

# Equidistribution over function fields

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## Abstract

We prove equidistribution of a generic net of small points in a projective variety  $X$  over a function field  $K$ . For an algebraic dynamical system over  $K$ , we generalize this equidistribution theorem to a small generic net of subvarieties. For number fields, these results were proved by Yuan and we transfer here his methods to function fields. If  $X$  is a closed subvariety of an abelian variety, then we can describe the equidistribution measure explicitly in terms of convex geometry.

## 1 Introduction

Equidistribution plays an important role in various branches of science. In diophantine geometry, equidistribution of small points is the key in Ullmo's and Zhang's proof of the Bogomolov conjecture (see [Ullm], [Zh]). The goal of this article is to transfer Yuan's recent generalization of this equidistribution theorem to function fields. Since the Bogomolov conjecture over function fields is still open, this might have interesting applications.

In this paper, we consider the function field  $K = k(B)$  of an integral projective variety  $B$  over the field  $k$  with  $B$  regular in codimension 1. We fix an ample class  $\mathbf{c}$  on  $B$  to get an absolute height function  $h_{(L, \|\cdot\|)}$  on an irreducible  $d$ -dimensional projective variety  $X$  over  $K$  with respect to an admissibly metrized line bundle  $(L, \|\cdot\|)$  (see §3 for details). We fix also a place  $v$  of  $K$ . The Berkovich analytic space  $X_v^{\text{an}}$  is defined over the smallest algebraically closed field  $\mathbb{K}_v$  which is complete with respect to a valuation extending  $v$ . There is a regular Borel measure  $c_1(L, \|\cdot\|_v)^{\wedge d}$  on  $X_v^{\text{an}}$  which was introduced by Chambert-Loir and which is the non-archimedean analogue of the corresponding complex analytic wedge product of Chern forms (see §2).

We choose always an embedding  $\overline{K} \hookrightarrow \mathbb{K}_v$  to identify  $X(\overline{K})$  with a subset of  $X_v^{\text{an}}$ . The Galois group  $\text{Gal}(\overline{K}/K)$  acts on  $X$  and we denote the orbit of  $P \in X(\overline{K})$  by  $O(P)$ . Let  $\delta_P$  be the Dirac measure on  $X_v^{\text{an}}$  in  $P$ . We consider a *generic net*  $(P_m)_{m \in I}$  in  $X(\overline{K})$ , i.e.  $I$  is a directed set and for every proper closed subset  $Y$  of  $X$ , there is  $m_0 \in I$  with  $P_m \notin Y$  for all  $m \geq m_0$ . Our main result is the following *equidistribution theorem*:

**Theorem 1.1** *Let  $L$  be a big semiample line bundle on the irreducible  $d$ -dimensional projective variety  $X$  over the function field  $K$ . We endow  $L$  with a semi-positive admissible metric  $\|\cdot\|$ . We assume that  $(P_m)_{m \in I}$  is a generic net in  $X(\overline{K})$  with*

$$\lim_m h_{(L, \|\cdot\|)}(P_m) = \frac{1}{(d+1) \deg_L(X)} h_{(L, \|\cdot\|)}(X).$$

*For a place  $v$  of  $K$ , we have the following weak limit of regular probability measures on  $X_v^{\text{an}}$ :*

$$\frac{1}{|O(P_m)|} \sum_{P_m^\sigma \in O(P_m)} \delta_{P_m^\sigma} \xrightarrow{w} \frac{1}{\deg_L(X)} c_1(L, \|\cdot\|_v)^{\wedge d}.$$

For number fields, the history of equidistribution of small points starts with the theorem of Szpiro–Ullmo–Zhang [SUZ] which handles the case of an archimedean place  $v$  assuming positive curvature and  $X$  smooth. Smoothness was removed by Ullmo [Ullm] and Zhang [Zh] to prove the Bogomolov conjecture. Chambert-Loir has proved a non-archimedean version of these results assuming that the metric at the finite place  $v$  of the number field is induced by an ample model (see [Ch], Théorème 3.1). In case of abelian varieties and canonical metrics, this handles just the case of good reduction at  $v$  as the canonical metric is only semipositive at places of bad reduction. Moreover, Chambert-Loir proved equidistribution of small points for any admissible metric in the case of curves ([Ch], Théorème 4.2). Finally, Yuan introduced a variational principle to deal also with semipositive metrics and he proved Theorem 1.1 for any place  $v$  of the number field  $K$  (see [Yu], §5).

For function fields, a tropical version of Theorem 1.1 was proved in case of a closed subvariety of an abelian variety which is totally degenerate at the given place  $v$  (see [Gu4], Theorem 5.5). This was the key to prove Bogomolov’s conjecture for totally degenerate abelian varieties over function fields. Independently of the present paper, Faber ([Fa], Theorem 1.1) proved equidistribution of small points in case of algebraic dynamical systems over the function field of a curve. All these results assume  $L$  ample, but we will see that in our case, the arguments hold more generally for big semiample line bundles.

Yuan’s variational principle for function fields will be proved in §5 and leads to the fundamental inequality between the height of  $X$  and the heights  $h_{\overline{T}}(P_m)$ . It is based on Siu’s theorem in the theory of big line bundles which we recall in §4. Note that Siu’s inequality lies also at the root of Yuan’s article [Yu]. In fact, Yuan proves an arithmetic analogue in Arakelov geometry. In §6, we will first prove Theorem 1.1 from the fundamental inequality which is straightforward and which was used in all the articles mentioned above. Finally, we apply the methods to prove equidistribution of small subvarieties in algebraic dynamical systems.

To have useful applications of Theorem 1.1, a precise determination of the equidistribution measure  $c_1(L, \|\cdot\|_v)^{\wedge d}$  is indispensable. We restrict our attention now to an irreducible  $d$ -dimensional closed subvariety  $X$  of the abelian variety  $A$  over  $K$ . Here,  $K$  may be either a function field or a number field and we fix a non-archimedean place  $v$  of  $K$ . We assume that  $L$  is an even ample line bundle on  $A$  endowed with a canonical metric  $\|\cdot\|_{\text{can}}$  (see Remark 6.4 for the corresponding dynamical system). The height with respect to  $(L, \|\cdot\|_{\text{can}})$  is the famous Néron–Tate height.

In this situation, Theorem 6.7 of [Gu5] gives a completely explicit description of the equidistribution measure  $\mu := c_1(L|_X, \|\cdot\|_{\text{can},v})^{\wedge d}$  in terms of convex geometry. More precisely,  $\mu$  is supported in finitely many rational simplices of dimension at most  $d$  contained in  $X_v^{\text{an}}$  such that the restriction of  $\mu$  to each simplex is a positive multiple of the Lebesgue measure.

For applications, the following *tropical equidistribution theorem* is useful. We consider the Raynaud extension  $1 \rightarrow T \rightarrow E \rightarrow B \rightarrow 0$  of  $A_v^{\text{an}}$ . Here,  $E$  is the uniformization of  $A_v^{\text{an}}$  such that  $A_v^{\text{an}} = E/M$  for a lattice  $M$  in  $E$  and  $B$  is an abelian variety over  $\mathbb{K}_v$  of good reduction whose dimension is denoted by  $b$ . The split torus  $T$  induces a map  $\text{val} : E \rightarrow \mathbb{R}^n$  with  $\Lambda := \text{val}(M)$  a complete lattice in  $\mathbb{R}^n$  (see [Gu5], §4). Passing to the quotient, we obtain a continuous map  $\overline{\text{val}} : A_v^{\text{an}} \rightarrow \mathbb{R}^n/\Lambda$ . For a simplex  $\Delta$  in  $\mathbb{R}^n$ , let  $\delta_{\overline{\Delta}}$  be the Dirac measure in  $\overline{\Delta}$ , i.e. the push-forward of the relative Lebesgue measure on  $\Delta$  to  $\mathbb{R}^n/\Lambda$ .

