_+^{d-r}$ such that $\tau_d(Y^{\mathrm{an}} \cap \mathfrak{U}_\eta \setminus \mathfrak{H}'_\eta)$ is contained in lower dimensional faces of Σ . Since Δ_S is a face of $S(\mathfrak{U}, H') = \Delta(r, \pi) \times \mathbf{R}_+^{d-r}$, it is enough to show that the restriction of $-\log |f|$ to any maximal face σ of Σ is integral Γ -affine. By density of the value group Γ and by continuity, it is enough to prove that $(-\log |f|)|_\Delta$ is integral Γ -affine for any simplex Δ contained in the interior of σ . Here, Δ is assumed to be unimodular to $\Delta(d, \pi')$ for some non-zero $\pi' \in K$ with $|\pi'| \leq |\pi|$. It follows that $\tau_d^{-1}(\Delta)$ is the generic fibre of a strictly semistable formal open subscheme \mathfrak{U}_Δ of \mathfrak{U} . By construction, we have $S(\mathfrak{U}_\Delta, H'|_{\mathfrak{U}_\Delta}) = \Delta$. Since $\tau_d(Y)$ does not meet Δ , we conclude that f is invertible on $(\mathfrak{U}_\Delta)_\eta$. By Proposition 5.2 and Remark 5.4, the function $(-\log |f|)|_\Delta$ factorizes through the retraction to $S(\mathfrak{U}_\Delta, H'|_{\mathfrak{U}_\Delta}) = \Delta$ and the restriction of $-\log |f|$ to Δ is an integral Γ -affine function. \blacksquare

6. THE SLOPE FORMULA FOR SKELETONS

In this section we assume that (\mathcal{X}, H) is a strictly semistable pair with generic fibre X of dimension $d := \dim(X)$. We will give a generalization of the slope formula in [BPR13, Theorem 5.15] to this higher dimensional situation. In this section, we will use the divisorial intersection theory on \mathcal{X} recalled in Appendix A.

Let f be a non-zero rational function on X such that the support of $\mathrm{div}(f)$ is contained in the generic fibre of $\mathrm{supp}(H)$. As in 5.1, let $D_1 = V_1, \dots, D_R = V_R$ be the irreducible components of the special fibre \mathcal{X}_s , let $D_{R+1} = \mathcal{H}_1, \dots, D_{R+S} = \mathcal{H}_S$ be the prime components of the (horizontal) Weil divisor \mathcal{H} on \mathcal{X} associated to H and let $D = D_1 + \cdots + D_{R+S}$. We have

$$(6.0.3) \quad \mathrm{cyc}(f) = \sum_{i=1}^{R+S} \mathrm{ord}(f, D_i) D_i = \sum_{i=1}^R \mathrm{ord}(f, V_i) V_i + \sum_{j=1}^S \mathrm{ord}(f, \mathcal{H}_j) \mathcal{H}_j.$$

for the associated Weil divisor $\mathrm{cyc}(f)$ on \mathcal{X} . We recall from Proposition 5.2 that the restriction of $-\log |f|$ to the skeleton $S(\mathcal{X}, H)$ is a continuous function which is integral Γ -affine on any canonical polyhedron. The goal of this section is to study the slopes of this piecewise linear function.

6.1. The retraction of the divisor to the compactified skeleton. We recall from Remark 4.12 that we can compactify the skeleton $S(\mathcal{X}, H)$ to a subset $\widehat{S}(\mathcal{X}, H)$ of X^{an} whose boundary is the disjoint union of skeletons $S(\overline{T}, H|_{\overline{T}})$ for the strictly semistable pairs $(\overline{T}, H|_{\overline{T}})$ associated to the horizontal strata T of D . Any $(d-1)$ -dimensional canonical polyhedron Δ'_S of the boundary is contained in a unique skeleton $S(\mathcal{H}_k, H|_{\mathcal{H}_k})$ for some $k = 1, \dots, S$; for such Δ'_S we set

$m(\widehat{\tau}(f), \Delta'_S) := \text{ord}(f, \mathcal{H}_k)$. We define the *retraction* $\widehat{\tau}(f)$ of $\text{cyc}(f)$ to $S(\mathcal{X}, H)$ as the formal sum

$$\widehat{\tau}(f) := \sum_{\Delta'_S} m(\widehat{\tau}(f), \Delta'_S) \Delta'_S,$$

where the sum ranges over all $(d-1)$ -dimensional canonical polyhedra Δ'_S in the boundary of $\widehat{S}(\mathcal{X}, H)$. The support of $\widehat{\tau}(f)$ is either of pure dimension $d-1$ or empty.

Let Δ'_S be a $(d-1)$ -dimensional canonical polyhedron in the boundary of $\widehat{S}(\mathcal{X}, H)$. Then Δ'_S corresponds to a zero-dimensional vertical stratum S of $D|_{\mathcal{H}_k}$ for some k . Hence S is a vertical stratum of D contained in \mathcal{H}_k , so it is a component of the intersection of $(\mathcal{H}_k)_s$ with the closure of a unique one-dimensional vertical stratum T of D . We have $\Delta_S = \Delta_T \times \mathbf{R}_+$, with the direction $(0, 1)$ corresponding to the divisor H_k , and the canonical polyhedron Δ'_S is naturally identified with $\Delta_T \times \{\infty\}$. By Remark 5.3 we can recover the multiplicity $m(\widehat{\tau}(f), \Delta'_S)$ of Δ'_S in $\widehat{\tau}(f)$ as the slope of the restriction of $F = -\log |f|$ to $\Delta_S = \Delta_T \times \mathbf{R}_+$ in the direction $(0, 1)$.

6.2. We define the *retraction* $\tau(f)$ of $\text{cyc}(f)$ to $S(\mathcal{X})$ as the push-forward of $\widehat{\tau}(f)$ with respect to the canonical retraction $S(\mathcal{X}, H) \rightarrow S(\mathcal{X})$. The push-forward is defined as follows. If $\Delta'_S = \Delta_T \times \{\infty\}$ is a dimension- $(d-1)$ canonical polyhedron of the boundary of $\widehat{S}(\mathcal{X}, H)$ as in 6.1, we define its push-forward to be Δ_T if $\Delta_T \subset S(\mathcal{X})$, i.e. if Δ_T is bounded, and we define it to be zero otherwise. The corner locus $\tau(f)$ is then the formal sum of the push-forwards of the $(d-1)$ -dimensional canonical polyhedra of $\widehat{\tau}(f)$. We write

$$\tau(f) = \sum_{\Delta_T} m(\tau(f), \Delta_T) \Delta_T,$$

where the sum ranges over all $(d-1)$ -dimensional canonical polyhedra of $S(\mathcal{X})$. Note that, assuming Δ_T is bounded of dimension $d-1$, the coefficient $m(\tau(f), \Delta_T)$ of Δ_T in $\tau(f)$ is given by

$$m(\tau(f), \Delta_T) = \sum_k \text{ord}(f, \mathcal{H}_k) \cdot \#(\overline{T} \cap (\mathcal{H}_k)_s),$$

where k ranges over all numbers in $\{1, \dots, S\}$ such that \mathcal{H}_k does not contain T .

6.3. Cartwright's α -numbers. For the moment, we assume that we are in the following situation often occurring in number theory: Let R be a complete discrete valuation ring with uniformizer π . We assume that R is a subring of K° and that the discrete valuation of R extends to our given valuation v on the algebraically closed field K . Suppose that \mathcal{X} is the base change of a strictly semistable scheme over R in the sense of de Jong (see 3.2). In such a situation, the strictly semistable scheme \mathcal{X} can be covered by open subsets \mathcal{U} which admit an étale morphism

$$\psi : \mathfrak{U} \longrightarrow \text{Spf} \left(K^\circ \langle x_0, \dots, x_d \rangle / \langle x_0 \dots x_r - \pi \rangle \right).$$

Let u be a vertex of $S(\mathcal{X}, H)$ corresponding to the irreducible component V_u of \mathcal{X}_s by the stratum-face correspondence in Proposition 4.10. Since we have the same π for every chart \mathfrak{U} and since we assume $v(\pi) = 1$, there is a unique Cartier divisor C_u on \mathcal{X} with $\text{cyc}(C_u) = V_u$ (see Proposition 4.17).

Let T be a one-dimensional vertical stratum of D , so that the corresponding canonical polyhedron Δ_T has dimension $d-1$. Then we have the intersection number $C_u \cdot \overline{T}$ and we set

$$\alpha(u, \Delta_T) := -C_u \cdot \overline{T}.$$

Since everything is defined over R , this is a usual intersection number on a proper regular noetherian integral scheme over R and hence $\alpha(u, \Delta_T) \in \mathbf{Z}$.

Dustin Cartwright uses the numbers $\alpha(u, \Delta_T)$ to endow the compact skeleton $S(\mathcal{X})$ with the structure of a tropical complex in the sense of [Car13, Definition 1.1]. Note that he imposes an additional local Hodge condition which plays no role here.

6.4. Our next goal is to generalize Cartwright's α -numbers to any strictly semistable pair (\mathcal{X}, H) without additional assumptions. The resulting objects might be called weak tropical complexes. We have to deal with the problem that the irreducible component V_u of \mathcal{X}_s corresponding to the vertex u of $S(\mathcal{X}, H)$ is not necessarily the support of a Cartier divisor (see Proposition 4.17). As above, a one-dimensional vertical stratum T of D corresponds to a $(d-1)$ -dimensional canonical polyhedron Δ_T of $S(\mathcal{X}, H)$. Since \mathcal{X} is a proper scheme, the curve \overline{T} is projective. For every point $x \in \overline{T}(\tilde{K})$

there is a neighbourhood \mathfrak{U}_x of x in $\mathfrak{X} = \widehat{\mathcal{X}}$ which is a building block with distinguished stratum T or $\{x\}$. We let $\mathfrak{U} = \bigcup_{x \in \overline{T}(\tilde{K})} \mathfrak{U}_x$. This is a formal open subset of \mathfrak{X} containing \overline{T} . Note that \mathfrak{U} is a strictly semistable formal scheme and the Cartier divisor H on \mathcal{X} induces a formal strictly semistable pair $(\mathfrak{U}, \widehat{H}|_{\mathfrak{U}})$ (use Proposition 3.13). By construction, we have

$$\mathfrak{U}_\eta \cap S(\mathcal{X}, H) = S(\mathfrak{U}, \widehat{H}|_{\mathfrak{U}}) = \bigcup_{\Delta_S \succ \Delta_T} \Delta_S,$$

where Δ_S ranges over all canonical polyhedra of $S(\mathcal{X}, H)$ which contain Δ_T . In particular, the vertices u of $S(\mathcal{X}, H)$ contained in \mathfrak{U}_η correspond to the irreducible components V_u of \mathcal{X}_s intersecting \overline{T} .

Let Δ_S be a d -dimensional canonical polyhedron of $S(\mathcal{X}, H)$ containing Δ_T . Then S is a component of the intersection of \overline{T} with a vertical component V_u if and only if the finite part of Δ_S is strictly larger than the finite part of Δ_T , in which case we say that Δ_S *extends* Δ_T *in a bounded direction*. Otherwise S is a component of the intersection of \overline{T} with a horizontal component of D ; in this case we say that Δ_S *extends* Δ_T *in an unbounded direction*. We define $\deg_b(\Delta_T)$ (resp. $\deg_u(\Delta_T)$) to be the number of canonical polyhedra Δ_S extending Δ_T in a bounded (resp. unbounded) direction.

For each vertex $u \in \Delta_T$ we define an integer $\alpha(u, \Delta_T)$ as follows.

- (1) Suppose that Δ_T has positive-dimensional finite part, i.e. that T lies on at least two irreducible components of \mathcal{X}_s , and let $v(\pi)$ be the length of Δ_T . Then the same is true of any $\Delta_S \succ \Delta_T$, so by Proposition 4.17, for every vertex u of Δ_T , there is a unique effective Cartier divisor C_u on \mathfrak{U} with $\text{cyc}(C_u) = v(\pi)(V_u \cap \mathfrak{U}_s)$. For every such vertex u we let $-\alpha(u, \Delta_T) \in \mathbf{Z}$ denote the intersection number $C_u \cdot \overline{T}$. This is by definition the degree of the pull-back to \overline{T} of the line bundle on \mathfrak{U} associated to C_u .
- (2) If Δ_T has zero-dimensional finite part $\{u\}$, then we set $\alpha(u, \Delta_T) := \deg_b(\Delta_T)$.

Note that $\alpha(u, \Delta_T)$ can be calculated in any neighbourhood \mathfrak{U} of \overline{T} as above, hence is intrinsic to u and T . We will also need to remember intersection numbers with horizontal divisors, which we think of as “data at infinity”. The rays (unbounded one-dimensional faces) of the recession cone $\rho(\Delta_T)$ of Δ_T are in bijective correspondence with the horizontal components of D containing T in their special fibre. For such a ray r we let H_r be the corresponding horizontal component, and we define

$$\alpha(r, \Delta_T) := -H_r \cdot \overline{T}.$$

There is no problem computing this intersection product, as H_r is a Cartier divisor on all of \mathcal{X} .

Lemma 6.5. *Let Δ_T be a $(d-1)$ -dimensional canonical polyhedron of $S(\mathcal{X}, H)$. Then*

$$(6.5.1) \quad \deg_b(\Delta_T) = \sum_{u \in \Delta_T} \alpha(u, \Delta_T).$$

Proof. This is definitional if Δ_T has zero-dimensional finite part. Otherwise, since \overline{T} is a complete curve we have

$$0 = \text{div}(\pi) \cdot \overline{T} = \sum_u C_u \cdot \overline{T} = \sum_{u \notin \Delta_T} C_u \cdot \overline{T} - \sum_{u \in \Delta_T} \alpha(u, \Delta_T)$$

where the sums are taken over vertices of $S(\mathcal{X}, H)$ contained in \mathfrak{U}_η , and all intersection products are taken in \mathfrak{U} . Since all intersections are transverse, if $u \notin \Delta_T$ then $C_u \cdot \overline{T}$ is the number of points in $V_u \cap \overline{T}$, i.e. the number of d -dimensional canonical polyhedra containing both u and Δ_T . ■

6.6. The divisor of a piecewise-affine function. We will define the divisor of a piecewise integral Γ -affine function F on $S(\mathcal{X}, H)$ as a formal sum of $(d-1)$ -dimensional canonical polyhedra. In analogy with the slope formula for curves [BPR13, Theorem 5.15], we first define outgoing slopes along a d -dimensional canonical polyhedron.

Definition 6.7. Let Δ_T be a $(d-1)$ -dimensional canonical polyhedron of $S(\mathcal{X})$ and let Δ_S be a d -dimensional canonical polyhedron of $S(\mathcal{X}, H)$ containing Δ_T . Let $F : \Delta_S \rightarrow \mathbf{R}$ be an integral Γ -affine function.

- (1) If Δ_S extends Δ_T in a bounded direction then there is a unique vertex w of Δ_S not contained in Δ_T , and we define the *slope of F at Δ_T along Δ_S* to be the quantity

$$(6.7.1) \quad \text{slope}(F; \Delta_T, \Delta_S) := \frac{1}{v(\pi)} \left(F(w) - \frac{1}{\deg_b(\Delta_T)} \sum_{u \in \Delta_T} \alpha(u, \Delta_T) F(u) \right),$$

where the sum is over all vertices u of Δ_T and $v(\pi)$ is the length of Δ_S .

- (2) If Δ_S extends Δ_T in an unbounded direction then there is a unique ray s of $\rho(\Delta_S)$ not contained in $\rho(\Delta_T)$, and we define the *slope of F at Δ_T along Δ_S* to be

$$(6.7.2) \quad \text{slope}(F; \Delta_T, \Delta_S) := d_s F - \frac{1}{\deg_u(\Delta_T)} \sum_{r \subset \rho(\Delta_T)} \alpha(r, \Delta_T) d_r F.$$

Here the second sum is over all rays of the recession cone $\rho(\Delta_T)$, and for a ray r we denote by $d_r F$ the derivative of F along the primitive vector in the direction of r .

6.7.3. Definitions (6.7.1) and (6.7.2) require some explanation. First we treat (6.7.1). If X is a curve, or more generally if Δ_T has zero-dimensional finite part $\{u\}$, then $\alpha(u, \Delta_T) = \deg_b(\Delta_T)$ by definition, so in this case $\text{slope}(F; \Delta_T, \Delta_S) = (F(w) - F(u))/v(\pi)$. This is the difference between the values of F at the endpoints of the edge in Δ_S , divided by the length of Δ_S . If Δ_T has positive-dimensional finite part, suppose for simplicity that Δ_T is bounded. The problem with defining the slope in this situation is that the naïve slope $(F(w) - F(u))/v(\pi)$ may depend on the vertex $u \in \Delta_T$. If all of the quantities $\alpha(u, \Delta_T)$ were nonnegative then

$$(6.7.4) \quad m_T := \frac{1}{\deg_b(\Delta_T)} \sum_{u \in \Delta_T} \alpha(u, \Delta_T) u$$

would be a point of Δ_T by (6.5.1), so since F is affine-linear on Δ_T , we have $\text{slope}(F; \Delta_T, \Delta_S) = (F(v) - F(m_T))/v(\pi)$. Interpreting m_T as a weighted midpoint of Δ_T , and declaring that all of Δ_T has distance $v(\pi)$ from v , we are again able to interpret $\text{slope}(F; \Delta_T, \Delta_S)$ as a slope of F along a line segment. In general $\alpha(u, \Delta_T)$ need not be nonnegative, so m_T cannot be interpreted as a point of Δ_T , and hence this explanation is more of a heuristic.