**Theorem 1.2** *Let  $X$  be a geometrically integral  $d$ -dimensional closed subvariety of  $A$ . Then there is  $e \in \{0, \dots, \min\{b, d\}\}$  such that the tropical variety  $\overline{\text{val}}(X_v^{\text{an}})$  is the union of rational  $(d-e)$ -dimensional simplices  $\overline{\Delta}_1, \dots, \overline{\Delta}_M$  in  $\mathbb{R}^n/\Lambda$  and there is (a possibly empty) list of rational simplices  $\overline{\Delta}_{M+1}, \dots, \overline{\Delta}_{M+N}$  contained in  $\overline{\text{val}}(X_v^{\text{an}})$*

with  $\dim(\overline{\Delta}_j) \in \{d-b, \dots, d-e-1\}$  satisfying the following properties:

For every ample even line bundle  $L$  of  $A$ , there are  $r_1, \dots, r_{M+N} \in (0, \infty)$  such that for every generic net  $(P_m)_{m \in I}$  in  $X(\overline{K})$  with

$$\lim_m \hat{h}_L(P_m) = \frac{\hat{h}_L(X)}{(d+1) \deg_L(X)},$$

we have the following weak limit of regular probability measures on  $\mathbb{R}^n/\Lambda$ :

$$\frac{1}{|O(P_m)|} \sum_{P_m^\sigma \in O(P_m)} \delta_{\overline{\text{val}}(P_m^\sigma)} \xrightarrow{w} \sum_{j=1}^{M+N} r_j \delta_{\overline{\Delta}_j}$$

This tropical equidistribution theorem follows immediately from Theorem 1.1 (resp. its number theoretic analogue in [Yu], Theorem 5.1) and from [Gu5], Theorem 1.1.

The skeleton  $S(A_v^{\text{an}})$  of  $A$  is a canonical subset of  $A_v^{\text{an}}$  which is a proper deformation retraction of  $A_v^{\text{an}}$  and which is homeomorphic to  $\mathbb{R}^n/\Lambda$  by the restriction of  $\overline{\text{val}}$  (see [Ber1], §6.5). Let  $\overline{\mu}$  be the unique regular Borel measure on  $A_v^{\text{an}}$  with  $\text{supp}(\overline{\mu}) = S(A_v^{\text{an}})$  such that  $\overline{\text{val}}_*(\overline{\mu})$  is the Haar probability measure of  $\mathbb{R}^n/\Lambda$ .

**Theorem 1.3** *Let  $K$  be either a function field or a number field and let  $L$  be an ample even line bundle on the abelian variety  $A$  over  $K$ . Let  $(P_m)_{m \in I}$  be a generic net in  $A(\overline{K})$  such that  $\lim_m \hat{h}_L(P_m) = 0$ . If  $v$  is a non-archimedean place of  $K$ , then we have the following weak limit of regular probability measures on  $A_v^{\text{an}}$ :*

$$\frac{1}{|O(P_m)|} \sum_{P_m^\sigma \in O(P_m)} \delta_{P_m^\sigma} \xrightarrow{w} \overline{\mu}.$$

We will see in Theorem 6.3 and Remark 6.4 that  $\hat{h}_L(X) = 0$ . Then Theorem 1.3 is a consequence of Theorem 1.1 (resp. [Yu], Theorem 5.1) and [Gu5], Corollary 7.3. Hence in the special case  $X = A$ , no lower dimensional simplices occur in Theorem 1.2. On the other hand, the quite natural Example 7.4 in [Gu5] shows that lower dimensional simplices are possible in all dimensions  $d-b, \dots, d-e-1$ .

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### Terminology

In  $A \subset B$ ,  $A$  may be equal to  $B$ . The complement of  $A$  in  $B$  is denoted by  $B \setminus A$  as we reserve  $-$  for algebraic purposes. The zero is included in  $\mathbb{N}$ . We use  $\overline{K}$  for an algebraic closure of a field  $K$ .

A variety over  $K$  is a separated reduced scheme of finite type. The group of cycles of pure dimension  $t$  is denoted by  $Z_t(X)$ . If  $L$  is a line bundle on a projective variety  $X$ , then  $\deg_L(X)$  is the degree of  $X$  with respect to  $L$ . If some positive tensor power of  $L$  is generated by global sections, then  $L$  is called *semiample*.

## 2 Chambert-Loir's measures

In this section,  $K_v$  is a field complete with respect to the discrete valuation  $v$ . For analytic considerations, we will work over the completion  $\mathbb{K}_v$  of an algebraic closure of  $K_v$ . This is an algebraically closed field with algebraically closed residue field (see [BGR], §3.4). The field  $\mathbb{K}_v$  plays a similar role as  $\mathbb{C}$  for archimedean places of number fields. The unique extension of  $v$  to a valuation of  $\mathbb{K}_v$  is also denoted by  $v$ . We fix a constant  $c \in (0, 1)$  and we use the absolute value  $|\cdot|_v := c^{-v}$  to define  $v$ -norms  $\|\cdot\|_v$  on a  $\mathbb{K}_v$ -vector space. We denote the valuation ring of  $\mathbb{K}_v$  by  $\mathbb{K}_v^\circ$  and similarly we proceed for subfields of  $\mathbb{K}_v$ .

**2.1** In the following, we consider a projective scheme  $X$  over  $K_v$ . We denote the *associated analytic space* by  $X_v^{\text{an}}$ . Here, we use Berkovich's construction which behaves similar as the complex analytique analogue. Most algebraic properties hold also analytically and conversely, there is a GAGA principle. For details, we refer to [Ber1], 3.4.

**2.2** Let  $L$  be a line bundle on  $X$ . A  $v$ -metric  $\|\cdot\|_v$  on  $L$  is a continuous family of  $v$ -norms on the fibres of  $L_v^{\text{an}}$ . Two  $v$ -metrics  $\|\cdot\|_v, \|\cdot\|_{v'}$  on  $L$  induce a metric  $\|\cdot\|_{v'/\|\cdot\|_v}$  on  $O_X$  and evaluation at the constant section 1 leads to a continuous nowhere vanishing function  $g := \|\cdot\|_{v'/\|\cdot\|_v}(1)$  on  $X_v^{\text{an}}$ . The *distance of uniform convergence* is defined by

$$d_v(\|\cdot\|_v, \|\cdot\|_{v'}) := \sup_{x \in X_v^{\text{an}}} |\log(g(x))|.$$

**2.3** A projective scheme  $\mathcal{X}$  over the valuation ring  $K_v^\circ$  with generic fibre  $X$  is called an *algebraic  $K_v^\circ$ -model* of  $X$ . If the line bundle  $\mathcal{L}$  on  $\mathcal{X}$  is an algebraic  $K_v^\circ$ -model of  $L$ , then we get a natural  $v$ -metric  $\|\cdot\|_{\mathcal{L}}$  on  $L$  by setting  $\|s(x)\|_{\mathcal{L}} = 1$  for any local trivialization  $s$  of  $\mathcal{L}$ . We call  $\|\cdot\|_{\mathcal{L}}$  the *algebraic  $v$ -metric* associated to  $\mathcal{L}$ .

More generally, we may consider models in the category of admissible formal schemes over  $\mathbb{K}_v^\circ$  leading to *formal  $v$ -metrics* on  $L$ . A formal  $v$ -metric  $\|\cdot\|_{\mathcal{L}}$  associated to the formal  $\mathbb{K}_v^\circ$ -model  $\mathcal{L}$  is called *semipositive* if the reduction  $\tilde{\mathcal{L}}$  modulo  $v$  is a numerically effective line bundle. This formal point of view is suitable in the analytic context. We refer to [Gu1], §7, for more details. We will see in Proposition 3.4 that every formal  $v$ -metric is algebraic over a finite base change of  $K_v$ .

If  $\|\cdot\|_v$  is any formal  $v$ -metric on  $O_X$ , then  $\log\|1\|_v$  is called a *formal function* on  $X_v^{\text{an}}$ . The  $\mathbb{Q}$ -subspace  $\{\frac{1}{N}f \mid N \in \mathbb{N} \setminus \{0\}, f \text{ formal function}\}$  is dense in  $C(X_v^{\text{an}})$  (see [Gu1], Theorem 7.12).

A  $v$ -metric  $\|\cdot\|_v$  on  $L$  is called a *root of a formal  $v$ -metric* if some positive tensor power is a formal  $v$ -metric.

**2.4** A  $v$ -metric  $\|\cdot\|$  on  $L$  is called a *semipositive admissible  $v$ -metric* if  $\|\cdot\|$  is the uniform limit of roots of semipositive formal  $v$ -metrics on  $L$ . A  $v$ -metric  $\|\cdot\|_v$  of  $L$  is called *admissible* if there are line bundles  $L_1, L_2$  on  $X$  with  $\varphi^*(L) \cong L_1 \otimes L_2^{-1}$  such that  $\|\cdot\|_v = \|\cdot\|_1 / \|\cdot\|_2$  for semipositive admissible  $v$ -metrics  $\|\cdot\|_i$  of  $L_i$ .

We note that admissible  $v$ -metrics are closed under pull-back and tensor product. Every formal  $v$ -metric is admissible (see [Gu2], Proposition 10.4). Moreover, the canonical metrics on line bundles of an abelian variety over  $K_v$  are admissible. This is proved by a variant of Tate's limit argument and is the very reason why we have allowed uniform limits and roots in the definition of semipositive admissible  $v$ -metrics (see [Gu5], Example 3.7).

For details, we refer to [Gu5], §3.

**2.5** In non-archimedean analysis, no good definition of Chern forms of  $v$ -metrized line bundles is known. However, Chambert-Loir [Ch] has introduced a measure which is the analogue of top-dimensional wedge products of such Chern forms. Using a slight generalization of his construction and  $d := \dim(X)$ , we get a regular Borel measure  $c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d})$  on  $X_v^{\text{an}}$  with respect to admissibly  $v$ -metrized line bundles  $\overline{L_1}, \dots, \overline{L_d}$ , at least if  $X$  is geometrically integral (see [Gu5], Proposition 3.8). Passing to a finite base extension and proceeding by linearity in the components, we deduce the following result:

**Proposition 2.6** *Let  $L_1, \dots, L_t$  be line bundles on the projective scheme  $X$  over  $K_v$  endowed with admissible  $v$ -metrics  $\|\cdot\|_1, \dots, \|\cdot\|_t$ . For every cycle  $Z \in Z_t(X)$*

and every continuous function  $g$  on  $X_v^{\text{an}}$ , we have a real integral  $\int_{Z_v^{\text{an}}} g c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_t)$  with the following properties:

- (a) The integral defines a bounded linear functional on  $C(X_v^{\text{an}})$  and hence it is induced by a regular Borel measure on  $X_v^{\text{an}}$  with support in  $\text{supp}(Z)$ .
- (b) The integral is a multilinear symmetric function of  $\overline{L}_1 := (L_1, \|\cdot\|_1), \dots, \overline{L}_t := (L_t, \|\cdot\|_t)$  and linear in  $Z$ .
- (c) If  $\varphi : Y \rightarrow X$  is a morphism projective schemes over  $K_v$  and  $W \in Z_t(Y)$ , then

$$\int_{\varphi_*(W)_v^{\text{an}}} g c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_t) = \int_{W_v^{\text{an}}} g \circ \varphi c_1(\varphi^* \overline{L}_1) \wedge \cdots \wedge c_1(\varphi^* \overline{L}_t).$$

- (d) If  $\|\cdot\|_1, \dots, \|\cdot\|_t$  are semipositive,  $Z \geq 0$  and  $\mu := c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_t)$ , then

$$\left| \int_{Z_v^{\text{an}}} g \mu \right| \leq \int_{Z_v^{\text{an}}} |g| \mu \leq |g|_{\text{sup}} \deg_{L_1, \dots, L_t}(Z).$$

- (e) We have  $\int_{Z_v^{\text{an}}} c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_t) = \deg_{L_1, \dots, L_t}(Z)$ .
- (f) For  $j \in \{1, \dots, t\}$ , let  $\|\cdot\|_{j,n}$  be a sequence of semipositive admissible  $v$ -metrics on  $L_j$  converging uniformly to  $\|\cdot\|_j$ . Then

$$\lim_{n \rightarrow \infty} \int_{Z_v^{\text{an}}} g c_1(L_1, \|\cdot\|_{1,n}) \wedge \cdots \wedge c_1(L_t, \|\cdot\|_{t,n}) = \int_{Z_v^{\text{an}}} g c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_t).$$

### 3 Heights

Let  $K = k(B)$  be the function field of an integral projective variety  $B$  over the field  $k$  such that  $B$  is regular in codimension 1. A place  $v$  on  $K$  corresponds to a prime divisor  $Y$  on  $B$  and  $v$  is the vanishing order along  $Y$ . By abuse of notation, we use  $v$  also for the generic point of  $Y$  and hence the set  $M_B$  of places on  $K$  may be viewed as a subset of  $B$ . We fix an ample class  $\mathbf{c} \in \text{Pic}(B)$ . If we count every  $v \in M_B$  with multiplicity  $\deg_{\mathbf{c}}(\overline{v})$ , then we get a product formula on  $K$  for the discrete absolute values  $|\cdot|_v := e^{-\text{ord}_v(\cdot)}$  and hence a theory of heights which we now briefly describe in this section.

**3.1** Let  $L$  be a line bundle on a projective scheme  $X$  over  $K$ . For  $v \in M_B$ , we apply the local theory from §2 to the line bundle  $L_v := L \otimes_K K_v$  over  $X_v := X \otimes_K K_v$ . We define an *admissible  $M_B$ -metric*  $\|\cdot\|$  on  $L$  as a family  $\|\cdot\| := (\|\cdot\|_v)_{v \in M_B}$  of admissible  $v$ -metrics  $\|\cdot\|_v$  on  $L_v := L \otimes_K K_v$  satisfying the following hypothesis: There is an open dense subset  $V$  of  $B$ , a projective scheme  $\mathcal{X}$  over  $V$  with generic fibre  $X$ ,  $N \in \mathbb{N} \setminus \{0\}$  and a line bundle  $\mathcal{L}$  on  $\mathcal{X}$  such that  $L^{\otimes N} = \mathcal{L}|_X$  and such that  $\|\cdot\|_v^{\otimes N}$  is the algebraic  $v$ -metric associated to  $\mathcal{L}_v := \mathcal{L} \otimes_{K_v} K_v^\circ$  at all places  $v \in M_B \cap V$ .

The admissible  $M_B$ -metric  $\|\cdot\|$  on  $L$  is called *semipositive* if  $\|\cdot\|_v$  is a semipositive admissible  $v$ -metric for all  $v \in M_B$ . Note that this definition of semipositivity is weaker than the one given in [Gu4], 3.2.

It is clear that the pull-back and the tensor product of (semipositive) admissible  $M_B$ -metrics are again (semipositive) admissible  $M_B$ -metrics. We note that for any non-zero element  $s$  in a fibre of  $L$ , there are only finitely many  $v \in M_B$  with  $\|s\|_v \neq 1$ .

The canonical metric of a line bundle on an abelian variety over  $K$  is  $M_B$ -admissible. This is proved in [Gu4], 3.5, for ample symmetric line bundles. For odd line bundles, this follows similarly using [Gu5], Example 3.7, and the general case follows by linearity. The arguments will be generalized in 6.1 for arbitrary dynamical systems.

**3.2** An admissible  $M_B$ -metric  $\|\cdot\|$  on  $L$  is called *algebraic* if  $V = B$  and  $N = 1$  in the above definition. A *formal  $M_B$ -metric* is an admissible  $M_B$ -metric  $\|\cdot\|$  on  $L$  such that  $\|\cdot\|_v$  is a formal  $v$ -metric for every  $v \in M_B$ .

Every line bundle  $L$  is isomorphic to the difference of two very ample line bundles and hence we deduce that  $L$  has an algebraic  $M_B$ -metric.