Now consider (6.7.2). Again if all of the $\alpha(r, \Delta_T)$ were nonnegative then the vector

$$u_T := \frac{1}{\deg_u(\Delta_T)} \sum_{r \subset \rho(\Delta_T)} \alpha(r, \Delta_T) r_0$$

would be contained in $\rho(\Delta_T)$, where r_0 denotes the primitive vector on the ray r . In this case $\text{slope}(F; \Delta_T, \Delta_S)$ would be the derivative of F in the direction $s_0 - u_T$. The primary argument for the reasonableness of this definition is that it is the obvious “linearized” analogue of (6.7.1).

Note that in either case, $\text{slope}(F; \Delta_T, \Delta_S)$ does not change if we replace F by $F + c$ for $c \in \mathbf{R}$.

Remark 6.7.5. When X is a curve, $\text{slope}(F; \Delta_T, \Delta_S)$ is always an integer. In the bounded case this follows from the fact that F is integral Γ -affine on each canonical polyhedron, and in the unbounded case it follows from the fact that $d_s F \in \mathbf{Z}$. In higher dimensions the slopes need not be integers. See Remark 7.12.

Definition 6.8. Let $F : S(\mathcal{X}, H) \rightarrow \mathbf{R}$ be a continuous function which is integral Γ -affine on each canonical polyhedron. For every $(d - 1)$ -dimensional polyhedron Δ_T of $S(\mathcal{X}, H)$ we define

$$(6.8.1) \quad m(\text{div}(F), \Delta_T) := \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{bounded}}} \text{slope}(F; \Delta_T, \Delta_S)$$

$$(6.8.2) \quad m(\widehat{\text{div}}(F), \Delta_T) := \sum_{\Delta_S \succ \Delta_T} \text{slope}(F; \Delta_T, \Delta_S),$$

where the first (resp. second) sum ranges over all bounded (resp. bounded and unbounded) d -dimensional canonical polyhedra Δ_S of $S(\mathcal{X}, H)$ containing Δ_T . Note that $m(\text{div}(F), \Delta_T) = 0$ if

Δ_T is unbounded. We define

$$\begin{aligned} \operatorname{div}(F) &:= \sum_{\Delta_T} m(\operatorname{div}(F), \Delta_T) \Delta_T \\ \widehat{\operatorname{div}}(F) &:= \sum_{\Delta_T} m(\widehat{\operatorname{div}}(F), \Delta_T) \Delta_T, \end{aligned}$$

where the first (resp. second) sum ranges over the bounded (resp. bounded and unbounded) $(d-1)$ -dimensional canonical polyhedra Δ_T of $S(\mathcal{X}, H)$.

Let Δ_T be a $(d-1)$ -dimensional canonical polyhedron of $S(\mathcal{X}, H)$. Substituting (6.7.1) into (6.8.1) gives

$$(6.8.3) \quad v(\pi) m(\operatorname{div}(F), \Delta_T) = \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{bounded}}} F(w_S) - \sum_{u \in \Delta_T} \alpha(u, \Delta_T) F(u),$$

where w_S is the unique vertex of Δ_S not contained in Δ_T . In Cartwright's situation, we have $v(\pi) = 1$ and Δ_T is a canonical simplex of $S(\mathcal{X})$. Then $m(\operatorname{div}(F), \Delta_T)$ agrees with [Car13, Definition 1.5].

We can now state the **slope formula** for the skeleton $S(\mathcal{X}, H)$.

Theorem 6.9. (Slope formula) *Let $f \in K(X)^\times$ be a rational function such that $\operatorname{supp}(\operatorname{div}(f)) \subset \operatorname{supp}(H)_\eta$ and let $F = -\log |f|_{S(\mathcal{X}, H)}$. Then F is continuous and integral Γ -affine on each canonical polyhedron of $S(\mathcal{X}, H)$, and we have*

$$\widehat{\operatorname{div}}(F) = 0.$$

The identity $\widehat{\operatorname{div}}(F) = 0$ is a kind of balancing condition for F on $S(\mathcal{X}, H)$, which is strongly analogous to the balancing condition for tropical varieties. We require one lemma before proving Theorem 6.9.

Lemma 6.10. *We use the notation in 6.4. Suppose that Δ_T has positive-dimensional finite part and length $v(\pi)$. Then there exists $\lambda \in K^\times$ such that the Cartier divisor $(\lambda^{-1}f)|_{\mathfrak{U}}$ is an integer linear combination of $\{C_u \mid u \in \mathfrak{U}_\eta \text{ a vertex of } S(\mathcal{X}, H)\}$ and $\{H_i|_{\mathfrak{U}} \mid i = 1, \dots, S\}$. More precisely,*

$$(6.10.1) \quad \operatorname{div}(\lambda^{-1}f)|_{\mathfrak{U}} = \sum_u n_u C_u + \sum_{i=1}^S \operatorname{ord}(f, \mathcal{H}_i) H_i|_{\mathfrak{U}}$$

where $n_u = \frac{1}{v(\pi)}(\operatorname{ord}(f, V_u) - v(\lambda)) \in \mathbf{Z}$ and u ranges over all vertices of $S(\mathcal{X}, H)$ contained in \mathfrak{U}_η .

Proof. Let $\mathfrak{U}_x \subset \mathfrak{U}$ be a building block as in 6.4. By the proof of Proposition 5.2 and Remark 5.3, we see that there exists $\lambda \in K^\times$ such that $\frac{1}{v(\pi)}(\operatorname{ord}(f, V_u) - v(\lambda)) \in \mathbf{Z}$ for all vertices u of $S(\mathcal{X}, H)$ contained in $(\mathfrak{U}_x)_\eta$. It follows that for any two such vertices u, u' we have $\operatorname{ord}(f, V_u) - \operatorname{ord}(f, V_{u'}) \in v(\pi)\mathbf{Z}$. This last statement is independent of the choice of building block containing u, u' , so since any two vertices of $S(\mathcal{X}, H)$ contained in \mathfrak{U}_η are connected by finite faces of positive dimension, we see that $\operatorname{ord}(f, V_u) - \operatorname{ord}(f, V_{u'}) \in v(\pi)\mathbf{Z}$ for any two vertices u, u' of $S(\mathcal{X}, H)$ contained in \mathfrak{U}_η . Choosing λ as above (with respect to any choice of building block), we have $\operatorname{ord}(\lambda^{-1}f, V_u) \in v(\pi)\mathbf{Z}$ for all vertices u of $S(\mathcal{X}, H)$ contained in \mathfrak{U}_η . Letting $n_u = \frac{1}{v(\pi)} \operatorname{ord}(\lambda^{-1}f, V_u)$, we have the equality (6.10.1), as both sides have the same Weil divisor (use Proposition A.7). \blacksquare

Proof of Theorem 6.9. We showed that F is continuous and integral Γ -affine on canonical polyhedra in Proposition 5.2. We have to prove that $m(\widehat{\operatorname{div}}(F), \Delta_T) = 0$ for all $(d-1)$ -dimensional canonical polyhedra Δ_T of $S(\mathcal{X}, H)$. First suppose that Δ_T has positive-dimensional finite part and length $v(\pi)$. We use the notation in 6.4. Choose $\lambda \in K^\times$ as in Lemma 6.10. Multiplying f by a non-zero scalar does not change $\widehat{\operatorname{div}}(F)$, so we may replace f with $\lambda^{-1}f$ to assume that we have an equality of Cartier divisors

$$\operatorname{div}(f)|_{\mathfrak{U}} = \sum_u n_u C_u + \sum_{i=1}^S \operatorname{ord}(f, \mathcal{H}_i) H_i|_{\mathfrak{U}}$$

where $n_u = \frac{1}{v(\pi)} \text{ord}(f, V_u) \in \mathbf{Z}$ and u ranges over all vertices of $S(\mathcal{X}, H)$ contained in \mathfrak{U}_η . As $\text{div}(f)$ is a principal Cartier divisor, we have

$$(6.10.2) \quad 0 = \text{div}(f)|_{\mathfrak{U}} \cdot \bar{T} = \sum_u n_u C_u \cdot \bar{T} + \sum_{i=1}^S \text{ord}(f, \mathcal{H}_i) H_i \cdot \bar{T}.$$

For u a vertex of $S(\mathcal{X}, H)$ contained in \mathfrak{U}_η , by definition we have $F(u) = \text{ord}(f, V_u) = v(\pi) n_u$. Substituting into (6.10.2), we have

$$(6.10.3) \quad 0 = \frac{1}{v(\pi)} \sum_u F(u) C_u \cdot \bar{T} + \sum_{i=1}^S \text{ord}(f, \mathcal{H}_i) H_i \cdot \bar{T}.$$

If $u \notin \Delta_T$ then $C_u \cdot \bar{T}$ is equal to the number of canonical polyhedra Δ_S containing Δ_T and u . If $u \in \Delta_T$ then $C_u \cdot \bar{T} = -\alpha(u, \Delta_T)$, so

$$(6.10.4) \quad \begin{aligned} \frac{1}{v(\pi)} \sum_u F(u) C_u \cdot \bar{T} &= \frac{1}{v(\pi)} \left(\sum_{\substack{\Delta_S \succ \Delta_T \\ \text{bounded}}} F(w_S) - \sum_{u \in \Delta_T} \alpha(u, \Delta_T) F(u) \right) \\ &= \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{bounded}}} \frac{1}{v(\pi)} \left(F(w_S) - \frac{1}{\text{deg}_b(\Delta_T)} \sum_{u \in \Delta_T} \alpha(u, \Delta_T) F(u) \right) \\ &= \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{bounded}}} \text{slope}(F; \Delta_T, \Delta_S), \end{aligned}$$

where the first sum runs over all canonical polyhedra Δ_S extending Δ_T in a bounded direction, and w_S is the vertex of Δ_S not contained in Δ_T .

Recall that a ray r contained in the recession cone $\rho(\Delta_R)$ of a canonical polyhedron Δ_R of $S(\mathcal{X}, H)$ corresponds to a horizontal component \mathcal{H}_r containing the stratum R in the special fibre. Moreover, the multiplicity $\text{ord}(f, \mathcal{H}_r)$ is equal to the derivative of F along the primitive vector on the ray of $\rho(\Delta_T)$ corresponding to H_r : see 6.1 and 6.4. If T is not contained in \mathcal{H}_i for some $i \in \{1, \dots, R+S\}$, then either $T \cap \mathcal{H}_i = \emptyset$ or there is a d -dimensional canonical polyhedron Δ_S which extends Δ_T in the unbounded direction r_i for a ray r_i with $\mathcal{H}_{r_i} = \mathcal{H}_i$. The first case is equivalent to $H_i \cdot \bar{T} = 0$. In the second case, $H_i \cdot \bar{T}$ is equal to the number of d -dimensional canonical polyhedra Δ_S extending Δ_T in an unbounded direction corresponding to \mathcal{H}_i . If T is contained in \mathcal{H}_i , then there is a ray $r_i \subset \rho(\Delta_T)$ with $\mathcal{H}_{r_i} = \mathcal{H}_i$ and $H_i \cdot \bar{T} = -\alpha(r_i, \Delta_T)$. It follows that

$$(6.10.5) \quad \begin{aligned} \sum_{i=1}^S \text{ord}(f, \mathcal{H}_i) H_i \cdot \bar{T} &= \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{unbounded}}} d_s F - \sum_{r \subset \rho(\Delta_T)} \alpha(r, \Delta_T) d_r F \\ &= \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{unbounded}}} \left(d_s F - \frac{1}{\text{deg}_u(\Delta_T)} \sum_{r \subset \rho(\Delta_T)} \alpha(r, \Delta_T) d_r F \right) \\ &= \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{unbounded}}} \text{slope}(F; \Delta_T, \Delta_S), \end{aligned}$$

where the first sum runs over the canonical polyhedra Δ_S extending Δ_T in an unbounded direction, the second runs over all rays of $\rho(\Delta_T)$, and s is the ray of $\rho(\Delta_S)$ not contained in $\rho(\Delta_T)$. Combining (6.10.3) with (6.10.4) and (6.10.5), we obtain

$$(6.10.6) \quad 0 = \sum_{\Delta_S \succ \Delta_T} \text{slope}(F; \Delta_T, \Delta_S) = m(\widehat{\text{div}}(F), \Delta_T).$$

Now suppose that Δ_T has zero-dimensional finite part $\{u\}$. A separate argument is needed as the vertical components of \mathcal{X}_s are not necessarily Cartier on a formal open subscheme containing \bar{T} . In this case $\bar{T} = V_u \cap (\mathcal{H}_{i_2})_s \cap \dots \cap (\mathcal{H}_{i_d})_s$ for some $i_2, \dots, i_d \in \{1, \dots, S\}$ and for a vertex u of $S(\mathcal{X}, H)$. Replacing f by $\lambda^{-1}f$ for a suitable non-zero $\lambda \in K$, we may assume that $\text{ord}(f, V_u) = 0$.

Let us consider the Cartier divisor $E := \sum_{j=1}^S \text{ord}(f, \mathcal{H}_j) H_j$ on \mathcal{X} . Then the Cartier divisor $D := \text{div}(f) - E$ has support in the special fibre of \mathcal{X} , and we have

$$\text{ord}(D, V_u) = \text{ord}(f, V_u) - F(u) = 0.$$

We conclude that D intersects \overline{T} properly which means that the intersection of \overline{T} with the support of D is zero-dimensional. Then the intersection product $D \cdot \overline{T}$ is a well-defined cycle on \overline{T} supported in the union of all zero-dimensional strata S such that Δ_S extends Δ_T in a bounded direction.

The multiplicity m_S of $D \cdot \overline{T}$ in S may be computed on a building block \mathfrak{U} with distinguished stratum S . The finite part of Δ_S is an edge from u to another vertex w of length $v(\pi)$. By Proposition 4.17, there is a unique effective Cartier divisor C on \mathfrak{U} with $\text{cyc}(C) = v(\pi) \cdot (V_w \cap \mathfrak{U}_S)$. We have

$$\text{cyc}(D|_{\mathfrak{U}}) = \text{ord}(f, V_w) \cdot (V_w \cap \mathfrak{U}_S)$$

and hence we deduce from Proposition A.7 that

$$D|_{\mathfrak{U}} = \frac{\text{ord}(f, V_w)}{v(\pi)} \cdot C.$$

Using that \mathfrak{U} is a strictly semistable formal scheme, the multiplicity m_S of $D \cdot \overline{T}$ in S is

$$m_S = \frac{\text{ord}(f, V_w)}{v(\pi)} = \frac{1}{v(\pi)} (F(w) - F(u)) = \text{slope}(F; \Delta_T, \Delta_S).$$

We conclude that

$$(6.10.7) \quad D \cdot \overline{T} = \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{bounded}}} \text{slope}(F; \Delta_T, \Delta_S).$$

On the other hand, we have

$$0 = \text{div}(f) \cdot \overline{T} = D \cdot \overline{T} + E \cdot \overline{T}.$$

We substitute (6.10.7) for $D \cdot \overline{T}$. Moreover, $E \cdot \overline{T}$ can be calculated as in (6.10.5) as we have not used positive dimensionality of the finite part of Δ_T in the argument. Now the claim follows as in (6.10.6). \blacksquare

6.11. One could imagine defining $\widehat{\text{div}}(F)$ as a formal sum which includes the $(d-1)$ -dimensional canonical polyhedra Δ'_S in the boundary of the compactified skeleton $\widehat{S}(\mathcal{X}, H)$, with the multiplicity $m(\widehat{\text{div}}(F), \Delta'_S)$ being determined by an outgoing slope of F . In this case the correct statement would be $\widehat{\text{div}}(F) + \widehat{\tau}(f) = 0$, where $\widehat{\tau}(f)$ is the retraction of the principal Weil divisor $\text{cyc}(f)$ to $\widehat{S}(\mathcal{X}, H)$ (see 6.1). We leave this reformulation to the interested reader.

As a consequence of Theorem 6.9, we derive the **slope formula** for the skeleton $S(\mathcal{X})$.

Theorem 6.12. *Let $f \in K(X)^\times$ be a rational function such that $\text{supp}(\text{div}(f)) \subset \text{supp}(H)_\eta$ and let $F = -\log|f|_{S(\mathcal{X})}$. Then F is continuous and integral Γ -affine on each canonical polyhedron of $S(\mathcal{X})$, and we have*

$$\text{div}(F) + \tau(f) = 0,$$

where $\tau(f)$ is the retraction of the principal Weil divisor $\text{cyc}(f)$ to $S(\mathcal{X})$ from 6.2.