**3.3** We are interested in the absolute height for points or more generally for cycles defined over an algebraic closure  $\overline{K}$ . Every finite subextension  $K'/K$  is a function field  $K' = k(B')$ , where  $B'$  is an irreducible normal projective variety with a finite surjective morphism  $p : B' \rightarrow B$  leading to the finite extension  $K'/K$  (see [BG], Lemma 1.4.10). On  $K'$ , we use the discrete absolute values extending those of  $K$ , i.e. for  $w \in M_{K'}$  with ramification index  $e(w/v)$  over  $v \in M_K$ , we consider the absolute value

$$|f|_w := e^{-\text{ord}_w(f)/e(w/v)}$$

on  $K'$ . We consider the ample class  $\mathbf{c}' := p^*(\mathbf{c})$  on  $B'$  and we count every  $w \in M_{K'}$  with multiplicity

$$\mu(w) := \frac{e(w/v) \deg_{\mathbf{c}'}(\overline{w})}{[K' : K]} = \frac{[K'_w : K_v] \deg_{\mathbf{c}}(\overline{w})}{[K' : K]},$$

where  $[K'_w : K_v]$  is the degree of the completions (see [BG], Example 1.4.13).

We show next that every admissible  $M_B$ -metric  $\|\cdot\|$  on  $L$  induces a canonical  $M_{B'}$ -metric on  $L' := L \otimes_K K'$  which we call the *base change* of  $\|\cdot\|$  to  $L'$ . Every  $w \in M_{B'}$  is lying over a unique  $v \in M_B$ . Using the identification  $\mathbb{K}_v = \mathbb{K}_w$ , the  $v$ -metric on  $L$  is a  $w$ -metric. The metric  $\|\cdot\|_w$  is induced by the base change of the algebraic model for  $v$  in a dense open subset and hence we get a canonical  $M_{B'}$ -admissible metric on  $L'$ . If  $\|\cdot\|$  is semipositive, then it is clear that the base change of  $\|\cdot\|$  to  $L'$  is also semipositive.

The following result clarifies the relation between formal and algebraic metrics. The arguments are analogous to Yuan's Lemma 5.5. Since  $\mathbb{K}_v^\circ$  is not noetherian, additional care is needed here and we give the full proof.

**Proposition 3.4** *On a projective scheme  $X$  over  $K$ , every formal metric is algebraic after a suitable finite base extension  $K'/K$ .*

**Proof:** Let  $\|\cdot\|$  be an admissible  $M_B$ -metric on the line bundle  $L$  of  $X$ . Since  $L$  has an algebraic metric (see 3.2), we may assume that  $L = O_X$ . There is a dense open subset  $V$  of  $B$  such that  $\|\cdot\|_v$  is the trivial metric on  $O_X$  for every  $v \in M_B \cap V$ . For  $v \notin V$ ,  $\|\cdot\|_v$  is the formal  $v$ -metric on  $O_X$  induced by a formal  $\mathbb{K}_v^\circ$ -model  $\mathcal{L}_v$  of  $O_X$ . The line bundle  $\mathcal{L}_v$  is living on a formal  $\mathbb{K}_v^\circ$ -model  $\mathcal{X}_v$  of  $X_v^{\text{an}}$ . We choose a projective  $B$ -model  $\mathcal{Y}$  of  $X$ . By [BL], §4, we may assume that  $\mathcal{X}_v$  is the admissible formal blowing up of  $\mathcal{Y}_v := \mathcal{Y} \times_B \mathbb{K}_v^\circ$  in a coherent ideal sheaf  $\mathcal{I}_v$  supported in the special fibre over  $v$ . By the formal GAGA principle ([Ullr], Theorem 6.8),  $\mathcal{I}_v$  is indeed defined algebraically and hence  $\mathcal{X}_v$  is the formal completion of a projective  $\mathbb{K}_v^\circ$ -model  $\mathcal{X}_v^{\text{alg}}$  along the special fibre. The formal GAGA principle again shows that  $\mathcal{L}_v$  is the formal completion of an algebraic  $\mathbb{K}_v^\circ$ -model  $\mathcal{L}_v^{\text{alg}}$  of  $L \otimes_K \mathbb{K}_v$  on  $\mathcal{X}_v^{\text{alg}}$ .

We choose a  $K$ -embedding  $\sigma : \overline{K} \rightarrow \mathbb{K}_v$  and hence we get a place  $u$  of  $\overline{K}$ . We note that  $\mathbb{K}_v$  is the completion of  $\overline{K}^\sigma := \sigma(\overline{K})$  ([BGR], Proposition 3.4.1/3). Since  $\mathcal{I}_v$  is supported in the special fibre,  $\mathcal{I}_v$  contains a non-zero  $\beta \in \overline{K}^\sigma$  with  $|\beta|_v < 1$ . We easily deduce that  $\mathcal{I}_v$  may be generated by homogeneous polynomials with coefficients in  $\overline{K}^\sigma \cap \mathbb{K}_v^\circ$  and hence  $\mathcal{X}_v^{\text{alg}}$  is defined over  $\overline{K}^\sigma \cap \mathbb{K}_v^\circ$ . The line bundle  $\mathcal{L}_v^{\text{alg}}$  is given by the Cartier divisor  $D_v$  on  $\mathcal{X}_v^{\text{alg}}$  associated to the meromorphic section 1. Multiplying  $D_v$  by a suitable non-zero element in  $\overline{K}^\sigma$ , we may assume that  $D_v$  is an effective Cartier divisor on  $\mathcal{X}_v^{\text{alg}}$ . Then  $O(-D_v)$  is a coherent ideal sheaf on  $\mathcal{X}_v^{\text{alg}}$  containing a sufficiently small  $\beta' \in \overline{K}^\sigma \setminus \{0\}$ . As above, we deduce that  $O(-D_v)$  is a coherent ideal sheaf defined over  $\overline{K}^\sigma \cap \mathbb{K}_v^\circ$ . Passing to a blowing up in this ideal sheaf supported in the special fibre over  $v$ , we conclude that  $\|\cdot\|_v$  is induced by an algebraic model of  $O_X$  defined over  $\overline{K}^\sigma \cap \mathbb{K}_v^\circ$ .

For a finite normal subextension  $K'/K$  of  $\overline{K}/K$ , let  $X' := X \otimes_K K'$ , let  $R_w$  be the valuation ring of the place  $w := u|_{K'}$  of  $K'$  and let  $\|\cdot\|_w$  be the  $w$ -metric of  $O_{X'}$  obtained from  $\|\cdot\|_v$  by base change. Using  $\sigma$ -transfer of the above model to  $\overline{K}$  and choosing  $K'/K$  sufficiently large, we get an algebraic model  $\mathcal{L}'_w$  of  $O_{X'}$  over  $R_w$  inducing  $\|\cdot\|_w$ . Note that every place  $w$  of  $K'$  over  $v$  is of this form for a suitable  $\sigma$  and that  $K'$  is independent of  $\sigma$  by normality. Since there are only finitely many  $v \in M_B \setminus V$ , we may assume that  $K'$  works for every such  $v$ .

Now  $K'$  is the function field of an irreducible regular projective variety  $B'$  over  $k$  such that the extension  $K'/K$  is induced by a finite morphism  $p : B' \rightarrow B$ . For a place  $w$  in the dense open subset  $V' := p^{-1}(V)$  of  $B'$ , the metric  $\|\cdot\|_w$  is induced by the trivial bundle on  $\mathcal{Y}' := \mathcal{Y} \times_B V'$ . For  $w \notin V'$ , the model  $\mathcal{L}'_w$  is defined and agrees with  $\mathcal{O}_{\mathcal{Y}'}$  over a dense open subset of  $V'$ . By glueing, we obtain an algebraic  $B'$ -model  $\mathcal{L}'$  of  $O_{X'}$  inducing the base change of  $\|\cdot\|$  to  $O_{X'}$ .  $\square$

**Theorem 3.5** *Let  $K' = k(B')$  be a finite extension of  $K$  as in 3.3. For admissibly  $M_{B'}$ -metrized line bundles  $\overline{L}_0, \dots, \overline{L}_t$  on a projective scheme  $X$  over  $K' = k(B')$ , there is a unique function  $h_{\overline{L}_0, \dots, \overline{L}_t} : Z_t(X) \rightarrow \mathbb{R}$  with the following properties:*

- (a)  $h_{\overline{L}_0, \dots, \overline{L}_t}(Z)$  is multilinear and symmetric in  $\overline{L}_0, \dots, \overline{L}_t$ , and linear in  $Z \in Z_t(X)$ .
- (b) If  $\varphi : X' \rightarrow X$  is a morphism of projective schemes over  $K'$ , then we have functoriality

$$h_{\varphi^* \overline{L}_0, \dots, \varphi^* \overline{L}_t} = h_{\overline{L}_0, \dots, \overline{L}_t} \circ \varphi_*.$$

- (c) Let us replace the admissible  $v$ -metric  $\|\cdot\|_{0,v}$  at one place  $v \in M_{B'}$  by the admissible  $v$ -metric  $\|\cdot\|'_{0,v}$  and let  $\overline{L}'_0$  be the resulting  $M_{B'}$ -admissibly metrized line bundle. Then  $\|\cdot\|'_{0,v} / \|\cdot\|_{0,v}$  is an admissible  $v$ -metric on  $O_X$  defining a continuous function  $g$  on  $X_v^{\text{an}}$  as in 2.2 and we have

$$\begin{aligned} h_{\overline{L}'_0, \overline{L}_1, \dots, \overline{L}_t}(Z) - h_{\overline{L}_0, \overline{L}_1, \dots, \overline{L}_t}(Z) \\ = -\mu(v) \int_{Z_v^{\text{an}}} \log(g) c_1(L_1, \|\cdot\|_{1,v}) \wedge \cdots \wedge c_1(L_t, \|\cdot\|_{t,v}). \end{aligned}$$

- (d) If the  $M_{B'}$ -metrics of  $\overline{L}_0, \dots, \overline{L}_t$  are algebraic, induced by the line bundles  $\mathcal{L}_0, \dots, \mathcal{L}_t$  on the projective scheme  $\pi : \mathcal{X} \rightarrow B'$  with generic fibre  $X$ , then the height of any  $t$ -dimensional prime cycle  $Z$  of  $X$  with closure  $\overline{Z}$  in  $\mathcal{X}$  is given as an intersection number on  $\mathcal{X}$  by

$$h_{\overline{L}_0, \dots, \overline{L}_t}(Z) = \frac{1}{[K' : K]} \deg_{c'}(\pi_* (c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_t) \overline{Z})).$$

(e) If  $P \in X(K')$  and  $\bar{L}$  is an admissibly  $M_{B'}$ -metrized line bundle on  $X$ , then

$$h_{\bar{L}}(P) = - \sum_{v \in M_{B'}} \mu(v) \log \|s(P)\|_v$$

for any non-zero  $s(P)$  in the fibre of  $L$  over  $P$ .

**Proof:** By multilinearity, we may assume that  $L_0, \dots, L_t$  are generated by global sections. In §11 of [Gu2],  $h_{\bar{L}_0, \dots, \bar{L}_t}(Z)$  was defined for any field with product formula. It is shown that (a),(b) and (e) hold. Property (c) is true by linearity in  $\bar{L}_0$  and by definition of the right hand side as a local height of  $Z$  (see [Gu3], 3.8). For algebraic metrics, the local heights in [Gu2] are intersection numbers and (d) follows from the normalizations in 3.3.

Every line bundle has an algebraic  $M_{B'}$ -metric. By (a) and (c), we conclude that it is enough to consider algebraic  $M_{B'}$ -metrics and hence uniqueness follows from (d).  $\square$

**Definition 3.6** For  $Z \in Z_t(X)$ , we call  $h_{\bar{L}_0, \dots, \bar{L}_t}(Z)$  the *height* of  $Z$  with respect to the  $M_{B'}$ -metrized line bundles  $\bar{L}_0, \dots, \bar{L}_t$ . If  $\bar{L} = \bar{L}_0 = \dots = \bar{L}_t$ , then the height of  $Z$  is denoted by  $h_{\bar{L}}(Z)$ .

**Remark 3.7** We note that  $h_{\bar{L}_0, \dots, \bar{L}_t}(Z)$  is an absolute height, i.e. it is invariant under base change and hence well-defined on  $Z_t(X \otimes_{K'} \bar{K}')$ . To see this, let us consider the finite subextension  $K'' = k(B'')/K'$  of  $\bar{K}'/K'$ . Every  $w \in M_{B''}$  is lying over a unique  $v \in M_{B'}$ . Then the  $v$ -metric on  $L_j$  induces a canonical  $w$ -metric and hence we get an  $M_{B''}$ -admissible metric on  $L_j \otimes_{K'} K''$ . By (d) and the projection formula, the heights of  $Z$  and  $Z \otimes_{K'} K''$  agree with respect to algebraic  $M_{B'}$ -metrics. By uniqueness, they agree with respect to any  $M_{B'}$ -admissible metrics.

In particular, we get the height  $h_{\bar{L}}(P)$  for every  $P \in X(\bar{K}')$  with respect to an  $M_{B'}$ -metrized line bundle  $L$  on  $X$ . The distance of  $M_{B'}$ -metrics  $\|\cdot\|, \|\cdot\|'$  on  $L$  is measured by

$$d(\|\cdot\|, \|\cdot\|') := \sum_{v \in M_{B'}} \mu(v) d_v(\|\cdot\|_v, \|\cdot\|'_v)$$

using the local distance  $d_v$  of uniform convergence from 2.2. Since the metrics agree up to finitely many  $v \in M_{B'}$ , the sum is finite. By projection formula, the distance is absolute, i.e. invariant under base change of the metrics.

Recall that Weil's theorem says that the height of points is determined by the line bundle up to bounded functions. We have the following generalization for heights of cycles:

**Corollary 3.8** For  $j \in \{0, \dots, t\}$ , let  $\|\cdot\|_j, \|\cdot\|'_j$  be semipositive admissible  $M_{B'}$ -metrics on the line bundle  $L_j$  of the projective scheme  $X$  over  $K'$ . For every effective cycle  $Z \in Z_t(X)$ , we have

$$\begin{aligned} h_{(L_0, \|\cdot\|'_0), \dots, (L_t, \|\cdot\|'_t)}(Z) - h_{(L_0, \|\cdot\|_0), \dots, (L_t, \|\cdot\|_t)}(Z) \\ \leq \sum_{j=0}^t d(\|\cdot\|_j, \|\cdot\|'_j) \deg_{L_0, \dots, L_{j-1}, L_{j+1}, \dots, L_t}(Z) \end{aligned}$$

and all occurring degrees are non-negative.

**Proof:** This follows from a  $(t+1)$ -fold application of Theorem 3.5(d) and Proposition 2.6(c).  $\square$

**Example 3.9** Let  $\overline{L}_0, \dots, \overline{L}_t$  be canonically metrized line bundles on an abelian variety  $A$  over  $K$ . We have seen in 3.1 that canonical metrics are admissible  $M_B$ -metrics. Since they are determined by the choice of a rigidification, they are unique up to positive rational multiples. The product formula and Theorem 3.5 show that  $h_{\overline{L}_0, \dots, \overline{L}_t}(Z)$  does not depend on the choice of the canonical metrics. We call

$$\widehat{h}_{L_0, \dots, L_t}(Z) := h_{\overline{L}_0, \dots, \overline{L}_t}(Z)$$

the *Néron–Tate height* of  $Z \in Z_t(A_{\overline{K}})$  with respect to  $L_0, \dots, L_t$ . In particular, we get a Néron–Tate height on  $A(\overline{K})$  with respect to  $L \in \text{Pic}(A)$ . We refer to [Gu2] for the properties of  $\widehat{h}_{L_0, \dots, L_t}(Z)$  which can be easily deduced from Theorem 3.5.

**3.10** All results obtained so far hold also for a number field  $K$  using models over the ring of integers and arithmetic intersection theory. In the following, we consider special features of the function field  $K = k(B)$ .

First, we deal with the extension of the constant field  $k$ . Algebraic extensions are covered by 3.3 and since the heights are absolute, we may assume in the following that  $k$  is algebraically closed.