Proof. Let Δ_T be a bounded $(d-1)$ -dimensional canonical polyhedron. It remains to prove that $m(\text{div}(F), \Delta_T) + m(\tau(f), \Delta_T) = 0$. We have

$$\begin{aligned} 0 &= m(\widehat{\text{div}}(F), \Delta_T) = \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{bounded}}} \text{slope}(F; \Delta_T, \Delta_S) + \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{unbounded}}} \text{slope}(F; \Delta_T, \Delta_S) \\ &= m(\text{div}(F), \Delta_T) + \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{unbounded}}} \text{slope}(F; \Delta_T, \Delta_S), \end{aligned}$$

so we need to argue that the last sum is equal to $m(\tau(f), \Delta_T)$. If Δ_S extends Δ_T in an unbounded direction then since $\rho(\Delta_T)$ contains no rays, we have $\text{slope}(F; \Delta_T, \Delta_S) = d_S F = \text{ord}(f, \mathcal{H}_i)$,

where s is the ray of $\rho(\Delta_S)$ and \mathcal{H}_i is the horizontal component containing S . Using 6.2 and (6.10.5), we have

$$m(\tau(f), \Delta_T) = \sum_k \text{ord}(f, \mathcal{H}_k) \cdot \#(\overline{T} \cap (\mathcal{H}_k)_s) = \sum_k \text{ord}(f, \mathcal{H}_k) H_k \cdot \overline{T} = \sum_{\substack{\Delta_S \succ \Delta_T \\ \text{unbounded}}} \text{slope}(F; \Delta_T, \Delta_S),$$

which finishes the proof. \blacksquare

Example 6.13. Suppose that $\dim(X) = 1$, i.e. that X is a curve. In this case $\widehat{S}(\mathcal{X}, H) = S(\mathcal{X}, H) \cup \text{supp}(H)$, and identifying canonical polyhedra on the boundary of $\widehat{S}(\mathcal{X}, H)$ with points of $X(K)$ identifies $\widehat{\tau}(f)$ with $\text{div}(f)$. If $\text{div}(f) = \sum n_x \cdot x$ then $\tau(f) = \sum n_x \cdot \tau(x)$, which is the retraction $\tau_*(\text{div}(f))$ of the divisor $\text{div}(f)$ to the skeleton $S(\mathcal{X}, H)$. Hence the identity $\text{div}(F) + \tau(f) = 0$ of Theorem 6.12 reads

$$\text{div}(-\log|f|) = -\tau_*(\text{div}(f)).$$

As explained in 6.7.3, if $\Delta_T = \{x\} \subset S(\mathcal{X}, H)$ is a zero-dimensional canonical polyhedron then for every segment or ray Δ_S containing x , the slope of F at Δ_T along Δ_S is simply the outgoing slope of F in the direction of Δ_T . The statement $\widehat{\text{div}}(F) = 0$ of Theorem 6.9 therefore says that the sum of the outgoing slopes of F at x is zero. This, along with the other results we have proved, essentially recovers the slope formula for curves [BPR13, Theorem 5.15].

Remark 6.14. It is not hard to show that in dimension 1, the divisor $\tau(f) = \tau_*\text{div}(f)$ and the harmonicity condition $\text{div}(F) + \tau(f) = 0$ of Theorem 6.9 uniquely determine the piecewise linear function F on $S(\mathcal{X})$ up to additive translation (see [Thu05, §1.2.3] or [BR10, Proposition 3.2(A)]). In higher dimensions, this Neumann problem in polyhedral geometry need not have a unique solution: there exist piecewise linear spaces S with α -numbers $\alpha(\cdot, \cdot)$ and non-constant piecewise linear functions $F : S \rightarrow \mathbf{R}$ as in Definition 6.8 such that $\text{div}(F) = 0$. Such an example can be found in [Car13, Example 6.11]. Dustin Cartwright has pointed out to the authors that his example arises as a skeleton $S(\mathcal{X})$ of a strictly semistable pair (\mathcal{X}, H) , where \mathcal{X} is a toric scheme over a (discrete) valuation ring. Moreover, his non-constant piecewise linear functions F with $\text{div}(F) = 0$ is the restriction of $-\log|f|$ to $S(\mathcal{X})$ for a non-constant rational function f on \mathcal{X} .

Note that this cannot happen on $S(\mathcal{X}, H)$ for any strictly semistable pair (\mathcal{X}, H) . If f is a non-zero rational function on \mathcal{X} and if F is the restriction of $-\log|f|$ to $S(\mathcal{X}, H)$, then $\widehat{\text{div}}(F) = 0$ yields that f is constant as f is determined up to multiples by $\text{cyc}(f)$ and hence also by $\widehat{\tau}(f)$.

7. A TWO-DIMENSIONAL EXAMPLE

In this section, we give an example to illustrate the slope formula for skeletons. We choose the square of a Tate elliptic curve which fits well for analytic purposes. As before, K is an algebraically closed field complete with respect to the valuation v and with non-zero value group $\Gamma \subset \mathbf{R}$. We assume that the residue field \widetilde{K} of K has characteristic $\neq 2$.

7.1. Recall that an abelian variety A over K is *totally degenerate* if $A^{\text{an}} = T^{\text{an}}/P$ where $T = \text{Spec}(K[M])$ is a multiplicative torus and P is a lattice in T^{an} . Here, a lattice means a discrete subgroup P of T^{an} contained in $T(K)$ such that $\text{trop} : T^{\text{an}} \rightarrow N_{\mathbf{R}}$ maps P isomorphically onto a complete lattice Λ of $N_{\mathbf{R}}$, where N is the dual of the character lattice M of T . Passing to the quotient, we get a continuous proper map $\overline{\text{trop}} : A^{\text{an}} \rightarrow N_{\mathbf{R}}/\Lambda$.

Example 7.2. The totally degenerate abelian varieties of dimension 1 are called Tate elliptic curves. For every $q \in K^\times$ with $v(q) > 0$, we have a Tate elliptic curve E given analytically by $T_1^{\text{an}}/q^{\mathbf{Z}}$ with torus $T_1 = \text{Spec}(K[\mathbf{Z}])$ and lattice $q^{\mathbf{Z}}$. Algebraically, the elliptic curve E is given by the generalized Weierstrass equation

$$y^2 + xy = x^3 + a_4x + a_6,$$

where a_4, a_6 are given by the following convergent power series

$$a_4 = -5 \sum_{n=1}^{\infty} n^3 q^n / (1 - q^n), \quad a_6 = -\frac{1}{12} \sum_{n=1}^{\infty} (7n^5 + 5n^3) q^n / (1 - q^n).$$

The isomorphism $T_1^{\text{an}}/q^{\mathbf{Z}} \rightarrow E^{\text{an}}$ is given by

$$(7.2.1) \quad \begin{aligned} x(\zeta) &= \sum_{n=-\infty}^{\infty} q^n \zeta / (1 - q^n \zeta)^2 - 2 \sum_{n=1}^{\infty} n q^n / (1 - q^n) \\ y(\zeta) &= \sum_{n=-\infty}^{\infty} q^{2n} \zeta^2 (1 - q^n \zeta)^3 + \sum_{n=1}^{\infty} n q^n / (1 - q^n), \end{aligned}$$

where ζ is the torus coordinate on T_1 . In the following, we will identify E^{an} with $T_1^{\text{an}}/q^{\mathbf{Z}}$ using this isomorphism of analytic groups. This is due to Tate [Tat95] (see also [Sil09, Theorem C14.1]).

7.3. Let A be a totally degenerate abelian variety over K as in 7.1. We define a *polytope* in $N_{\mathbf{R}}/\Lambda$ as a subset $\overline{\Delta}$ of $N_{\mathbf{R}}/\Lambda$ given as the bijective image of a polytope Δ in $N_{\mathbf{R}}$ with respect to the quotient homomorphism. Recall that a *polytopal decomposition* of a set S in $N_{\mathbf{R}}$ is a polytopal complex with support S . A *polytopal decomposition* $\overline{\mathcal{C}}$ of $N_{\mathbf{R}}/\Lambda$ is a finite collection of polytopes in $N_{\mathbf{R}}/\Lambda$ induced by an infinite Λ -periodic polytopal decomposition \mathcal{C} of $N_{\mathbf{R}}$. We will assume always that $\overline{\mathcal{C}}$ is integral Γ -affine which means that all polytopes of \mathcal{C} are integral Γ -affine polytopes in $N_{\mathbf{R}}$.

For $\Delta \in \mathcal{C}$, we have a polytopal subdomain $U_{\Delta} = \text{trop}^{-1}(\Delta)$ of T^{an} which is the Berkovich spectrum of the strictly affinoid algebra

$$K\langle U_{\Delta} \rangle = \left\{ \sum_{u \in M} \alpha_u \chi^u \mid \lim_{|u| \rightarrow \infty} v(\alpha_u) + \langle u, \omega \rangle = \infty \forall \omega \in \Delta \right\},$$

where χ^u is the character of T corresponding to $u \in M$, the coefficients α_u of the power series are in K and where $|u|$ uses any norm on $M_{\mathbf{R}}$. The supremum norm is given here by

$$\left\| \sum_{u \in M} \alpha_u \chi^u \right\|_{\text{sup}} = \max_{\omega \in \Delta, u \in M} |\alpha_u| e^{-\langle u, \omega \rangle}.$$

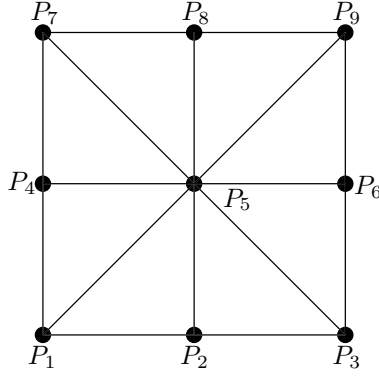
Recall that $K\langle U_{\Delta} \rangle^{\circ}$ is the subalgebra of $K\langle U_{\Delta} \rangle$ given by the Laurent series of supremum norm ≤ 1 . The canonical formal K° -model of U_{Δ} is $\mathfrak{U}_{\Delta} := \text{Spf}(K\langle U_{\Delta} \rangle^{\circ})$. This is an affine admissible formal K° -scheme, as mentioned in 2.1.

The admissible formal schemes $(\mathfrak{U}_{\Delta})_{\Delta \in \mathcal{C}}$ glue together to an admissible formal scheme \mathfrak{E} which is a K° -model of T^{an} . Passing to the quotient $\mathfrak{A} := \mathfrak{E}/P$, we get a K° -model of A . It is called the *Mumford model associated to the polytopal decomposition* $\overline{\mathcal{C}}$. For more details about this construction, we refer the reader to [Gub07, §4, §6].

7.4. A simplex Δ in $N_{\mathbf{R}}$ is called *regular* if there is a basis of N and $a > 0, a \in \Gamma$ such that Δ is a translate of $\{\omega \in N_{\mathbf{R}} \mid \omega_i \geq 0, \omega_1 + \dots + \omega_n \leq a\}$, where $\omega_1, \dots, \omega_n$ are the coordinates of ω . If Δ is regular then $\mathfrak{U}_{\Delta} \cong \mathfrak{U}_{\Delta(n+1, \pi)}$, where $v(\pi) = a$. Suppose that $\overline{\mathcal{C}}$ is induced by an infinite Λ -periodic polytopal decomposition \mathcal{C} of $N_{\mathbf{R}}$ as above. If \mathcal{C} consists of regular simplices, then it is clear from the definitions that the Mumford model \mathfrak{A} associated to $\overline{\mathcal{C}}$ is a strictly semistable formal scheme. Then the skeleton $S(\mathfrak{A})$ may be identified with $N_{\mathbf{R}}/\Lambda$, the triangulation $\overline{\mathcal{C}}$ induces the decomposition of $S(\mathfrak{A})$ into canonical simplices and $\overline{\text{trop}}$ is the canonical retraction $\tau : A^{\text{an}} \rightarrow S(\mathfrak{A})$. This follows from [Gub07, Proposition 6.3] and the definitions.

7.5. In the rest of this section, we will focus on the following special case of the above construction. We consider the Tate elliptic curve E associated to $q \in K^{\times}$ which means $E^{\text{an}} = T_1^{\text{an}}/q^{\mathbf{Z}}$ for the one-dimensional torus $T_1 = \text{Spec}(K[\zeta^{\pm 1}])$. For simplicity, we assume $v(q) = 1$. Then $A := E^2$ is a totally degenerate abelian variety over K given analytically by T^{an}/P , where $T = T_1 \times T_1$ and $P = q^{\mathbf{Z}} \times q^{\mathbf{Z}}$. For the above lattices, we have $M = \Lambda = \mathbf{Z} \times \mathbf{Z}$. Referring to 7.2, we denote by ζ_1 (resp. ζ_2) the pull-back of the torus coordinate ζ and by x_1, y_1 (resp. x_2, y_2) the pull-back of the algebraic coordinates x, y of the generalized Weierstrass equation with respect to the projection to the first (resp. second) factor of E^2 .

7.6. We choose the regular triangulation \mathcal{C} of \mathbf{R}^2 obtained by Λ -translations from the natural unit square by dividing it in four squares and drawing the two diagonals in the original unit square. See Figure 1.

FIGURE 1. The triangulation \mathcal{C} of 7.6.

Let \mathfrak{A} be the Mumford model of A associated to \mathcal{C} . We have seen in 7.3 that \mathfrak{A} is a strictly semistable K° -model of $A = E^2$. By 7.4, the skeleton and its canonical simplices are induced by the decomposition $\overline{\mathcal{C}}$ of \mathbf{R}^2/Λ . The unit square serves as a fundamental lattice and we number its vertices by $P_1 = (0, 0), \dots, P_9 = (1, 1)$, where we identify for example the vertices $P_1, P_3 = (1, 0)$, $P_7 = (0, 1)$ and P_9 according to Λ -translation. An edge in $\overline{\mathcal{C}}$ with the two vertices P_i and P_j is denoted by e_{ij} and a face in $\overline{\mathcal{C}}$ with the three vertices P_i, P_j, P_k is denoted by Δ_{ijk} . By the stratum–face correspondence in Proposition 4.10, $S(\mathfrak{A})$ has 4 two-dimensional strata, 12 one-dimensional strata and 8 zero-dimensional strata.

Proposition 7.7. *The Mumford model \mathfrak{A} obtained from the regular triangulation above is algebraic.*

Proof. Let L be an ample line bundle on $A = E^2$. By [Gub07, 6.5], L induces a positive definite symmetric bilinear form b on Λ and a cocycle $\lambda \mapsto z_\lambda$ of $H^1(\Lambda, C(\mathbf{R}^2))$ with

$$z_\lambda(\omega) = z_\lambda(0) + b(\omega, \lambda)$$

for $\lambda \in \Gamma$ and $\omega \in \mathbf{R}^2$. It follows from [Gub07, Proposition 6.6] that every $f \in C(\mathbf{R}^2)$ with the following two conditions (a) and (b) induces a line bundle \mathfrak{L} on \mathfrak{A} with generic fibre L :

- (a) The restriction of f to $\Delta \in \mathcal{C}$ is integral Γ -affine.
- (b) $f(\omega + \lambda) = f(\omega) + z_\lambda(\omega)$ for all $\omega \in \mathbf{R}^2$ and all $\lambda \in \Lambda$.

We choose a root $\sqrt{q} \in K^\times$ leading to a 2-torsion point $P := [\sqrt{q}]$ of E . Then $L_0 := \mathcal{O}(P \times E + E \times P)$ is an ample line bundle on $A = E^2$. It is shown in §3.3 of Christensen’s thesis [Chr13] that there exists a strictly convex, piecewise linear, continuous real function $f_0 \in C(\mathbf{R}^2)$ satisfying (b) and a non-zero $m \in \mathbf{N}$ such that $f := mf_0$ satisfies (a). Moreover, the maximal domains of linearity for Christensen’s f_0 are the two-dimensional simplices of \mathcal{C} . We conclude that $L := L_0^{\otimes m}$ is an ample line bundle with a formal K° -model \mathfrak{L} on the Mumford model \mathfrak{A} . By [Gub07, Corollary 6.7], the restriction of \mathfrak{L} to the special fibre \mathfrak{A}_s is an ample line bundle. It follows from Grothendieck’s algebraization criterion (see [EGAIII₁, Theorem 5.4.5] in the case of discrete valuations and the generalization to arbitrary real valuations by Ullrich in [Ull95, Proposition 6.9] that \mathfrak{A} is algebraic. \blacksquare

Let \mathcal{A} be the algebraization of \mathfrak{A} . This is a strictly semistable algebraic K° -model of A .

7.8. In our running example $A = E^2$, we choose the rational function $f = x_1 - x_2$ using the algebraic coordinates from 7.5. Then the divisor of f on A is equal to the sum of the diagonal and the anti-diagonal in $A = E \times E$ minus $E \times 0 + 0 \times E$. If we consider the horizontal divisor H' given as the sum of the closures of the diagonal, the anti-diagonal, $E \times 0$ and $0 \times E$, then (\mathcal{A}, H') is not a strictly semistable pair on \mathcal{A} as the zero element lies in four components (at most two would be allowed). We blow up \mathcal{A} in the closure \mathcal{Y} of the zero element of A . This leads to a strictly semistable pair (\mathcal{X}, H) , where the horizontal components H_1, \dots, H_5 are given as follows: The strict transform of the closure of the diagonal (resp. anti-diagonal) with respect to the blow up is denoted by H_1 (resp. H_2). The strict transform of the closure of $E \times 0$ (resp. $0 \times E$) is denoted by H_3 (resp. H_4). The exceptional divisor of the blow up is denoted by H_5 .

The skeleton $S(\mathcal{X}, H)$ is obtained from $S(\mathcal{A})$ in the following way. First, we note that \mathcal{X}_s still has four vertical components V_1, V_2, V_4, V_5 lying over the irreducible components of \mathcal{A}_s corresponding to P_1, P_2, P_4, P_5 . We have described $S(\mathcal{A})$ as the quotient of \mathbf{R}^2 by the group action of Λ . Now we add to the plane \mathbf{R}^2 five new independent directions b_1, \dots, b_5 corresponding to the horizontal components H_1, \dots, H_5 . Then we expand the edges e_{15} and e_{59} (resp. e_{35} and e_{57}) in $S(\mathcal{A})$ to half-stripes in the b_1 -direction (resp. b_2 -direction). They correspond to the strata of D in the intersection of two vertical components with either H_1 or H_2 . Similarly, we expand the edges e_{12} and e_{23} (resp. e_{14} and e_{47}) to half-stripes in the b_3 -direction (resp. in the b_4 -direction). They correspond to the strata of D in the intersection of two vertical components with either H_3 or H_4 .

Over P_5 , we fill in two quadrants between b_1 and b_2 which both have the same two edges given by the halflines starting in P_1 in the directions b_1 and b_2 . This corresponds to the two strata points in the intersection of H_1, H_2 and V_5 . Note that we use here that the residue characteristic is not 2. Over P_1 , we add the 5 quadrants filling in between $(b_1, b_5), (b_2, b_5), (b_3, b_5), (b_4, b_5)$ and (b_1, b_2) . The first four quadrants correspond to the single point in the intersection of H_i, H_5 and V_1 for $i \in \{1, \dots, 4\}$. We note that H_1, H_2 and V_1 intersect only in one point and this corresponds to the last quadrant. There are no other intersections over 0 as the blow up separates H_i and H_j for $i \neq j$ in $\{1, \dots, 4\}$.

7.9. Our goal is to illustrate the slope formula (Theorem 6.12) for $F := -\log|f|$ on the skeleton $S(\mathcal{A}) = S(\mathcal{X})$. By 6.2 and 7.8, the retraction $\tau(f)$ to $S(\mathcal{X})$ is given by

$$(7.9.1) \quad \tau(f) = e_{15} + e_{59} + e_{35} + e_{57} - e_{12} - e_{23} - e_{14} - e_{47}.$$

This is the only part where we use the strictly semistable pair (\mathcal{X}, H) . The remaining computations can be done solely on the Mumford model \mathcal{A} and on the skeleton $S(\mathcal{A}) = S(\mathcal{X})$. In particular, the projection formula in Proposition A.12 shows that we may compute the occurring intersection numbers on the model \mathcal{A} . The vertices P_1, P_2, P_4, P_5 correspond to the irreducible components of \mathcal{A}_s which we denote by Y_1, Y_2, Y_4, Y_5 . Let $D_i := D_{P_i}$ be the Cartier divisor associated to the vertex P_i by Proposition 4.17.

We illustrate the slope formula by showing that $-m(\operatorname{div}(F), e_{15}) = m(\tau(f), e_{15}) = 1$. Here the edge e_{15} corresponds to the one-dimensional stratum T_{15} . By (6.8.3) we have

$$(7.9.2) \quad \begin{aligned} \frac{1}{2}m(\operatorname{div}(F), e_{15}) &= F(P_2) + F(P_4) - \alpha(P_1, e_{15})F(P_1) - \alpha(P_5, e_{15})F(P_5) \\ &= F(P_2) + F(P_4) + (D_1 \cdot \overline{T_{15}})F(P_1) + (D_5 \cdot \overline{T_{15}})F(P_5). \end{aligned}$$

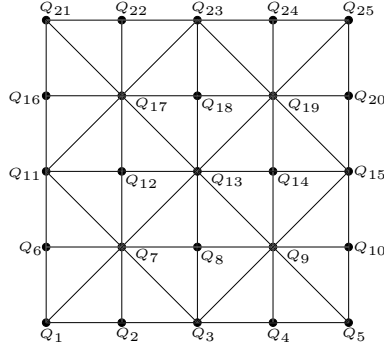
We must show that the right side of this equation is $-\frac{1}{2}$.

7.10. We compute now the quantities on the right side of (7.9.2). Consider a point ξ in the skeleton $S(T_1^{\text{an}})$ of the torus $T_1 = \operatorname{Spec}(K[\zeta^{\pm 1}])$. If $|q|^{1/2} < |\zeta(\xi)| < 1$ then the unique summand in the Laurent expansion (7.2.1) of $x(\zeta(\xi))$ with maximal absolute value is $\zeta(\xi)$. In particular, $|x(\zeta(\xi))| = |\zeta(\xi)|$. The ultrametric inequality and continuity of F then imply that $F(P_1) = F(P_2) = F(P_4) = 0$ and $F(P_5) = 1/2$. Therefore we only need to prove that $D_5 \cdot \overline{T_{15}} = -1$.

7.11. To compute the intersection number $D_5 \cdot \overline{T_{15}}$ from (7.9.2), we are going to use Kolb's relations (see Proposition A.17). The problem is that the canonical simplices of $S(\mathcal{A})$ are not determined by their vertices: for instance, Δ_{125} and Δ_{578} have the same vertices. To deal with that, we pass to a covering $\widehat{\varphi} : \mathfrak{A}' \rightarrow \mathfrak{A}$, where \mathfrak{A}' is the Mumford model of $A = E^2$ induced by the regular triangulation $\frac{1}{2}\mathcal{C}$ of \mathbf{R}^2 and where $\widehat{\varphi}$ on the generic fibres is multiplication by 2. The fundamental lattice is still the unit square, but it is now divided up into 16 squares. We number the vertices by $Q_1 = (0, 0), \dots, Q_{25} = (1, 1)$ as in 7.6. See Figure 2. Similarly as in Proposition 7.7, we can show that \mathfrak{A}' is algebraic. Indeed, the same proof works using the convex function $f_0 \circ [2]$. We denote the strictly semistable algebraic K° -model by \mathcal{A}' and let $\varphi : \mathcal{A}' \rightarrow \mathcal{A}$ be the underlying algebraic morphism.

Let D'_j (resp. Y'_j) be the Cartier divisor (resp. the irreducible component of \mathcal{A}'_s) associated to Q_j . Since we may use $\pi = q^{1/4}$ for the strictly semistable model \mathcal{A}' , the Weil divisor associated to D'_j is

$$(7.11.1) \quad \operatorname{cyc}(D'_j) = v(q^{1/4})Y'_j = \frac{1}{4}Y'_j.$$

FIGURE 2. The triangulation $\frac{1}{2}\mathcal{C}$ of 7.11.

Here and in the following, we use the refined intersection theory with Cartier divisors which we recall in Appendix A. It follows easily from the definition of the Cartier divisors D_i and D'_j that

$$(7.11.2) \quad \varphi^*(D_5) = 2(D'_7 + D'_9 + D'_{17} + D'_{19}).$$

Let T'_{17} be the stratum of \mathcal{A}' corresponding to the edge between Q_1 and Q_7 . Note that φ induces a surjective morphism from the orbit T'_{17} onto the orbit T_{15} of degree 2 (see [Gub07, Proposition 6.4]). We deduce that

$$(7.11.3) \quad \varphi_*(\overline{T'_{17}}) = 2\overline{T_{15}}.$$

The projection formula in Proposition A.12, (7.11.2) and (7.11.3) show that

$$D_5 \cdot \overline{T_{15}} = (D'_7 + D'_9 + D'_{17} + D'_{19}) \cdot \overline{T'_{17}}.$$

The combinatorial nature of the triangulation shows that D'_9, D'_{17}, D'_{19} do not intersect $\overline{T'_{17}}$; since we deal with normal crossing divisors we have $\overline{T'_{17}} = D'_7 \cdot Y'_1$, so

$$D_5 \cdot \overline{T_{15}} = D'_7 \cdot \overline{T'_{17}} = D'_7 \cdot D'_7 \cdot Y'_1 = 4D'_7 \cdot D'_7 \cdot D'_1,$$

where the last equality is from (7.11.1). Using Kolb's relation (b) in Proposition A.17, we have

$$D'_7 \cdot D'_7 \cdot D'_1 = -D'_7 \cdot (D'_2 + D'_{12} + D'_{17}) \cdot D'_1 = -D'_7 \cdot D'_2 \cdot D'_1,$$

where we have used again the combinatorial nature of the triangulation in the last step. By (7.11.1) we have $-4D'_7 \cdot D'_2 \cdot D'_1 = -D'_7 \cdot D'_2 \cdot Y'_1$, so

$$(7.11.4) \quad D_5 \cdot \overline{T_{15}} = -D'_7 \cdot D'_2 \cdot Y'_1 = -1,$$

where in the last step we have used that $D'_2 + D'_7$ restricts to a normal crossing divisor on Y'_1 .

Conclusion: We have shown that $-m(\operatorname{div}(F), e_{15}) = 1 = m(\tau(f), e_{15})$.

Remark 7.12. Equation (7.11.4) says that $\alpha(P_5, e_{15}) = 1$; by Lemma 6.5.1 we have $\alpha(P_1, e_{15}) = 1$ as well. Therefore the “weighted midpoint”

$$m = \frac{1}{2}(P_1 + P_5)$$

of (6.7.4) makes sense, and is equal to the midpoint of e_{15} . We have $F(P_1) = 0$ and $F(P_5) = \frac{1}{2}$, so $F(m) = \frac{1}{4}$. Since $v(\pi) = \frac{1}{2}$ and $F(P_2) = F(P_4) = 0$, we have

$$\begin{aligned} \operatorname{slope}(F; e_{15}, \Delta_{125}) &= \frac{1}{v(\pi)}(F(P_2) - F(m)) = -2 \cdot \frac{1}{4} = -\frac{1}{2} \\ \operatorname{slope}(F; e_{15}, \Delta_{145}) &= \frac{1}{v(\pi)}(F(P_4) - F(m)) = -2 \cdot \frac{1}{4} = -\frac{1}{2}. \end{aligned}$$

Thus we have

$$m(\operatorname{div}(F), e_{15}) = \operatorname{slope}(F; e_{15}, \Delta_{125}) + \operatorname{slope}(F; e_{15}, \Delta_{145}) = -\frac{1}{2} - \frac{1}{2} = -1,$$

as above. Notice that the slopes are not integers in this case.

8. THE STURMFELS–TEVELEV FORMULA

The original Sturmfels–Tevelev multiplicity formula relates tropical multiplicities of maximal cones of tropicalizations of closed subvarieties of tori under a torus homomorphism. It is proved in [ST08, Theorem 1.1] for fields with a trivial valuation and in [BPR11, Corollary 8.4] in general. A “skeletal” variant was proved for a smooth curve embedded as a closed subscheme of torus in [BPR11, Corollary 6.9]. In the special case of a trivially valued field in characteristic 0, a higher dimensional variant was also proved by Cueto [Cue12, Theorem 2.5].

In this section we will prove a generalization of the “skeletal” variant which works in any dimension and also for varieties equipped with a map to a torus which is generically finite onto its image, but not necessarily a closed immersion. As our formula is formally very similar to the ones mentioned above, we also call it a Sturmfels–Tevelev multiplicity formula.

8.1. We fix a strictly semistable pair (\mathcal{X}, H) . Let $T = \text{Spec}(K[M])$ be an algebraic torus, let $N = \text{Hom}(M, \mathbf{Z})$ be the group of one-parameter subgroups of T , and let $\text{trop} : T^{\text{an}} \rightarrow N_{\mathbf{R}}$ be the tropicalization map, as in 2.3. Let $U = X \setminus \text{supp}(H)$ and let $\varphi : U \rightarrow T$ be a morphism. Let $U' \subset T$ be the schematic image of φ .

Proposition 8.2. *The map $\text{trop} \circ \varphi : U^{\text{an}} \rightarrow N_{\mathbf{R}}$ factors through the retraction $\tau : U^{\text{an}} \rightarrow S(\mathcal{X}, H)$, and the restriction of $\text{trop} \circ \varphi$ to any canonical polyhedron of $S(\mathcal{X}, H)$ is an integral Γ -affine map. Moreover, $\text{Trop}(U') = \text{trop} \circ \varphi(S(\mathcal{X}, H))$.*

Proof. Choosing a basis for $M \cong \mathbf{Z}^n$, we may write φ as a tuple $(\varphi_1, \dots, \varphi_n) : U \rightarrow \mathbf{G}_{m,K}^n$; we may regard each φ_i as a non-zero rational function on X such that $\text{supp}(\text{div}(\varphi_i)) \subset \text{supp}(H)_\eta$. The first assertions follow by applying Proposition 5.2 to each φ_i . The difficulty in the final assertion is that the map $\varphi : U \rightarrow U'$ needs not be surjective, but it follows from [Gub13a, Lemma 4.9] that $\text{Trop}(U') = \text{trop}(\varphi(U^{\text{an}}))$. We conclude that $\text{Trop}(U') = \text{trop} \circ \varphi(\tau(U^{\text{an}})) = \text{trop} \circ \varphi(S(\mathcal{X}, H))$. ■

8.3. Suppose now that $\varphi : U \rightarrow U'$ is generically finite, so $d := \dim(X) = \dim(U')$. We denote the degree of this map by $[U : U']$. As explained in 2.3, $\text{Trop}(U')$ is the support of an integral Γ -affine polyhedral complex Σ_1 of pure dimension d . Recall that for every maximal (i.e. d -dimensional) polyhedron $\Delta \in \Sigma_1$ we have defined a tropical multiplicity $m_{\text{Trop}}(\Delta) \in \mathbf{N} \setminus \{0\}$.

Let $\varphi_{\text{aff}} : S(\mathcal{X}, H) \rightarrow \text{Trop}(U')$ denote the restriction of $\text{trop} \circ \varphi$ to $S(\mathcal{X}, H)$. This is an integral Γ -affine map on each canonical polyhedron of $S(\mathcal{X}, H)$. Consider a polyhedron $\Delta \in \Sigma_1$ of dimension $d = \dim(X)$. Choose a Γ -rational point $\omega \in \text{relint}(\Delta)$ not contained in a polyhedron $\varphi_{\text{aff}}(\Delta_S)$ of dimension $< d$ for any canonical polyhedron Δ_S of $S(\mathcal{X}, H)$. Such points are dense in $\text{relint}(\Delta)$. Clearly $\varphi_{\text{aff}}^{-1}(\omega)$ is finite, with each point contained in (the relative interior of) a unique canonical polyhedron Δ_S of dimension d . Let Δ_S be such a canonical polyhedron. The image of Δ_S under φ_{aff} is contained in the affine span of Δ , so we get a lattice index $[N_\Delta : N_{\Delta_S}]$ as in 2.2.

Theorem 8.4. (Skeletal Sturmfels–Tevelev multiplicity formula) *Using the notations and hypotheses above, we have the identity*

$$[U : U'] m_{\text{Trop}}(\Delta) = \sum_{\Delta_S} [N_\Delta : N_{\Delta_S}],$$

where the sum ranges over all canonical polyhedra Δ_S of the skeleton $S(\mathcal{X}, H)$ with $\text{relint}(\Delta_S) \cap \varphi_{\text{aff}}^{-1}(\omega) \neq \emptyset$.

Proof. The proof is similar to the proof of Theorem 8.2 in [BPR11]. To simplify the notation we set $\text{trop}_\varphi := \text{trop} \circ \varphi : U^{\text{an}} \rightarrow N_{\mathbf{R}}$. Consider the affinoid space $U'_\omega := \text{trop}^{-1}(\omega) \cap U'^{\text{an}}$. By Proposition 8.2 the map trop_φ factors through the retraction to the skeleton, so $X_\omega := \varphi^{-1}(U'_\omega) = \text{trop}_\varphi^{-1}(\omega)$ is the finite disjoint union of the analytic domains $X_{\omega'} = \tau^{-1}(\omega')$ with ω' ranging over the finite set $\varphi_{\text{aff}}^{-1}(\omega)$. By Corollary 4.16 each $X_{\omega'}$ is affinoid, hence X_ω is affinoid.

We claim that $\varphi : X_\omega \rightarrow U'_\omega$ is finite. By [Ber90, Corollary 2.5.13], it suffices to show that the boundary $\partial(X_\omega/U'_\omega)$ is empty. We have $\partial(U^{\text{an}}/U'^{\text{an}}) = \emptyset$ because the analytification of any scheme is boundaryless [Ber90, Theorem 3.4.1] (or *closed* in Berkovich’s terminology); therefore $\partial(X_\omega/U'_\omega) = \emptyset$ by pullback [Ber90, Proposition 3.1.3]. This proves that $\varphi : X_\omega \rightarrow U'_\omega$ is finite, so the induced morphism of canonical models $\varphi_\omega : \mathfrak{X}_\omega \rightarrow \mathfrak{U}'_\omega$ is finite by [BPR11, Theorem 3.17 and Proposition 3.13].