Let us consider a field extension  $k'/k$  and the corresponding function field  $K' := k'(B)$  of the variety  $B' := B \otimes_k k'$ . We will use always the ample class  $\mathbf{c}'$  on  $B'$  obtained from  $\mathbf{c}$  by base change. We consider a line bundle  $L$  on the projective scheme  $X$  over  $K$  endowed with an admissible  $M_B$ -metric  $\|\cdot\| = (\|\cdot\|_v)_{v \in M_B}$ . We claim that the line bundle  $L' := L \otimes_K K'$  on  $X' := X \otimes_K K'$  has a natural admissible  $M_{B'}$ -metric.

Indeed, there is an open dense subset  $V$  of  $B$  as in 3.1, where the metric  $\|\cdot\|$  is induced by the  $N$ -th root of a metric associated to a line bundle  $\mathcal{L}$  defined over  $V$ . By base change, we can use the model  $\mathcal{L}' := \mathcal{L} \otimes_k k'$  on the open dense subset  $V' := V \otimes_k k'$  to define  $\|\cdot\|'_w := \|\cdot\|_{\mathcal{L}', w}^{1/N}$  for all  $w \in M_{B'} \cap V'$ . Note that  $M_{B'} \setminus V'$  is a finite set of prime divisors  $v$  defined over  $k$  and hence there is a unique extension of  $\|\cdot\|_v$  to a  $v$ -metric on  $L'$ . It is immediate from the definitions that  $\|\cdot\|'$  is  $M_{B'}$ -admissible.

The same argument shows that  $\|\cdot\|'$  is semipositive if  $\|\cdot\|$  is semipositive. It follows from Theorem 3.5 and the compatibility of Chambert-Loir's measures with base change ([Gu3], Remark 3.10) that the height remains invariant under base change to  $K'$ .

**3.11** In the remaining part of this section, we show how we can reduce heights over the function field  $K = k(B)$  with  $\delta := \dim(B) \geq 2$  to the case of the function field of a curve. This is a classical process using a generic curve in  $B$ .

Again, we may assume that  $k$  is algebraically closed. Replacing  $\mathbf{c}$  by a tensor power, we may assume that  $\mathbf{c}$  is very ample. We choose a basis  $t_0, \dots, t_N$  of global sections of a line bundle representing  $\mathbf{c}$ . Let  $\xi = (\xi_j^{(i)})$  with algebraically independent entries  $\xi_j^{(i)}$ ,  $i = 1, \dots, \delta - 1$ ,  $j = 0, \dots, N$ , and let  $\eta$  be the vector obtained from  $\xi$  by omitting  $\xi_0^{(i)}$ ,  $i = 1, \dots, \delta - 1$ . Let  $B'$  (resp.  $B''$ ) be the base change of  $B$  to  $k' := k(\eta)$  (resp.  $k'' := k(\xi)$ ). Then  $B'$  and  $B''$  are geometrically integral projective varieties which are geometrically regular in codimension 1. The *generic curve*  $B_{\mathbf{c}}$  is obtained by

$$B_{\mathbf{c}} := \text{div} \left( \xi_0^{(1)} t_0 + \dots + \xi_N^{(1)} t_N \right) \dots \text{div} \left( \xi_0^{(\delta-1)} t_0 + \dots + \xi_N^{(\delta-1)} t_N \right) \cdot B''.$$

By [Lan], Section VII.6,  $B_{\mathbf{c}}$  is a geometrically irreducible smooth projective curve over  $k''$ . By construction, we have

$$k'(B') = k''(B_{\mathbf{c}}). \quad (1)$$

Another way to see this is by considering the generic projection  $\pi : B' \dashrightarrow \mathbb{P}_{k'}^{\delta-1}$ , given by

$$\pi[x_0 : \cdots : x_N] = [-x_0 : \xi_1^{(1)}x_1 + \cdots + \xi_N^{(1)}x_N : \cdots : \xi_1^{(\delta-1)}x_1 + \cdots + \xi_N^{(\delta-1)}x_N].$$

Then the fibre over the generic point  $[1 : \xi_0^{(1)} : \cdots : \xi_0^{(\delta-1)}]$  of  $\mathbb{P}_{k'}^{\delta-1}$  is  $B_{\mathbf{c}}$ . Hence  $B_{\mathbf{c}}$  is a dense subset of  $B'$  with the same function field. Moreover, we get  $M_{B_{\mathbf{c}}} = M_{B'} \cap B_{\mathbf{c}}$ .

**3.12** Now we apply 3.10 and 3.11 to a line bundle  $L$  on the projective scheme  $X$  over  $K$  endowed with an admissible  $M_B$ -metric  $\|\cdot\|$ . We consider the base change to the function field  $K' := k'(B')$ , where  $k'$  and  $B'$  are as in 3.11, leading to a line bundle  $L' := L \otimes_K K'$  endowed with a natural metric  $\|\cdot\|'$  (see 3.10). By (1),  $L'$  and  $X'$  are defined over  $k''(B_{\mathbf{c}})$  and by restriction, we get a natural  $M_{B_{\mathbf{c}}}$ -metric  $\|\cdot\|_{\mathbf{c}}$  on  $L'$ . Since  $\pi$  is a generic projection, the base change  $w \in M_{B'}$  of  $v \in M_B$  is contained in  $B_{\mathbf{c}}$  and hence  $w$  is a closed point of  $B_{\mathbf{c}}$  with valuation ring  $\mathcal{O}_{B_{\mathbf{c}},w} = \mathcal{O}_{B',w} = \mathcal{O}_{B,v} \otimes_k k'$  in  $k''(B_{\mathbf{c}})$ . We claim that

$$h_{(L_0, \|\cdot\|_0), \dots, (L_t, \|\cdot\|_t)}(Z) = h_{(L'_0, \|\cdot\|_{0,\mathbf{c}}), \dots, (L'_t, \|\cdot\|_{t,\mathbf{c}})}(Z') \quad (2)$$

for every  $Z \in Z_t(X)$  with  $Z' := Z \otimes_K K'$  and line bundles  $L_0, \dots, L_t$  with admissible  $M_B$ -metrics  $\|\cdot\|_0, \dots, \|\cdot\|_t$ .

**Proof:** We deduce from 3.10 that the height of  $Z$  is equal to the height of  $Z'$  with respect to the function field of  $B'$ . We note that only places  $w$  of  $B'$  coming from places  $v$  of  $B$  contribute to the height of  $Z'$  and hence we get (2). More precisely, the height is given as a sum of local heights  $\lambda(Z, v)$  with respect to suitable meromorphic sections  $s_0, \dots, s_t$  (see [Gu2], §11). Over a dense open subset  $V$  of  $B$ ,  $\lambda(Z, v)$  is given as an intersection number on an algebraic model. Using base change to the dense open subset  $V' := V \otimes_k k'$  of  $B'$ , it is clear that  $\lambda(Z, w) \neq 0$  only for places  $w$  of  $V'$  which are induced by places  $v$  of  $V$ . Since the complement of  $V'$  in  $B'$  is also defined over  $k$  and every irreducible component of  $B' \setminus V'$  is the base change of an irreducible component of  $B \setminus V$ , we get the claim.  $\square$

## 4 Big line bundles

In this section, we recall some facts about big line bundles on a  $t$ -dimensional irreducible projective variety  $\mathcal{X}$  over any field  $k$ . For proofs, we refer to [Laz], §2.2. Note that Lazarsfeld states the results for projective varieties over an algebraically closed field of characteristic 0, but the arguments hold in the more general setting.

**Definition 4.1** *A line bundle  $\mathcal{L}$  on  $\mathcal{X}$  is called big if there is  $C > 0$  such that*

$$h^0(\mathcal{L}^{\otimes m}) := \dim H^0(\mathcal{X}, \mathcal{L}^{\otimes m}) \geq Cm^t$$

for all sufficiently large  $m \in \mathbb{N}$ .

**4.2** Let  $\varphi : \mathcal{X}' \rightarrow \mathcal{X}$  be a surjective morphism of irreducible projective varieties over  $k$ . We suppose that  $\varphi$  is generically finite. If  $\mathcal{L}$  is big on  $\mathcal{X}$ , then  $\varphi^*(\mathcal{L})$  is big on  $\mathcal{X}'$ . The converse holds for  $\varphi$  birational. By passing to the normalization, we may always reduce to the case of normal varieties.