The canonical model \mathfrak{X}_ω of X_ω is the disjoint union of the canonical models $\mathfrak{X}_{\omega'}$ of $X_{\omega'}$ for $\omega' \in \varphi_{\text{aff}}^{-1}(\omega)$. By Corollary 4.16, the special fibre of $\mathfrak{X}_{\omega'}$ is isomorphic to $\mathbf{G}_{m, \tilde{K}}^d$. By the projection formula [Gub98, Proposition 4.5] applied to the Cartier divisor $\text{div}(\nu)$ on \mathfrak{U}'_ω for some non-zero ν in the maximal ideal $K^{\circ\circ}$ of the valuation ring of K , we get

$$(8.4.1) \quad \deg(\varphi_\omega) = \sum_{\omega' \rightarrow Y} [(\mathfrak{X}_{\omega'})_s : Y]$$

for every irreducible component Y of the special fibre of \mathfrak{U}'_ω , where ω' ranges over all elements in $\varphi_{\text{aff}}^{-1}(\omega)$ with $\varphi((\mathfrak{X}_{\omega'})_s) = Y$. By [BPR11, Lemma 8.3], since $X_\omega \rightarrow U'_\omega$ is finite we have $\deg(\varphi_\omega) = [U : U']$.

Let \mathfrak{U}^{ω} be the polyhedral formal model of U'_ω as in [BPR11, Definition 4.14]. It is the closure of U'_ω in the canonical model of $\text{trop}^{-1}(\omega)$. Its special fibre is $\text{in}_\omega(U')$. We have a canonical finite surjective morphism $\mathfrak{U}'_\omega \rightarrow \mathfrak{U}^{\omega}$ which is an isomorphism on generic fibres (see [BPR11, Corollary 3.16]). Since the special fibre of \mathfrak{U}'_ω is reduced, for each irreducible component Z of $\text{in}_\omega(U')$ we have

$$(8.4.2) \quad \sum_{Y \rightarrow Z} [Y : Z] = m_Z(\text{in}_\omega(U')),$$

where the sum runs over all irreducible components Y of $(\mathfrak{U}'_\omega)_s$ mapping onto Z , and $m_Z(\text{in}_\omega(U'))$ is the multiplicity of Z in $\text{in}_\omega(U')$. This follows again from the projection formula; see [BPR11, 3.34(2)].

Composing $\varphi_\omega : \mathfrak{X}_\omega \rightarrow \mathfrak{U}'_\omega$ with $\mathfrak{U}'_\omega \rightarrow \mathfrak{U}^{\omega}$ gives a finite surjective morphism $\mathfrak{X}_\omega \rightarrow \mathfrak{U}^{\omega}$. As explained in 2.3, (the reduced scheme underlying) an irreducible component Z of $\text{in}_\omega(U')$ is isomorphic to the multiplicative torus of rank d over \tilde{K} . As the same is true for $(\mathfrak{X}_{\omega'})_s$ for $\omega' \in \varphi_{\text{aff}}^{-1}(\omega)$, one checks as in the proof of [BPR11, Corollary 8.4] that

$$(8.4.3) \quad [(\mathfrak{X}_{\omega'})_s : Z] = [N_\Delta : N_{\Delta_{S(\omega')}}]$$

for every ω' with $(\mathfrak{X}_{\omega'})_s$ lying over Z , where $\Delta_{S(\omega')}$ is the unique canonical polyhedron of $S(\mathcal{X}, H)$ with $\omega' \in \text{relint}(\Delta_{S(\omega')})$.

Recall that $m_{\text{Trop}}(\Delta)$ is the number of irreducible components of $\text{in}_\omega(U')$ counted with multiplicities. Since $\deg(\varphi_\omega) = [U : U']$ we have

$$(8.4.4) \quad [U : U'] m_{\text{Trop}}(\Delta) = \sum_Z \deg(\varphi_\omega) m_Z(\text{in}_\omega(U')),$$

where Z runs over all irreducible components of $\text{in}_\omega(U')$. Combining this with (8.4.1), (8.4.2), and (8.4.3) lead to

$$[U : U'] m_{\text{Trop}}(\Delta) = \sum_Z \sum_{Y \rightarrow Z} \sum_{\omega' \rightarrow Y} [(\mathfrak{X}_{\omega'})_s : Y] [Y : Z] = \sum_{\omega'} [N_\Delta : N_{\Delta_{S(\omega')}}],$$

where Y ranges over all irreducible components of $(\mathfrak{U}'_\omega)_s$ lying over Z and where ω' ranges over all elements in $\varphi_{\text{aff}}^{-1}(\omega)$ with $\varphi((\mathfrak{X}_{\omega'})_s) = Y$. Since $\omega' \mapsto \Delta_{S(\omega')}$ is a bijection from $\varphi_{\text{aff}}^{-1}(\omega)$ onto the set of canonical polyhedra Δ_S of $S(\mathcal{X}, H)$ with $\text{relint}(\Delta_S) \cap \varphi_{\text{aff}}^{-1}(\omega) \neq \emptyset$, we get the claim. \blacksquare

Remark 8.5. It follows from the considerations in 8.3 that when φ is generically finite onto its image then $S(\mathcal{X}, H)$ necessarily has dimension $d = \dim(X)$. This non-trivial condition on the strictly semistable pair (\mathcal{X}, H) is not obvious from the definitions. As $\text{Trop}(U')$ has pure dimension d and is connected in codimension one, one might wonder if there exist natural additional conditions on φ which guarantee that $S(\mathcal{X}, H)$ has the same properties.

9. FAITHFUL TROPICALIZATION

In this section we fix a strictly semistable pair (\mathcal{X}, H) . As always we use the associated notation 3.3. For $n \geq 0$ we let $\text{trop} : \mathbf{G}_{m, K}^{n, \text{an}} = \text{Spec}(K[x_1^{\pm 1}, \dots, x_n^{\pm n}])^{\text{an}} \rightarrow \mathbf{R}^n$ denote the tropicalization map as in 2.3, defined by

$$\text{trop}(p) = (-\log |x_1(p)|, \dots, -\log |x_n(p)|).$$

Our goal is to prove that there is a rational map φ from $X = \mathcal{X}_\eta$ to a torus $T \cong \mathbf{G}_{m, K}^n$ which takes $S(\mathcal{X}, H)$ isomorphically onto its image. Note that a rational map φ is always defined on

$S(\mathcal{X}, H)$ since the points of $S(\mathcal{X}, H)$ are norms on the function field of X : see Remark 4.7. By “isomorphically” we mean that we want φ to be injective on $S(\mathcal{X}, H)$ and we want it to preserve the integral affine structure. Roughly, in this situation one “sees” the entire skeleton in the tropicalization of (the image of) X ; this is an important compatibility between the intrinsic and embedded polyhedral structures of X . (See however Remark 9.6.)

We start with the following basic property which is an immediate consequence of Proposition 5.6.

Proposition 9.1. *Let $f = (f_1, \dots, f_n) : X \dashrightarrow \mathbf{G}_{m,K}^n$ be a rational map. Then $S(\mathcal{X}, H)$ can be covered by finitely many integral Γ -affine polyhedra Δ such that $\text{trop} \circ f|_{\Delta}$ is an integral Γ -affine map.*

Definition 9.2. A rational map $f = (f_1, \dots, f_n) : X \dashrightarrow \mathbf{G}_{m,K}^n$ is said to be *unimodular* on a canonical polyhedron Δ_S of $S(\mathcal{X}, H)$ provided that Δ_S can be covered by finitely many integral Γ -affine polyhedra Δ such that $\text{trop} \circ f|_{\Delta}$ is a unimodular integral Γ -affine map on Δ (see 2.2). We call f *unimodular on $S(\mathcal{X}, H)$* if f is unimodular on any canonical polyhedron of $S(\mathcal{X}, H)$. We say that f is a *faithful tropicalization* of $S(\mathcal{X}, H)$ if f is unimodular and $\text{trop} \circ f$ is injective on $S(\mathcal{X}, H)$.

The next lemma is essentially [BPR11, Lemma 6.17].

Lemma 9.3. *Let f_1, \dots, f_n, g be non-zero rational functions on X , and suppose that the rational map $f = (f_1, \dots, f_n) : X \dashrightarrow \mathbf{G}_{m,K}^n$ is unimodular on the canonical polyhedron Δ_S of $S(\mathcal{X}, H)$. Then $(f_1, \dots, f_n, g) : X \dashrightarrow \mathbf{G}_{m,K}^{n+1}$ is also unimodular on Δ_S .*

Proof. It follows from Proposition 5.6 that the skeleton $S(\mathcal{X}, H)$ has a covering by finitely many integral Γ -affine polyhedra Δ such that $\text{trop} \circ h|_{\Delta}$ is an integral Γ -affine map $\Delta \rightarrow \mathbf{R}^{n+1}$ for $h := (f_1, \dots, f_n, g)$. Since $\text{trop} \circ h|_{\Delta}$ factors through $\text{trop} \circ f|_{\Delta}$, transitivity of lattice indices shows easily that h is unimodular. \blacksquare

Proposition 9.4. *For every canonical polyhedron $\Delta_S \subset S(\mathcal{X}, H)$, there exist non-zero rational functions $f_1, \dots, f_n \in K(X)$ such that $\text{trop} \circ (f_1, \dots, f_n)|_{\Delta_S} : \Delta_S \rightarrow \mathbf{R}^n$ is a unimodular integral Γ -affine map (and therefore injective). In particular, $f = (f_1, \dots, f_n)$ is unimodular on Δ_S .*

Proof. Let S be the corresponding vertical stratum of D . Every point of S has a neighbourhood \mathcal{U} that admits an étale morphism $\psi : \mathcal{U} \rightarrow \mathcal{S} = \text{Spec}(K^\circ[x_0, \dots, x_d]/\langle x_0 \cdots x_r - \pi \rangle)$ as in (3.1.1). We can shrink \mathcal{U} so that $\mathfrak{U} = \widehat{\mathcal{U}}$ is a building block with distinguished stratum S . In particular, S is defined by $\psi^*(x_0) = \cdots = \psi^*(x_{r+s}) = 0$. The canonical polyhedron Δ_S of the skeleton $S(\mathcal{X}, H)$ is contained in $\mathfrak{U}_\eta \subset \mathcal{U}^{\text{an}}$, and

$$\text{Val}_S(p) = (-\log |\psi^* x_0(p)|, \dots, -\log |\psi^* x_{r+s}(p)|)$$

maps Δ_S homeomorphically onto $\Delta(r, \pi) \times \mathbf{R}_+^s$ by 4.3. In fact, the structure of integral Γ -affine polyhedron on Δ_S is defined by the map Val_S , so $\text{Val}_S|_{\Delta_S}$ is by definition a unimodular integral Γ -affine map. Interpreting $\psi^*(x_0), \dots, \psi^*(x_{r+s})$ as rational functions on X and Val_S as the composition of $(\psi^*(x_0), \dots, \psi^*(x_{r+s})) : X \dashrightarrow \mathbf{G}_{m,K}^{r+s+1}$ with $\text{trop} : \mathbf{G}_{m,K}^{r+s+1, \text{an}} \rightarrow \mathbf{R}^{r+s+1}$, we obtain the claim. \blacksquare

Theorem 9.5. *Let (\mathcal{X}, H) be a strictly semistable pair. Then there exists a finite collection f_1, \dots, f_n of non-zero rational functions on X such that the associated rational map $X \dashrightarrow \mathbf{G}_{m,K}^n$ is a faithful tropicalization of $S(\mathcal{X}, H)$.*

Proof. By Proposition 9.4 and Lemma 9.3, we can find $f_1, \dots, f_n \in K(X)^\times$ such that the rational map $f = (f_1, \dots, f_n)$ is unimodular on every canonical polyhedron Δ_S : indeed, we may take any collection (f_1, \dots, f_n) which includes all rational functions from Proposition 9.4 for each Δ_S . By construction, $\text{trop} \circ f$ is injective on every Δ_S . It remains to enlarge the collection (f_1, \dots, f_n) so that $\text{trop} \circ f$ is injective on $S(\mathcal{X}, H)$.

By Chow’s lemma [EGAII, Theorem 5.6.1], there is a birational surjective morphism $\varphi : \mathcal{X}' \rightarrow \mathcal{X}$ for a projective variety \mathcal{X}' over K° . There are open dense subsets \mathcal{U} of \mathcal{X} and \mathcal{U}' of \mathcal{X}' such that φ restricts to an isomorphism $\mathcal{U}' \rightarrow \mathcal{U}$. For simplicity, we use this to identify \mathcal{U} with \mathcal{U}' and hence we have an identification $K(X) = K(X')$ of the function fields of the generic fibres

X, X' of \mathcal{X} and \mathcal{X}' . Since any element of the skeleton is an Abhyankar point (see Remark 4.7), we have $S(\mathcal{X}, H) \subset \mathcal{U}_\eta^{\text{an}} = \mathcal{U}'_\eta^{\text{an}}$.

We have seen in 3.15 that a strictly semistable pair has a canonical stratification $\text{str}(\mathcal{X}_s, H)$ of the special fibre \mathcal{X}_s . The preimage $\varphi^{-1}(S)$ of $S \in \text{str}(\mathcal{X}_s, H)$ is not necessarily irreducible, but it contains only finitely many generic points. Let \mathcal{F} be the collection of all such generic points for all strata S . Note that \mathcal{F} is a finite subset of \mathcal{X}' . Since \mathcal{X}' is projective, any two points of \mathcal{X}' are contained in a common affine open subset. Using that \mathcal{X}' is quasicompact, we conclude that for every $\zeta' \in \mathcal{F}$, there are finitely many affine open subsets $\mathcal{U}'_{\zeta', j}$ containing ζ' and covering \mathcal{X}' . On every such $\mathcal{U}'_{\zeta', j}$, there are finitely many regular functions $f_{\zeta', jk} \in \mathcal{O}(\mathcal{U}'_{\zeta', j})$ whose reductions to the special fibre have zero set $\overline{\zeta'} \cap (\mathcal{U}'_{\zeta', j})_s$. This means that for every $x' \in (X')^{\text{an}}$ with reduction $\text{red}_{\mathcal{X}'}(x') \in (\mathcal{U}'_{\zeta', j})_s$, we have $|f_{\zeta', jk}(x')| = 1$ for some k if $\text{red}_{\mathcal{X}'}(x') \notin \overline{\zeta'}$ and $|f_{\zeta', jk}(x')| < 1$ for all k if $\text{red}_{\mathcal{X}'}(x') \in \overline{\zeta'}$. Note that we have $|f_{\zeta', jk'}(x')| \leq 1$ for all k' .

These finitely many functions $f_{\zeta', jk}$ may be viewed as rational functions on X and we add them to the collection (f_1, \dots, f_n) considered at the beginning. We claim that the resulting tropicalization is faithful. As remarked above, it is enough to show that this tropicalization is injective. We consider points $x \neq y$ from $S(\mathcal{X}, H)$ and we have to look for a function from our extended collection such that the absolute value of the function separates x and y . By Corollary 4.11, x is contained in the relative interior of a unique canonical polyhedron Δ_S and the reduction $\text{red}_{\mathcal{X}}(x)$ is the generic point ζ_S of the vertical stratum $S \in \text{str}(\mathcal{X}_s, H)$. A similar statement holds for y and we denote the corresponding vertical stratum by T and reduction by ζ_T .

If $T \subset \overline{S}$, then Δ_S is a closed face of Δ_T by 4.6. We conclude that $x, y \in \Delta_T$ and $x = y$ follows from injectivity of $\text{trop} \circ f$ on Δ_T . So we may assume that $T \not\subset \overline{S}$ and hence $\zeta_T \notin \overline{S}$. Using that $x, y \in S(\mathcal{X}, H) \subset \mathcal{U}_\eta^{\text{an}} = \mathcal{U}'_\eta^{\text{an}}$, we may view x and y as points of $(X')^{\text{an}}$. Note that

$$\varphi \circ \text{red}_{\mathcal{X}'}(x) = \text{red}_{\mathcal{X}}(x) = \zeta_S \in S.$$

There is a generic point ζ' of $\varphi^{-1}(S)$ with $\text{red}_{\mathcal{X}'}(x) \in \overline{\zeta'}$. By construction, there is a j such that $\text{red}_{\mathcal{X}'}(x) \in \mathcal{U}'_{\zeta', j}$. From $\zeta_S = \varphi \circ \text{red}_{\mathcal{X}'}(x)$, we deduce that $\zeta_S = \varphi \circ \text{red}_{\mathcal{X}'}(\zeta')$. Similarly, we show

$$\varphi \circ \text{red}_{\mathcal{X}'}(y) = \text{red}_{\mathcal{X}}(y) = \zeta_T \notin \overline{S}$$

and hence $\text{red}_{\mathcal{X}'}(y) \notin \overline{\zeta'}$. We conclude from the above considerations that $|f_{\zeta', jk}(y)| = 1$ for some k . Using $\text{red}_{\mathcal{X}'}(x) \in \overline{\zeta'}$, we have $|f_{\zeta', jk}(x)| < 1$. This proves the claim. \blacksquare

Remark 9.6. In the statement of Theorem 9.5 it is not assumed that the divisor of each f_i has support contained in $\text{supp}(H)$. In particular, the tropicalization map will not generally factor through retraction to the skeleton, so its image in \mathbf{R}^n may be much larger than the image of $S(\mathcal{X}, H)$.