The next result is called *Kodaira's lemma*.

**Lemma 4.3** *Let  $D$  be a Cartier divisor on the irreducible projective variety  $\mathcal{X}$  over  $k$ . Then the following statements are equivalent:*

- (a)  $O(D)$  is big.
- (b) For all ample divisors  $H$  on  $\mathcal{X}$ , there is a non-zero  $m \in \mathbb{N}$  and an effective Cartier divisor  $E$  on  $\mathcal{X}$  such that  $mD$  is rationally equivalent to the Cartier divisor  $H + E$ .
- (c) There is an ample divisor  $H$  on  $\mathcal{X}$ , a non-zero  $m \in \mathbb{N}$  and an effective Cartier divisor  $E$  on  $\mathcal{X}$  such that  $mD$  is numerically equivalent to  $H + E$ .

**4.4** If  $\mathcal{L}$  is a big line bundle on the projective variety  $\mathcal{X}$ , then there is a proper closed subset  $\mathcal{V}$  of  $\mathcal{X}$  such that for all irreducible closed subvarieties  $\mathcal{Y}$  of  $\mathcal{X}$  not contained in  $\mathcal{V}$ , the restriction  $\mathcal{L}|_{\mathcal{Y}}$  is big.

The main result of this section is *Siu's theorem*:

**Theorem 4.5** For nef Cartier divisors  $D, E$  on  $\mathcal{X}$  and  $m \in \mathbb{N}$ , we have

$$h^0(O(m(D - E))) \geq \frac{1}{t!}(D^t - tD^{t-1} \cdot E)m^t + o(m^t).$$

Of course, this is most useful if the intersection number  $D^t - tD^{t-1} \cdot E$  is positive. Then it follows that  $O(D - E)$  is big.

**Theorem 4.6** Let  $L$  be a numerically effective line bundle on  $\mathcal{X}$ . Then  $L$  is big if and only if  $\deg_L(\mathcal{X}) > 0$ .

In particular, the degree of  $\mathcal{X}$  with respect to a big semiample line bundle is positive.

## 5 The fundamental inequality

In this section, we prove the fundamental inequality which is a central tool to prove the equidistribution theorem in the next section. Through almost the whole section, we assume that  $K$  is the function field of an irreducible regular projective curve  $C$  over the algebraically closed field  $k$ . Only at the end, we will generalize the fundamental inequality to arbitrary function fields using reduction to the generic curve.

We fix a big semiample line bundle  $L$  on the  $d$ -dimensional irreducible projective variety  $X$  over  $K$ .

**Definition 5.1** The *essential minimum* of  $X$  with respect to the  $M_C$ -admissibly metrized line bundle  $\bar{L}$  is defined by

$$e_1(X, \bar{L}) := \sup_Y \inf_{P \in X(\bar{K}) \setminus Y} h_{\bar{L}}(P),$$

where  $Y$  ranges over all closed subsets of codimension 1 in  $X$ .

**5.2** The semipositive admissible  $M_C$ -metric  $\|\cdot\|$  of  $\bar{L}$  is called *nef* if  $h_{\bar{L}}(P) \geq 0$  for all  $P \in X(\bar{K})$ . If  $\|\cdot\|$  is an algebraic metric induced by a line bundle  $\mathcal{L}$  on a projective  $C$ -model  $\mathcal{X}$  of  $X$ , then this means that the restrictions of  $\mathcal{L}$  to all horizontal curves in  $\mathcal{X}$  have non-negative degrees. By semipositivity, the same holds for vertical curves in  $\mathcal{X}$  and hence  $\|\cdot\|_{\mathcal{L}}$  is nef if and only if  $\mathcal{L}$  is nef.

**Lemma 5.3** Let  $\|\cdot\|_0$  be a semipositive admissible  $M_C$ -metric on  $L$ . Then there is an ample line bundle  $\mathcal{M}$  on  $C$  such that  $\|\cdot\|_0 \otimes \pi^* \|\cdot\|_{\mathcal{M}}$  is a nef metric on  $L$ , where  $\pi : X \rightarrow \text{Spec}(K)$  is the morphism of structure.

**Proof:** Passing to a positive tensor power, we may assume that  $L$  is generated by global sections and that there is an open dense affine subset  $V$  of  $C$  and a projective model  $\pi : \mathcal{X} \rightarrow V$  of  $X$  with a line bundle  $\mathcal{L}$  on  $\mathcal{X}$  such that  $\mathcal{L}$  is a model of  $L$  with  $\| \cdot \|_{0,v} = \| \cdot \|_{\mathcal{L},v}$  for all  $v \in V$ . Let  $s_1, \dots, s_n$  be a basis of  $H^0(X, L)$ . Using that  $X = \mathcal{X} \otimes_V K$  is obtained by flat base change, we get

$$H^0(\pi^{-1}(V), \mathcal{L}) \otimes_{\mathcal{O}(V)} K = H^0(X, L).$$

By passing to a smaller  $V$ , we may also assume that  $s_1, \dots, s_n \in H^0(\pi^{-1}(V), \mathcal{L})$ . For a closed point  $v$  of  $C$ ,  $\|s_j\|_{0,v}$  is bounded. There is  $R > 0$  such that  $\|s_j\|_{0,v} \leq R$  for all  $v \in C \setminus V$  and  $j = 1, \dots, n$ . We consider the divisor

$$D := \sum_{v \in C \setminus V} m \cdot [v]$$

for a multiplicity  $m \geq \log(R)$ . Then  $s_D$  is a global section of  $\mathcal{M} := \mathcal{O}(D)$  with  $m = -\log \|s_D\|_{\mathcal{M},v}$  for all  $v \in C \setminus V$ . For  $P \in X(\overline{K})$ , there is  $j \in \{1, \dots, n\}$  with  $s_j(P) \neq 0$ . There is an irreducible regular projective curve  $C'$  and a finite morphism  $p : C' \rightarrow C$  such that  $P \in X(K')$  for the function field  $K' = k(C')$ . By Theorem 3.5(e), we get

$$\begin{aligned} h_{(L, \| \cdot \|_{0 \otimes \pi^*} \| \cdot \|_{\mathcal{M}})}(P) &= - \sum_{w \in C'} \mu(w) (\log \|s_j(P)\|_{0,w} + \log \|s_D\|_{\mathcal{M},w}) \\ &\geq - \sum_{w \in C' \setminus p^{-1}(V)} \mu(w) (\log \|s_j(P)\|_{0,w} - m) \geq 0 \end{aligned}$$

and hence  $\| \cdot \|_{0 \otimes \pi^*} \| \cdot \|_{\mathcal{M}}$  is nef.  $\square$

Using the distance  $d_v$  of uniform convergence from 2.2, we have the following uniform version of the above result:

**Corollary 5.4** *Let  $\| \cdot \|_0$  be a semipositive admissible  $M_C$ -metric on  $L$ . Then there is an ample line bundle  $\mathcal{M}$  on  $C$  such that for every semipositive admissible  $M_C$ -metric  $\| \cdot \|$  on  $L$ , we get a semipositive admissible  $M_C$ -metric*

$$\| \cdot \|' := \| \cdot \|_{0 \otimes \pi^*} \| \cdot \|_{\mathcal{M}} \otimes \prod_{v \in M_C} \pi^* \| \cdot \|_{\mathcal{O}(v)}^{d_v(\| \cdot \|_v, \| \cdot \|_{0,v})} \quad (3)$$

on  $L$  which is nef.