Example 9.7. When $\dim(X) = 1$ the skeleton $S(\mathcal{X}, H)$ is a metric graph. Theorem 9.5 says that there exists a rational map φ from X to a torus whose restriction to $S(\mathcal{X}, H)$ is an isometry onto its image, where the metric on the image is defined by the lattice length. Increasing the dimension of the torus, we may even assume that φ is a closed embedding on an open subscheme. Therefore this extends the faithful tropicalization result of [BPR11, Theorem 6.22], as well as considerably simplifying the proof (especially since our Chow's lemma argument is not needed).

10. SECTIONS OF TROPICALIZATIONS

10.1. Let $\varphi : U \hookrightarrow T$ be a closed immersion of a very affine variety U into an algebraic K -torus $T = \text{Spec}(K[M])$. As U is a variety, it is an integral scheme. We set $\text{trop}_\varphi = \text{trop} \circ \varphi^{\text{an}} : U^{\text{an}} \rightarrow N_{\mathbf{R}}$, where $N = \text{Hom}(M, \mathbf{Z})$, so $\text{Trop}(U) = \text{trop}_\varphi(U^{\text{an}})$ is the corresponding tropicalization as defined in section 2.3. Fixing a basis of the character group M of T , we identify T with the torus $\text{Spec}(K[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ and $M_{\mathbf{R}}, N_{\mathbf{R}}$ with \mathbf{R}^n . Let \mathfrak{a} be the ideal defining U as a closed subscheme of T . Fix $\omega = (\omega_1, \dots, \omega_n) \in \text{Trop}(U)$, and put $r_i = \exp(-\omega_i) \in \mathbf{R}$. Then U_ω is the spectrum of the affinoid algebra

$$\mathcal{A}_\omega = K\langle r^{-1}x, rx^{-1} \rangle / \mathfrak{a}K\langle r^{-1}x, rx^{-1} \rangle,$$

which carries the residue norm

$$(10.1.1) \quad \|f\|_{\text{res}} = \inf_{\pi_\omega(g)=f} \|g\|_r = \min_{\pi_\omega(g)=f} \|g\|_r,$$

where $\pi_\omega : K\langle r^{-1}x, rx^{-1} \rangle \rightarrow \mathcal{A}_\omega$ is the quotient map and $\|\cdot\|_r$ denotes the spectral norm on $K\langle r^{-1}x, rx^{-1} \rangle$, i.e.

$$\left\| \sum_{I \in \mathbb{Z}^n} a_I x^I \right\|_r = \max_I \{|a_I| r^I\}.$$

When $\omega \in N_\Gamma$, then \mathcal{A}_ω is strictly affinoid, so the infimum in (10.1.1) is a minimum by [BGR84, Corollary 5.2.7/8]. In general, we can use the base change trick from [Ber90, Proof of Proposition 2.1.3] to reduce to the strictly affinoid case.

10.2. Now let $\omega \in \text{Trop}(U) \cap N_\Gamma$. Then the analytic domain $U_\omega = \text{trop}_\varphi^{-1}(\omega) = \mathcal{M}(\mathcal{A}_\omega)$ is strictly affinoid. Let \mathfrak{U}^ω be the *polyhedral formal model* of U_ω defined by

$$\mathfrak{U}^\omega = \text{Spf} \left(K\langle r^{-1}x, rx^{-1} \rangle^\circ / (\mathfrak{a}K\langle r^{-1}x, rx^{-1} \rangle \cap K\langle r^{-1}x, rx^{-1} \rangle^\circ) \right).$$

It is an admissible formal scheme over K° with special fibre equal to the initial degeneration $\text{in}_\omega(U)$ (see [BPR11, Proposition 4.17]). Let $\mathfrak{U}_\omega^{\text{can}} = \text{Spf}(\mathcal{A}_\omega^\circ)$ be the *canonical model* of the affinoid space U_ω , as defined in 2.1; its special fibre is the *canonical reduction* of U_ω . As noted in the remark after [BPR11, Proposition 4.17], we have a canonical finite morphism $\mathfrak{U}_\omega^{\text{can}} \rightarrow \mathfrak{U}^\omega$, which is the identity on generic fibres.

By [Ber90, Proposition 2.4.4(iii)] and its proof, the supremum seminorm $\|\cdot\|_{\text{sup}}$ on \mathcal{A}_ω is equal to the maximum of the norms contained in the Shilov boundary of U_ω , which correspond to the irreducible components of the canonical reduction $\text{Spec}(\mathcal{A}_\omega^\circ \otimes_{K^\circ} \tilde{K})$ of U_ω . The canonical reduction is always a reduced scheme. Therefore, the following are equivalent:

- (1) U_ω has a unique Shilov boundary point;
- (2) the canonical reduction of U_ω is an integral scheme; and
- (3) the supremum seminorm $\|\cdot\|_{\text{sup}}$ on \mathcal{A}_ω is multiplicative.

In particular, when the above conditions hold then the supremum seminorm is the unique Shilov boundary point of U_ω . We will leverage this fact to construct a section of trop_φ on the locus $\{\omega \in \text{Trop}(U) : m_{\text{Trop}}(\omega) = 1\}$.

Lemma 10.3. *Let $\varphi : U \hookrightarrow T$ be a closed immersion of a very affine variety U into an algebraic K -torus $T = \text{Spec}(K[M])$. For $\omega \in \text{Trop}(U) \cap N_\Gamma$, if $m_{\text{Trop}}(\omega) = 1$ then $U_\omega = \text{trop}_\varphi^{-1}(\omega)$ contains a unique Shilov boundary point.*

Proof. As noted above, we have a canonical finite morphism $\mathfrak{U}_\omega^{\text{can}} \rightarrow \mathfrak{U}^\omega$, which is the identity on generic fibres. By [BPR11, Corollary 3.16], the canonical morphism induces a surjective finite map between the equidimensional special fibres $(\mathfrak{U}_\omega^{\text{can}})_s = \text{Spec}(\mathcal{A}_\omega^\circ \otimes_{K^\circ} \tilde{K})$ and $\text{in}_\omega(U)$. By hypothesis, $\text{in}_\omega(U)$ is irreducible and generically reduced. Therefore, by the projection formula [BPR11, 3.34(2)], there is a unique irreducible component of $(\mathfrak{U}_\omega^{\text{can}})_s$, and it maps birationally onto the reduction of $\text{in}_\omega(U)$. We conclude that $(\mathfrak{U}_\omega^{\text{can}})_s$ is an integral scheme and the equivalence of (1) and (2) above shows the claim. \blacksquare

Remark 10.4. In Lemma 10.3 it is necessary to assume that U is irreducible, or at least equidimensional. As a counterexample, let U be the closed subscheme of $T = \text{Spec}(K[x_1^{\pm 1}, x_2^{\pm 1}])$ given by the ideal $\mathfrak{a} = \langle x_1 - 1, x_2 - 1 \rangle \cap \langle x_1 - 1 - \varpi \rangle$ for $\varpi \in K^\times$ with $|\varpi| < 1$. Then U is the disjoint union of the line $\{x_1 = 1 + \varpi\}$ with the point $(1, 1)$. The initial degeneration at $\omega = 0$ is defined by the ideal $\text{in}_\omega(\mathfrak{a}) = \langle (x_1 - 1)^2, (x_1 - 1)(x_2 - 1) \rangle$ over \tilde{K} . This is a generically reduced line with an associated point at $(1, 1)$. It has tropical multiplicity 1, but the canonical reduction is the disjoint union of a point and a line, so that U_ω has two Shilov boundary points.

Proposition 10.5. *For $\omega \in \text{Trop}(U) \cap N_\Gamma$, the following are equivalent:*

- (a) $\|\cdot\|_{\text{res}} = \|\cdot\|_{\text{sup}}$;
- (b) $\text{in}_\omega(U)$ is a reduced scheme.

If we assume additionally that $m_{\text{Trop}}(\omega) = 1$, i.e. if $\text{in}_\omega(U)$ is irreducible and generically reduced, then (a) and (b) are also equivalent to:

- (c) *the unique Shilov boundary point of U_ω is equal to $\|\cdot\|_{\text{res}}$;*

- (d) $\| \cdot \|_{\text{res}}$ is multiplicative;
- (e) $\text{in}_\omega(U)$ is an integral scheme;

Proof. The polyhedral formal model \mathfrak{U}^ω is $\text{Spf}(B_\omega)$ for $B_\omega = \pi_\omega(K\langle r^{-1}x, rx^{-1} \rangle^\circ) \subset \mathcal{A}_\omega^\circ$. For $f \in \mathcal{A}_\omega$ we have $f \in B_\omega$ if and only if there exists $g \in K\langle r^{-1}x, rx^{-1} \rangle^\circ$ such that $\pi_\omega(g) = f$, which is true if and only if $\|f\|_{\text{res}} \leq 1$. Since $\|f\|_{\text{sup}}$ and $\|f\|_{\text{res}}$ are in $|K^\times|$ in any case, we have $\| \cdot \|_{\text{res}} = \| \cdot \|_{\text{sup}}$ if and only if $\mathcal{A}_\omega^\circ = B_\omega$. Now the equivalence of (a) and (b) follows from [BPR11, Proposition 3.18].

If $m_{\text{Trop}}(\omega) = 1$, then Lemma 10.3 shows that U_ω has a unique Shilov boundary point. It follows from 10.2 that $\| \cdot \|_{\text{sup}}$ is multiplicative and equal to the unique Shilov boundary point. This proves the equivalence of (a), (c) and (d). Since $\text{in}_\omega(U)$ is irreducible, the equivalence of (b) and (e) is obvious. \blacksquare

The next result was proved for compact subsets of curves in [BPR11, Theorem 6.24]. We prove it here in a very general situation.

Theorem 10.6. *Let $\varphi : U \hookrightarrow T$ be a closed immersion of a very affine (integral) variety U into an algebraic K -torus $T = \text{Spec}(K[M])$, and let $\text{trop}_\varphi = \text{trop} \circ \varphi^{\text{an}} : U^{\text{an}} \rightarrow N_{\mathbf{R}}$. Let $Z \subset \text{Trop}(U)$ be a subset such that $m_{\text{Trop}}(\omega) = 1$ for all $\omega \in Z$. Then for every $\omega \in Z$, the affinoid space $U_\omega = \text{trop}_\varphi^{-1}(\omega)$ has a unique Shilov boundary point $s(\omega)$, and $\omega \mapsto s(\omega)$ defines a continuous partial section $s : Z \rightarrow U^{\text{an}}$ of the tropicalization map $\text{trop}_\varphi : U^{\text{an}} \rightarrow \text{Trop}(U)$ on the subset Z . Moreover, if Z is contained in the closure of its interior in $\text{Trop}(U)$, then s is the unique continuous section of trop_φ defined on Z .*

Proof. First we prove that $\text{trop}_\varphi^{-1}(\omega)$ has a unique Shilov boundary point for all $\omega \in Z$. When the valuation map $v : K^\times \rightarrow \mathbf{R}$ is surjective this follows from Lemma 10.3. In the general case let L be a non-archimedean extension field of K such that the valuation map $L^\times \rightarrow \mathbf{R}$ is surjective. Let U_L denote the base change of U to L , and let $p : U_L \rightarrow U$ be the projection. We have $\text{Trop}(U) = \text{Trop}(U_L)$, and the tropical multiplicities in the two tropicalizations coincide essentially by definition. See [Gub13b, Proposition 3.7, Definition 13.4]. By the above, the affinoid space $U_\omega \widehat{\otimes}_K L = \text{trop}_\varphi^{-1}(\omega) \widehat{\otimes}_K L = \text{trop}_{\varphi \widehat{\otimes}_K L}^{-1}(\omega) = (U_L)_\omega$ has a unique Shilov boundary point. It follows directly from the definition of the Shilov boundary that the image of the Shilov boundary of $U_\omega \widehat{\otimes}_K L$ with respect to p^{an} contains the Shilov boundary of U_ω as the former has more analytic functions than the latter. Hence p^{an} maps the unique Shilov boundary point $s_L(\omega)$ of $U_\omega \widehat{\otimes}_K L$ to the unique Shilov boundary point $s(\omega)$ of U_ω . Clearly $\omega \mapsto s(\omega)$ is a section of trop_φ . Note that we have in fact shown that the section s respects base extension, in that $s = p^{\text{an}} \circ s_L$, where $s_L : Z \rightarrow U_L^{\text{an}}$ is the partial section defined relative to L . In particular, if s_L is continuous, then s is.

Next we prove continuity and uniqueness when $U = T$ (and φ is the identity map). In this case $s(\omega)$ is the Gauss norm $\| \cdot \|_r : K[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow \mathbf{R}_+$ in the notation of 10.1, where $\omega = (\omega_1, \dots, \omega_n)$ and $r = (\exp(-\omega_1), \dots, \exp(-\omega_n))$. It is clear that s is continuous and is defined on all of \mathbf{R}^n ; its image is by definition the *skeleton* $S(T) = s(\mathbf{R}^n)$ of the torus T . We now turn to uniqueness. Let $\omega \in \mathbf{R}^n$, and suppose that there exists a continuous section $s' : Z \rightarrow T^{\text{an}}$ defined on an open neighbourhood Z of ω such that $s(\omega) \neq s'(\omega)$. Let $u = s(\omega)$. By hypothesis $u' := s'(\omega) \neq u$, so there exists a (non-zero) Laurent polynomial $h = \sum_{I \in \mathbf{Z}^n} a_I x^I \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that $|h(u')| < \|h\|_r$. Since $\text{trop}(u') = \omega$ we have $|x_i(u')| = r_i$ for all i . If there were a unique exponent I such that $|a_I| r^I = \|h\|_r$, then by the ultrametric inequality as applied to the seminorm corresponding to u' , we would have $|h(u')| = \|h\|_r$; therefore there are at least two exponents I such that $|a_I| r^I$ is maximal. In other words, the initial degeneration of h at ω is not a monomial, so $\omega \in \text{Trop}(h) := \text{trop}(V(h))$, where $V(h)$ is the zero set of h . The maps $w \mapsto |h(w)|$ and $t \mapsto \|h\|_t$ are continuous, so there exists a small open neighbourhood W of u' in T^{an} such that $|h(w)| < \|h\|_t$ for all $w \in W$, where $t = (|x_1(w)|, \dots, |x_n(w)|)$. By the above argument, then, we have $\text{trop}(W) \subset \text{Trop}(h)$. But $s'^{-1}(W) \subset \text{trop}(W)$ is an open neighbourhood of ω and $\text{Trop}(h)$ has codimension one in \mathbf{R}^n , a contradiction. Since the locus where two maps to a Hausdorff space coincide is closed, this implies that s is the unique continuous section defined on any subset Z which is contained in the closure of its interior.

We treat the general situation by reducing to the case of a torus settled above. Let $d = \dim(U)$; we may assume $d < n$. The tropicalization $\text{Trop}(U)$ is the support of an integral Γ -affine polyhedral complex Σ_1 of pure dimension d . It contains finitely many polyhedral faces Δ_i of maximal dimension d . Let L_i be the (d -dimensional) linear span of $\Delta_i - v$ for any $v \in \Delta_i$. Arguing by induction, one can show that for all $n > d$ there exists a $(d \times n)$ -matrix with entries in \mathbf{Z} such that the corresponding linear map $f : \mathbf{R}^n \rightarrow \mathbf{R}^d$ is injective on each L_i , and hence on each Δ_i . Let $\alpha : \mathbf{G}_m^n \rightarrow \mathbf{G}_m^d$ be the homomorphism of tori such that the associated cocharacter map is f . Consider the morphism $\psi = \alpha \circ \varphi : U \rightarrow \mathbf{G}_m^d$. The diagram

$$(10.6.1) \quad \begin{array}{ccc} U^{\text{an}} & \xrightarrow{\psi} & \mathbf{G}_m^{d,\text{an}} \\ \text{trop}_\varphi \downarrow & & \downarrow \text{trop} \\ \text{Trop}(U) & \xrightarrow{f} & \mathbf{R}^d \end{array}$$

is commutative, where we also write ψ for ψ^{an} . By construction, the map f is finite-to-one on the subset $\text{Trop}(U)$ of \mathbf{R}^n . Let $S(\mathbf{G}_m^d)$ be the skeleton of the torus \mathbf{G}_m^d , defined above. Fix $\omega' \in \mathbf{R}^d$ with coordinates in Γ , and write $\{\omega_1, \dots, \omega_\ell\} = f^{-1}(\omega')$. The affinoid domain $U'_{\omega'} := \text{trop}^{-1}(\omega') \subset \mathbf{G}_m^{d,\text{an}}$ has a unique Shilov boundary point, namely the unique point in the skeleton $S(\mathbf{G}_m^d)$ mapping to ω' . Now we use a similar argument as in the proof of the Sturmfels-Tevelev formula 8.4. By the commutativity of (10.6.1), $\psi^{-1}U'_{\omega'}$ is the disjoint union of the finitely many affinoid subdomains $U_{\omega_i} = \text{trop}_\varphi^{-1}(\omega_i)$ for $i = 1, \dots, \ell$. The analytification of ψ is boundaryless [Ber90, Theorem 3.4.1], hence by pullback [Ber90, Proposition 3.1.3] we find that $\partial(U_{\omega_i}/U'_{\omega'})$ is empty. By [Ber90, Corollary 2.5.13] this implies that $U_{\omega_i} \rightarrow U'_{\omega'}$ is finite. Therefore the associated map on reductions $\tilde{U}_{\omega_i} \rightarrow \tilde{U}'_{\omega'}$ is finite [BGR84, Theorem 6.3.4/2], where $U_{\omega_i} = \mathcal{M}(\mathcal{A}_{\omega_i})$ (resp. $U'_{\omega'} = \mathcal{M}(\mathcal{A}'_{\omega'})$) and $\tilde{U}_{\omega_i} = \text{Spec}(\tilde{\mathcal{A}}_{\omega_i})$ (resp. $\tilde{U}'_{\omega'} = \text{Spec}(\tilde{\mathcal{A}}'_{\omega'})$). Let $\omega = \omega_i$ for some i , and suppose that $m_{\text{Trop}}(\omega) = 1$. Then both reductions $\tilde{U}_\omega, \tilde{U}'_{\omega'}$ are irreducible, d -dimensional schemes over the residue field, so the generic point is mapped to the generic point. This implies that for such ω , the image of $s(\omega)$ under ψ lies in the skeleton $S(\mathbf{G}_m^d)$ and, conversely, if a point in $U_\omega = \text{trop}_\varphi^{-1}(\omega)$ is mapped to the skeleton $S(\mathbf{G}_m^d)$ under ψ , then it is equal to $s(\omega)$. In other words, $\{s(\omega)\} = \text{trop}_\varphi^{-1}(\omega) \cap \psi^{-1}(S(\mathbf{G}_m^d))$.

Now we prove that s is continuous. For this we may assume that $v : K^\times \rightarrow \mathbf{R}$ is surjective, as remarked above. It suffices to show that $s(Z)$ is closed in $\text{trop}_\varphi^{-1}(Z) \subset U^{\text{an}}$ (endowed with its relative topology), since $\text{trop}_\varphi : \text{trop}_\varphi^{-1}(Z) \rightarrow Z$ is a proper map to a metric space, and a proper map to a metric space is closed [Pal70]. In fact, since the image of a continuous section of a continuous map between Hausdorff spaces is necessarily closed, showing that s is continuous is equivalent to proving $s(Z)$ is closed; in particular, $S(\mathbf{G}_m^d)$ is closed in $\mathbf{G}_m^{d,\text{an}}$, being the image of the continuous section of $\text{trop} : \mathbf{G}_m^{d,\text{an}} \rightarrow \mathbf{R}^d$. When the valuation is surjective we have shown that $\{s(\omega)\} = \text{trop}_\varphi^{-1}(\omega) \cap \psi^{-1}(S(\mathbf{G}_m^d))$ for all $\omega \in Z$, so $s(Z) = \text{trop}_\varphi^{-1}(Z) \cap \psi^{-1}(S(\mathbf{G}_m^d))$ is indeed closed in $\text{trop}_\varphi^{-1}(Z)$.

Finally, we prove that s is unique when Z is contained in the closure of its interior in $\text{Trop}(U)$, no longer under the assumption that the valuation is surjective. Let $s' : Z \rightarrow U^{\text{an}}$ be another continuous partial section of trop_φ . Let $Z' \subset Z$ be an open subset of $\text{Trop}(U)$ contained in the relative interior of a d -dimensional integral Γ -affine polyhedron in $\text{Trop}(U)$. Then $f(Z')$ is open in \mathbf{R}^d and $f : \text{Trop}(U) \rightarrow \mathbf{R}^d$ is injective on Z' , so it has an inverse $g : f(Z') \rightarrow Z'$. Let $\omega \in Z'$ have Γ -rational coordinates. Define $\sigma, \sigma' : f(Z') \rightarrow \mathbf{G}_m^{d,\text{an}}$ by $\sigma = \psi \circ s \circ g$ and $\sigma' = \psi \circ s' \circ g$. These are both partial sections of $\text{trop} : \mathbf{G}_m^{d,\text{an}} \rightarrow \mathbf{R}^d$ defined on $f(Z')$, so by the torus case, they are equal. Hence $\psi(s'(\omega)) = \psi(s(\omega)) \in S(\mathbf{G}_m^d)$, so $s'(\omega) \in \text{trop}_\varphi^{-1}(\omega) \cap \psi^{-1}(S(\mathbf{G}_m^d)) = \{s(\omega)\}$. Therefore $s'(\omega) = s(\omega)$, so since such ω are dense in Z' , we conclude $s = s'$ on Z' . Because Z is contained in the closure of its interior in $\text{Trop}(U)$, the union of all such Z' is dense in Z , so since $s = s'$ on each Z' , we have $s = s'$ on Z . \blacksquare

Remark 10.7. Suppose that $\text{Trop}(U)$ has multiplicity one everywhere. With the notation in the proof of Theorem 10.6, we claim that $s(\text{Trop}(U)) = \psi^{-1}(S(\mathbf{G}_m^d))$. For $\omega \in \text{Trop}(U) \cap N_\Gamma$ we showed that $\{s(\omega)\} = \text{trop}_\varphi^{-1}(\omega) \cap \psi^{-1}(S(\mathbf{G}_m^d))$, which implies the claim if $\Gamma = \mathbf{R}$. One easily reduces to this case by extending scalars to a non-archimedean extension field L of K whose

valuation map $L^\times \rightarrow \mathbf{R}$ is surjective, and using the fact that $p^{-1}(S(\mathbf{G}_{m,K}^d)) = S(\mathbf{G}_{m,L}^d)$, where $p : \mathbf{G}_{m,L}^{d,\text{an}} \rightarrow \mathbf{G}_{m,K}^{d,\text{an}}$ is the structural morphism. Therefore $s(\text{Trop}(U))$ is a c -skeleton in the sense of [Duc03, Duc12]. See Theorem 5.1 of [Duc12].

To conclude this section, we show that the image of the section of tropicalization is contained in the skeleton in the case of a strictly semistable pair, and that the section preserves integral affine structures in a suitable sense.

Proposition 10.8. *Let (\mathcal{X}, H) be a strictly semistable pair of dimension d , let $U = X \setminus \text{supp}(H)_\eta$, and let $\varphi : U \rightarrow T \cong \mathbf{G}_m^n$ be a closed immersion into an algebraic K -torus. Let $Z \subset \text{Trop}(U)$ be a subset such that $m_{\text{Trop}}(\omega) = 1$ for all $\omega \in Z$. Then the image of the section $s : Z \rightarrow U^{\text{an}}$ defined in Theorem 10.6 is contained in the skeleton $S(\mathcal{X}, H)$.*

Moreover if Δ is an integral Γ -affine polyhedral face in $\text{Trop}(U)$ of dimension d which is contained in Z , then Δ is covered by finitely many integral Γ -affine polyhedra Δ_i such that s induces a unimodular integral Γ -affine map $\Delta_i \rightarrow \Delta_{S_i}$ for a canonical polyhedron Δ_{S_i} of $S(\mathcal{X}, H)$.

Proof. Define a partial ordering \leq on U^{an} by declaring that $x \leq y$ if $|f(x)| \leq |f(y)|$ for all $f \in K[M]$, where M is the character lattice of T . This is indeed a partial ordering because $U^{\text{an}} \subset T^{\text{an}}$ and T^{an} can be identified with a space of seminorms on $K[M]$. Let $z \in \text{Trop}(U)$ be a point of tropical multiplicity one, let $x = s(z) \in U^{\text{an}}$, and let $y = \tau(x) \in S(\mathcal{X}, H)$, where $\tau : U^{\text{an}} \rightarrow S(\mathcal{X}, H)$ is the retraction map. We want to show that $x = y$. Since trop_φ factors through τ by Proposition 5.2, we have $\text{trop}_\varphi^{-1}(z) = \tau^{-1}(\text{trop}_\varphi^{-1}(z) \cap S(\mathcal{X}, H))$, so $y \in \text{trop}_\varphi^{-1}(z)$ as well. Since x is by definition the Shilov boundary point of $\text{trop}_\varphi^{-1}(z)$, we have $y \leq x$.

By the ε -approximation argument used in the construction of the skeleton of a strictly semistable pair in §4, there exists an affinoid neighbourhood $X' \subset U^{\text{an}}$ of y of the form $X' = \tau^{-1}(X' \cap S(\mathcal{X}, H))$ which is the generic fibre of a strictly semistable formal scheme \mathfrak{X}' , such that the classical skeleton $S(\mathfrak{X}')$ coincides with $S(\mathcal{X}, H) \cap X'$. The retraction map $\tau : X' \rightarrow S(\mathfrak{X}')$ as defined by Berkovich also coincides with ours. Then [Ber99, Theorem 5.2(ii)] gives $x \leq y$, so $x = y$.

The unimodularity statement follows immediately from Proposition 5.2 and Theorem 8.4. \blacksquare

APPENDIX A. REFINED INTERSECTION THEORY WITH CARTIER DIVISORS

Let K be an algebraically closed field endowed with a non-trivial non-archimedean complete absolute value $|\cdot|$, corresponding valuation $v := -\log|\cdot|$, valuation ring K° , residue field \tilde{K} and value group $\Gamma := v(K^\times)$. We will first recall the construction from [Gub98] of the Weil divisor associated to a Cartier divisor on an admissible formal scheme over K° . This will be useful in the paper for several local considerations. Then we will study the refined intersection product of a Cartier divisor with a cycle on a proper flat variety \mathcal{X} over K° . Note that we cannot use the algebraic intersection theory as in [Ful98] since the valuation ring K° is not noetherian. Instead we pass to the formal completion $\widehat{\mathcal{X}}$ of \mathcal{X} along the special fibre to get an admissible formal scheme. Then the refined intersection product is an easy consequence of the above construction of the associated Weil divisor. The reference for the refined intersection product is [Gub03, §5].

We start with a Cartier divisor D on a quasicompact admissible formal scheme \mathfrak{X} over K° . Our first goal is the construction of the Weil divisor $\text{cyc}(D)$ on \mathfrak{X} associated to D .

A.1. Let X be the generic fibre of \mathfrak{X} . We define cycles on X as formal \mathbf{Z} -linear combinations of irreducible Zariski-closed subsets of X . By definition, a Zariski-closed subset is the image of a closed immersion of analytic spaces over K . In rigid geometry, Zariski-closed subsets are called closed analytic subsets. Usually in our paper, the generic fibre is algebraic and we have the basic operations for cycles as proper push-forward, flat pull-back and proper intersection with Cartier divisors from the first two chapters of Fulton's book [Ful98]. We note that this generalizes to quasicompact analytic spaces as they are covered by strictly affinoid subdomains $\mathcal{M}(\mathcal{A})$. Since \mathcal{A} is a noetherian K -algebra, we may use the algebraic intersection theory on $\text{Spec}(\mathcal{A})$ and glue to get the corresponding operations on X (see [Gub98, §2]). We should mention that \mathcal{A} is not of finite type over K as required in Fulton's book, but this assumption is not really necessary to develop the basic properties mentioned above (see [Tho90]). Another issue in the analytic setting is the definition of irreducible components of X which was handled in a paper by Conrad [Con99].

A.2. A *horizontal prime cycle* \mathfrak{V} on \mathfrak{X} is the closure of an irreducible Zariski-closed analytic subset of X in \mathfrak{X} as in [Gub98, Proposition 3.3]. A *horizontal cycle* on \mathfrak{X} is a formal \mathbf{Z} -linear combination of horizontal prime cycles. A *vertical prime cycle* is an irreducible closed subset of \mathfrak{X}_s . A *vertical cycle* is a formal Γ -linear combination of vertical prime cycles.

A cycle \mathfrak{Z} on \mathfrak{X} is a formal sum of a horizontal cycle \mathfrak{V} and a vertical cycle V . The prime cycles with non-zero coefficients are called the *prime components* of \mathfrak{Z} . A cycle is called *effective* if the multiplicities in its prime components are positive.

We say that a cycle \mathfrak{Z} on \mathfrak{X} is of codimension p if any horizontal prime component of \mathfrak{Z} is the closure of an irreducible closed analytic subset of X of codimension p and if any vertical prime component of \mathfrak{Z} has codimension $p - 1$ in \mathfrak{X}_s . A cycle on \mathfrak{X} of codimension 1 is called a *Weil divisor*.

Example A.3. To understand why we need Γ -coefficients for vertical cycles, we look at the simplest example $\mathfrak{X} = \text{Spec}(K^\circ)$ and $D = \text{div}(f)$ for a non-zero $f \in K$. Then the valuation v gives the multiplicity in the special fibre $\mathfrak{X}_s = \text{Spec}(\tilde{K})$ and we set $\text{cyc}(D) := v(f)\mathfrak{X}_s$.

In general, the construction of the Weil divisor $\text{cyc}(D)$ is based on the following local definition:

A.4. Let $\mathfrak{X} = \text{Spf}(\mathcal{A}^\circ)$ for a strictly affinoid algebra \mathcal{A} and $D = \text{div}(a/b)$ for $a, b \in \mathcal{A}^\circ$ which are not a zero-divisors. For an irreducible component Y of \mathfrak{X}_s , there is a unique ξ_Y in the generic fibre $X = \mathfrak{X}_\eta$ which reduces to the generic point of Y . Note that X is the Berkovich spectrum of \mathcal{A} , therefore we get existence and uniqueness of ξ_Y from [Ber90, Proposition 2.4.4]. Then we define the multiplicity of D in Y as $\text{ord}(D, Y) := \log |b(\xi_Y)| - \log |a(\xi_Y)|$.

A.5. We deal now with the general case. We assume first that the generic fibre X of \mathfrak{X} is irreducible and reduced. This assumption is satisfied in all our applications. To define the associated Weil divisor

$$\text{cyc}(D) = \sum_{\mathfrak{V}} \text{ord}(D, \mathfrak{V})\mathfrak{V}$$

on \mathfrak{X} , we have to define the multiplicity $\text{ord}(D, \mathfrak{V})$ of D in a prime cycle \mathfrak{V} of \mathfrak{X} of codimension 1. If \mathfrak{V} is horizontal, then \mathfrak{V} is the closure of an irreducible Zariski closed subset Y of the generic fibre X of codimension 1. The restriction D_η of D to X is a Cartier divisor and hence we may use A.1 to define $\text{ord}(D, \mathfrak{V}) := \text{ord}(D_\eta, Y)$.

If \mathfrak{V} is vertical, then \mathfrak{V} is equal to an irreducible component V of \mathfrak{X}_s . We choose a formal affine open subset $\mathfrak{U} = \text{Spf}(A)$ such that \mathfrak{U}_s is a non-empty subset of V and such that D is given on \mathfrak{U} by a/b for $a, b \in A$ which are not a zero-divisors. Then $\mathcal{A} := A \otimes_{K^\circ} K$ is a strictly affinoid algebra and we set $\mathfrak{X}' := \text{Spf}(\mathcal{A}^\circ)$. We have a canonical morphism $\mathfrak{X}' \rightarrow \text{Spf}(A)$ of admissible formal schemes with the same generic fibre which induces a finite surjective morphism on special fibres (see 4.13 of [Gub13b] for the argument). Let D' be the Cartier divisor on \mathfrak{X}' given by the pull-back of D . Then D' is given by a/b on \mathfrak{X}' as well. By using A.4, we define the multiplicity of D in V by

$$\text{ord}(D, V) := \sum_Y [\tilde{K}(Y) : \tilde{K}(V)] \text{ord}(D', Y),$$

where Y ranges over all irreducible components of \mathfrak{X}'_s (see [Gub98, §3] for more details).

If the generic fibre X is not irreducible or not reduced, then we use the cycle $\text{cyc}(X)$ associated to X from [Gub98, 2.7] and proceed by linearity in the prime components of $\text{cyc}(X)$ to define $\text{cyc}(D)$.

Remark A.6. If f is a meromorphic function on \mathfrak{X} which is invertible as a meromorphic function, then f defines a Cartier divisor $\text{div}(f)$ on \mathfrak{X} . This notion has to be distinguished from the associated Weil divisor which we denote by $\text{cyc}(f)$. We will use the same notation in the algebraic setting below.

Proposition A.7. *Let D be a vertical Cartier divisor on the admissible formal scheme \mathfrak{X} over K° ; that is, D is a Cartier divisor whose restriction to the generic fibre $X = \mathfrak{X}_\eta$ is trivial. We assume that the special fibre \mathfrak{X}_s is reduced. Then the following properties hold:*

- (a) *The union of the prime components of $\text{cyc}(D)$ is equal to $\text{supp}(D)$.*
- (b) *The Cartier divisor D is effective if and only if $\text{cyc}(D)$ is effective.*
- (c) *We have $D = 0$ if and only if $\text{cyc}(D) = 0$*

In particular, the map $D \mapsto \text{cyc}(D)$ is an injective homomorphism from the group of vertical Cartier divisors on \mathfrak{X} to the group of (vertical) Weil divisors on \mathfrak{X} .

Proof. Recall that the support of the vertical Cartier divisor D is the union of points of \mathfrak{X} for which the restriction of D to some neighbourhood is non-trivial. This gives a closed subset $\text{supp}(D)$ of \mathfrak{X} . It follows from the definition of $\text{cyc}(D)$ in A.5 that every prime component of $\text{cyc}(D)$ is contained in $\text{supp}(D)$. Moreover, if D is an effective Cartier divisor, then $\text{cyc}(D)$ is an effective Weil divisor.

First we prove (b). This claim is local and so we may assume that $\mathfrak{X} = \text{Spf}(A)$ for a K° -admissible algebra A and that D is given by $f = a/b$ for $a, b \in A$ which are not zero-divisors. Now we use that \mathfrak{X}_s is reduced. By a result of Bosch and Lütkebohmert (see [Gub98, Proposition 1.11]), this implies that $A = \mathcal{A}^\circ$ for a strictly affinoid algebra \mathcal{A} over K . To prove (b), it remains to show that $f \in \mathcal{A}^\circ$ if $\text{cyc}(f)$ is an effective Weil divisor. Since D is a vertical Cartier divisor, we know that f is an invertible element of \mathcal{A} . As we assume now that $\text{cyc}(f)$ is effective, we deduce from A.4 that $|f(\xi_Y)| \leq 1$ for every irreducible component Y of \mathfrak{X}_s . Since the supremum norm of f on X is equal to $\max_Y |f(\xi_Y)|$ [Ber90, Proposition 2.4.4], we conclude that $f \in \mathcal{A}^\circ$, proving (b).

Next we prove (a). Let $\text{supp}(\text{cyc}(D))$ be the union of the prime components of $\text{cyc}(D)$. We have seen at the beginning of the proof that $\text{supp}(\text{cyc}(D)) \subset \text{supp}(D)$; we have to show equality. By passing to the open subset $\mathfrak{X} \setminus \text{supp}(\text{cyc}(D))$, we may assume that $\text{cyc}(D) = 0$. Then (b) implies that D and $-D$ are both effective Cartier divisors, which means that D is trivial. This proves (a).

Finally, (c) is an easy consequence of (b). \blacksquare

Now we switch to the algebraic setting. Our goal is to define a refined intersection theory of Cartier divisors with cycles on a proper flat variety \mathcal{X} over K° with generic fibre $X = \mathcal{X}_\eta$. Let $Z_k(\mathcal{X}, \Gamma) = Z^p(\mathcal{X}, \Gamma)$ be the group of cycles on \mathcal{X} of topological dimension k (resp. of codimension p with $p = \dim(\mathcal{X}) - k$), where again the horizontal cycles have \mathbf{Z} -coefficients and the vertical cycles have coefficients in Γ .

A.8. We consider a Cartier divisor D on \mathcal{X} and $\mathcal{Z} \in Z_k(\mathcal{X}, \Gamma)$ which *intersect properly*. This means that no prime component of \mathcal{Z} is contained in the support $\text{supp}(D)$ of the Cartier divisor D . In this situation, the intersection product $D \cdot \mathcal{Z}$ is well-defined in $Z_{k-1}(\mathcal{X}, \Gamma)$ by the construction in A.5. Indeed, proceeding by linearity in the prime components of \mathcal{Z} , we may assume that \mathcal{Z} is a prime cycle; then:

If \mathcal{Z} is horizontal, then \mathcal{Z} is the closure of a closed subvariety Z of X . By properness of the intersection, $D|_Z$ is a well-defined Cartier divisor on Z and we define the horizontal part of $D \cdot \mathcal{Z}$ to be the cycle on \mathcal{X} induced from $\text{cyc}(D|_Z)$ by passing to the closures of the prime components. Let $\mathfrak{Z} := \widehat{\mathcal{Z}}$ be the formal completion along the special fibre. Then D induces a well-defined Cartier divisor $\widehat{D}|_{\mathfrak{Z}}$ on the admissible formal scheme \mathfrak{Z} over K° . Since \mathfrak{Z} and \mathcal{Z} have the same special fibre, it makes sense to define the vertical part of $D \cdot \mathcal{Z}$ as the vertical part of the Weil divisor associated to $\widehat{D}|_{\mathfrak{Z}}$. We end up with a cycle $D \cdot \mathcal{Z}$ on \mathcal{X} with support in $\text{supp}(\mathcal{Z}) \cap \text{supp}(D)$. It follows easily from the construction that $\text{cyc}(\widehat{D}|_{\mathfrak{Z}})$ is the formal completion of $D \cdot \mathcal{Z}$ defined componentwise.

If \mathcal{Z} is vertical, then it is a closed subvariety of \mathcal{X}_s and we define $D \cdot \mathcal{Z}$ as the Weil divisor associated to $D|_{\mathcal{Z}}$. This is a vertical cycle on \mathcal{X} with support in $\text{supp}(\mathcal{Z}) \cap \text{supp}(D)$.

Remark A.9. In the applications, \mathcal{X} is often normal. For example, every K° -toric variety is normal (see [Gub13b, Proposition 6.11]) and hence it follows from [EGAIV₄, Proposition 18.12.15] that every strictly semistable variety over K° is normal. If \mathcal{X} is normal, then the multiplicity $\text{ord}(D, V)$ in the irreducible component Y of \mathcal{X}_s has also an algebraic description: It follows from results of Knaf that the local ring $\mathcal{O}_{\mathcal{X}, \zeta_V}$ in the generic point ζ_V is a valuation ring for a unique real valued valuation w_V extending v . Then $\text{ord}(D, V) = w_V(a)$ for any local equation a of D in ζ_V . For details, we refer to [GS15], Proposition 2.11.

A.10. To get a refined intersection theory, we have to consider rational equivalence on a closed subset S of \mathcal{X} . Let $R(S, \Gamma)$ be the subgroup of $Z(\mathcal{X}, \Gamma)$ generated by all $\text{cyc}(f|_{\mathcal{Y}})$ and $\gamma \text{cyc}(g|_V)$, where \mathcal{Y} (resp. V) ranges over all horizontal (resp. vertical) closed subvarieties of S , where f (resp. g) are non-zero rational functions of \mathcal{Y} (resp. V) and where γ ranges over the value group

Γ . The local Chow group of \mathcal{X} with support in S is defined by

$$CH_S^*(\mathcal{X}, \Gamma) := Z^*(\mathcal{X}, \Gamma)/R^*(S, \Gamma)$$

and it will be graded by codimension.

Definition A.11. Let D be a Cartier divisor on the proper flat scheme \mathcal{X} over K° and let $\mathcal{Z} \in Z^p(\mathcal{X}, \Gamma)$. For a closed subset S of \mathcal{X} containing the support of \mathcal{Z} , we define the *refined intersection product*

$$D \cdot \mathcal{Z} \in CH_{\text{supp}(D) \cap S}^{p+1}(\mathcal{X}, \Gamma)$$

as follows: By linearity, we may assume that \mathcal{Z} is a prime cycle. If D intersects \mathcal{Z} properly, then $D \cdot \mathcal{Z}$ is even well-defined as a cycle of codimension 1 in \mathcal{Z} by A.8. If \mathcal{Z} is contained in $\text{supp}(D)$, then we choose a linearly equivalent Cartier divisor D' which intersects \mathcal{Z} properly and we define $D \cdot \mathcal{Z}$ as the class of $D' \cdot \mathcal{Z}$ in $CH_{\text{supp}(D) \cap S}^{p+1}(\mathcal{X}, \Gamma)$.

In the following result, we use proper push-forward and flat pull-back of cycles on flat varieties over K° . The definitions are the same as in Fulton's book [Ful98].

Proposition A.12. *The construction in Definition A.11 leads to a well-defined refined intersection product*

$$CH_S^p(\mathcal{X}, \Gamma) \rightarrow CH_{\text{supp}(D) \cap S}^{p+1}(\mathcal{X}, \Gamma)$$

which maps the class of a cycle \mathcal{Z} with support in S to $D \cdot \mathcal{Z}$. It has the following properties:

- (a) *The refined intersection product is bilinear using the union of supports.*
- (b) *If $\varphi : \mathcal{X}' \rightarrow \mathcal{X}$ is a morphism of flat proper varieties over K° and if S' is a closed subset of \mathcal{X}' with $\varphi(S') \subset S$, then the projection formula*

$$\varphi_*(\varphi^* D \cdot \alpha') = D \cdot \varphi_*(\alpha') \in CH_{\text{supp}(D) \cap S}^{p+1}(\mathcal{X}, \Gamma)$$

holds for every $\alpha' \in CH_{S'}^p(\mathcal{X}', \Gamma)$.

- (c) *For Cartier divisors D, E and $\alpha \in CH_S^p(\mathcal{X}, \Gamma)$ on \mathcal{X} , we have the commutativity law*

$$D \cdot E \cdot \alpha = E \cdot D \cdot \alpha \in CH_{\text{supp}(D) \cap \text{supp}(E) \cap S}^{p+2}(\mathcal{X}, \Gamma).$$

- (d) *If $\varphi : \mathcal{X}' \rightarrow \mathcal{X}$ is a flat morphism of flat proper varieties over K° and if $\alpha \in CH_S^p(\mathcal{X}, \Gamma)$, then*

$$\varphi^*(D \cdot \alpha) = \varphi^* D \cdot \varphi^* \alpha \in CH_{\varphi^{-1}(\text{supp}(D) \cap S)}(\mathcal{X}', \Gamma).$$

Proof. By passing to the formal completion of \mathcal{X} along the special fibre, this follows easily from [Gub03, Proposition 5.9]. ■

Remark A.13. The pull-back of the Cartier divisor D with respect to the morphism $\varphi : \mathcal{X}' \rightarrow \mathcal{X}$ is only well-defined as a Cartier divisor if $\varphi(\mathcal{X}')$ is not contained in $\text{supp}(D)$. However, the pull-back is well-defined as a pseudo-divisor in the sense of [Ful98, §2] and the refined intersection product makes sense in this more general setting (see [Gub03, §5] for details). We will not use it in our paper.

A.14. We have a degree map on 0-dimensional cycles of $Z(\mathcal{X}, \Gamma)$. It is compatible with vertical rational equivalence, i.e. it induces a homomorphism

$$\text{deg} : CH_{\mathcal{X}_s}^{d+1}(\mathcal{X}, \Gamma) \rightarrow \Gamma$$

where $d := \dim(X) = \dim(\mathcal{X}) - 1$. Let D_0, \dots, D_k be Cartier divisors on \mathcal{X} and let $\mathcal{Z} \in Z_{k+1}(\mathcal{X}, \Gamma)$ with horizontal part Z . We assume that $\text{supp}(D_0|_X) \cap \dots \cap \text{supp}(D_k|_X)$ does not intersect the support of Z . Then we get a well-defined *intersection number*

$$D_0 \cdots D_k \cdot \mathcal{Z} := \text{deg}(D_0 \cdots D_k \cdot \mathcal{Z}) \in \Gamma.$$

Often we will consider the special case of Cartier divisors D_0, \dots, D_d on \mathcal{X} with

$$\text{supp}(D_0|_X) \cap \dots \cap \text{supp}(D_d|_X) = \emptyset.$$

Using the cycle $\text{cyc}(\mathcal{X})$ induced by the cycle associated to X by passing to the closure of the components, we get a well-defined *intersection number*

$$D_0 \cdots D_d := D_0 \cdots D_d \cdot \text{cyc}(\mathcal{X}) := \text{deg}(D_0 \cdots D_d \cdot \text{cyc}(\mathcal{X})) \in \Gamma.$$

Remark A.15. The refined intersection theory considered above works also for an admissible formal scheme \mathfrak{X} over K° if the generic fibre is the analytification of a proper algebraic variety X . The same arguments apply (see [Gub03, §5] for details). Thus it is not necessary to check if a given formal K° -model of X is algebraic. This will be used in the following example.

Example A.16. Let C be a smooth projective curve over K with strictly semistable K° -model \mathcal{C} . For simplicity, we assume that every 1-dimensional canonical simplex has the same length $v(\pi)$ (see 4.6). Then $\mathcal{Y} := \mathcal{C} \times \mathcal{C}$ is a strictly polystable K° -model of $X := C \times C$ such that the canonical polyhedra of the skeleton $S(\mathcal{Y})$ are squares with edges of uniform length $v(\pi)$. We choose a diagonal in every square to get a triangulation of $S(\mathcal{Y})$. The preimages of the triangles with respect to the retraction $X^{\text{an}} \rightarrow S(\mathcal{Y})$ form a formal analytic atlas of X^{an} inducing a strictly semistable formal scheme \mathfrak{X} with skeleton $S(\mathfrak{X}) = S(\mathcal{Y})$ as a set, but with canonical simplices given by the chosen triangulation (see [Gub10, Proposition 5.5, Remark 5.6, Remark 5.19] for this construction). The triangulation yields that \mathfrak{X} lies over the formal completion of \mathcal{Y} . For $i = 1, 2$, the projection $p_i : X = C \times C \rightarrow C$ to the i -th factor extends to a morphism $p_i : \mathfrak{X} \rightarrow \mathfrak{C}$ for the formal completion \mathfrak{C} of \mathcal{C} . Let $(p_i)_{\text{aff}} : S(\mathfrak{X}) \rightarrow S(\mathcal{C})$ be the composition of $p_i : S(\mathfrak{X}) \rightarrow C^{\text{an}}$ with the retraction $C^{\text{an}} \rightarrow S(\mathcal{C})$. This is integral Γ -affine on every canonical simplex of $S(\mathfrak{X})$ (see Proposition 5.5). By the stratum–face correspondence in Proposition 4.10, we have a bijective correspondence between vertices u of the canonical triangulation of $S(\mathfrak{X})$ and irreducible components Y_u of \mathfrak{X}_s . Using Proposition 4.17, there is a unique effective vertical Cartier divisor D_u with $\text{cyc}(D_u) = v(\pi) \cdot V_u$.

Kolb has shown in [Kol14] that the intersection numbers of these vertical Cartier divisors D_u can be computed by the following relations:

Proposition A.17. *Let \mathfrak{X} be as in Example A.16 and assume that every 1-dimensional canonical simplex in $S(\mathcal{C})$ is determined by its two vertices, i.e. that $S(\mathcal{C})$ is a graph with no multiple edges. For vertices a, b of $S(\mathfrak{X})$, we have the following two relations:*

- (a) $D_a \cdot D_b \cdot \sum_c D_c = 0$, where c runs over all vertices of $S(\mathfrak{X})$.
- (b) If $(p_1)_{\text{aff}}(a) \neq (p_1)_{\text{aff}}(b)$ in $S(\mathcal{C})$, then

$$D_a \cdot D_b \cdot \sum_{(p_1)_{\text{aff}}(c) = (p_1)_{\text{aff}}(b)} D_c = 0,$$

where c runs over all vertices of $S(\mathfrak{X})$ with $(p_1)_{\text{aff}}(c) = (p_1)_{\text{aff}}(b)$.

Proof. As Kolb's Propositions 4.10 and 4.11 in [Kol14] are formulated algebraically and over a discrete valuation ring, we reproduce the argument for convenience. The first relation is obvious from the fact that $\sum_c D_c = \text{div}(\pi)$. By construction, $a_1 := (p_1)_{\text{aff}}(a)$ and $b_1 := (p_1)_{\text{aff}}(b)$ are two distinguished vertices of $S(\mathcal{C})$. It is clear that $p_1(Y_a)$ (resp. $p_1(Y_b)$) is the irreducible component of $\mathcal{C}_s = \mathfrak{C}_s$ corresponding to a_1 (resp. b_1). We may assume that $p_1(Y_a) \cap p_1(Y_b) \neq \emptyset$, as otherwise the intersection number in (b) is obviously zero. Hence we have an edge between a_1 and b_1 . By assumption, this edge is completely determined by its vertices a_1 and b_1 which means geometrically that $p_1(Y_a) \cap p_1(Y_b) = \{S\}$ for a single point $S \in \mathcal{C}_s(\tilde{K})$. Let D_{b_1} be the unique effective vertical Cartier divisor on \mathfrak{C} with $\text{cyc}(D_{b_1}) = v(\pi) \cdot Y_{b_1}$, where Y_{b_1} is the irreducible component of \mathcal{C}_s corresponding to the vertex b_1 of $S(\mathcal{C})$ (see Proposition 4.17). In a neighbourhood of S , the Cartier divisor D_{b_1} is given by a rational function f . We conclude that the Cartier divisor $D_{b_1} - \text{div}(f)$ has support in the complement of this neighbourhood of S showing

$$0 = D_a \cdot D_b \cdot p_1^* \text{div}(f) = D_a \cdot D_b \cdot p_1^* D_{b_1} \in CH_{\mathfrak{X}_s}^3(\mathfrak{X}, \Gamma).$$

It follows easily from the constructions that

$$p_1^*(D_{b_1}) = \sum_{(p_1)_{\text{aff}}(c) = (p_1)_{\text{aff}}(b)} D_c.$$

We conclude that

$$0 = D_a \cdot D_b \cdot \sum_{(p_1)_{\text{aff}}(c) = (p_1)_{\text{aff}}(b)} D_c \in CH_{\mathfrak{X}_s}^3(\mathfrak{X}, \Gamma),$$

proving the claim. ■

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