**Proof:** Since a positive tensor power of the metrics  $\| \cdot \|_0, \| \cdot \|$  is given by models over an open dense subset of  $C$ , the metrics agree on a smaller open dense subset of  $C$  and hence the product in (3) is finite. Using the notations from the proof of Lemma 5.3, we have

$$\begin{aligned} h_{(L, \| \cdot \|')} (P) &= - \sum_{w \in C'} \mu(w) \left( \log \|s_j(P)\|_w + \log \|s_D\|_{\mathcal{M},w} - d_w(\| \cdot \|_w, \| \cdot \|_{0,w}) \right) \\ &\geq - \sum_{w \in C'} \mu(w) (\log \|s_j(P)\|_{0,w} + \log \|s_D\|_{\mathcal{M},w}) \geq 0 \end{aligned}$$

and hence  $(L, \| \cdot \|')$  is nef.  $\square$

**5.5** Let  $\pi : \mathcal{X} \rightarrow C$  be a model of  $X$ , i.e.  $\pi$  is a projective flat morphism and  $X$  is the generic fibre of  $\mathcal{X}$ . Let  $\mathcal{N}$  be a line bundle on  $\mathcal{X}$  which is trivial on  $X$ . We consider models  $\mathcal{L}$  of  $L$  on  $\mathcal{X}$  which are vertically nef, i.e. the restriction of  $\mathcal{L}$  to the fibre over  $v$  is nef for all closed points  $v$  of  $C$ . For  $\varepsilon \in \mathbb{Q}$ , we may view  $\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}$

as an element of  $\text{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$  leading to well-defined heights and degrees using multilinearity. We will use  $h_{\mathcal{L}} := h_{(L, \|\cdot\|_{\mathcal{L}})}$  and  $e_1(X, \mathcal{L}) := e_1(X, (L, \|\cdot\|_{\mathcal{L}}))$ .

In the following lemma, we measure uniformity with respect to  $\mathcal{L}$  by  $d(\mathcal{L}) := d(\|\cdot\|_{\mathcal{L}}, \|\cdot\|_0)$  using the distance to a fixed semipositive admissible  $M_C$ -metric  $\|\cdot\|_0$  on  $L$  (see Remark 3.7).

**Lemma 5.6** *Under the hypothesis above and for  $|\varepsilon| \leq 1$ , we have*

$$\begin{aligned} & h^0(\mathcal{X}, (\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}) \\ & \geq \frac{1}{(d+1)!} (\deg_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(\mathcal{X}) + (d(\mathcal{L}) + 1)O(\varepsilon^2)) m^{d+1} + o_{\varepsilon, \mathcal{L}}(m^{d+1}) \end{aligned} \quad (4)$$

for sufficiently large and sufficiently divisible  $m \in \mathbb{N}$ , where the implicit constant in  $O(\varepsilon^2)$  may depend on  $L$  and  $\mathcal{N}$  but is independent of  $\mathcal{L}$ ,  $m$  and  $\varepsilon$ .

**Proof:** Replacing  $\mathcal{N}$  by  $\mathcal{N}^{-1}$  if necessary, we may assume that  $\varepsilon \geq 0$ . There are semiample line bundles  $\mathcal{L}_1, \mathcal{L}_2$  on  $\mathcal{X}$  with  $\mathcal{N} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ . As usual, we denote the generic fibre of  $\mathcal{L}_i$  by  $L_i$ . First, we prove the lemma under the additional assumption that  $\mathcal{L}$  is nef. Then we have the decomposition

$$\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon} \cong \mathcal{L}_+ \otimes (\mathcal{L}_-)^{-1} \quad (5)$$

for the numerically effective elements  $\mathcal{L}_+ := \mathcal{L} \otimes \mathcal{L}_1^{\otimes \varepsilon}$  and  $\mathcal{L}_- := \mathcal{L}_2^{\otimes \varepsilon}$  of  $\text{Pic}(\mathcal{X}) \otimes \mathbb{Q}$ . For  $m \in \mathbb{N}$  with  $m\varepsilon \in \mathbb{Z}$ , the  $m$ -th tensor power of (5) leads to a decomposition into numerically effective line bundles  $\mathcal{L}_+^{\otimes m}$  and  $\mathcal{L}_-^{\otimes m}$ . We note that

$$c_1(\mathcal{L}_+)^{d+1} - (d+1)c_1(\mathcal{L}_+)^d \cdot c_1(\mathcal{L}_-) = \deg_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(\mathcal{X}) + O_{\mathcal{L}}(\varepsilon^2).$$

In fact, we may choose the implicit constant in  $O_{\mathcal{L}}(\varepsilon^2)$  equal to

$$H_{\mathcal{L}} := 3^{d+1} \max |h_{\underbrace{\mathcal{L}, \dots, \mathcal{L}}_a, \underbrace{\mathcal{L}_1, \dots, \mathcal{L}_1}_b, \underbrace{\mathcal{L}_2, \dots, \mathcal{L}_2}_c}(X)|,$$

where the maximum is taken over all natural numbers with  $a + b + c = d + 1$ . Since  $\mathcal{L}$ ,  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are vertically nef, we may use Corollary 3.8 to change from  $\|\cdot\|_{\mathcal{L}}$  to  $\|\cdot\|_0$ . Then we get

$$H_{\mathcal{L}} \leq 3^{d+1} (H_0 + (d+1)d(\mathcal{L})D_L), \quad (6)$$

where  $H_0$  is defined as  $H_{\mathcal{L}}$  with  $(L, \|\cdot\|_0)$  replacing  $(L, \|\cdot\|_{\mathcal{L}})$  and where

$$D_L := \max \deg_{\underbrace{L, \dots, L}_a, \underbrace{L_1, \dots, L_1}_b, \underbrace{L_2, \dots, L_2}_c}(X),$$

with the maximum taken over all  $a + b + c = d$ . Hence we may apply Theorem 4.5 to prove (4).

Now we prove the claim in general. For  $v \in C$ , we choose positive rational approximations  $\delta_v \geq d_v(\|\cdot\|_{\mathcal{L}, v}, \|\cdot\|_{0, v})$ . We may assume that

$$\sum_{v \in C} \delta_v \leq d(\mathcal{L}) + d(\mathcal{M}), \quad (7)$$

where  $\mathcal{M}$  is the ample line bundle on  $C$  from Lemma 5.3 which is independent of  $\mathcal{L}$  and where  $d(\mathcal{M}) := d(\|\cdot\|_{\mathcal{M}}, \|\cdot\|_{O_C}) = \deg(\mathcal{M})$ . By Corollary 5.4,  $\mathcal{L}' := \mathcal{L} \otimes \pi^*(\mathcal{M}')$  is nef for

$$\mathcal{M}' := \mathcal{M} \otimes \left( \bigotimes_{v \in C} \mathcal{O}_C(\delta_v[v]) \right).$$

For  $m$  sufficiently large and sufficiently divisible, the first case shows

$$\begin{aligned} & h^0(\mathcal{X}, (\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}) \\ & \geq \frac{1}{(d+1)!} (\deg_{\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon}}(\mathcal{X}) - H_{\mathcal{L}'} \cdot \varepsilon^2) m^{d+1} + o_{\varepsilon, \mathcal{L}'}(m^{d+1}). \end{aligned} \quad (8)$$

Let  $m_0 \in \mathbb{N}$  such that  $(\mathcal{M}')^{\otimes m_0}$  is very ample. Recursively, we choose a sequence  $t_1, t_2, \dots$  of generic global sections of  $(\mathcal{M}')^{\otimes m_0}$ . We conclude that the closed subschemes  $E_i := \pi^*(\operatorname{div}(t_i))$  are disjoint on  $\mathcal{X}$ . For sufficiently large and sufficiently divisible  $m = m_0 m_1$ , we get an exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{X}, (\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}) & \rightarrow H^0(\mathcal{X}, (\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}) \\ & \rightarrow \bigoplus_{i=1}^{m_1} H^0(E_i, (\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}), \end{aligned} \quad (9)$$

where the second map is induced by tensoring with  $t_1 \otimes \dots \otimes t_{m_1}$  and the third map is induced by restriction. Since  $\mathcal{N}$  is trivial outside a finite set of fibres of  $\mathcal{X}$  over  $C$ , we conclude that  $\mathcal{N}|_{E_i}$  is trivial for all  $i$  by the generic choice of  $t_i$ . Since  $\pi^*(\mathcal{M}')|_{E_i}$  is trivial for all  $i$ , we conclude that

$$H^0(E_i, (\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}) = H^0(E_i, \mathcal{L}^{\otimes m}). \quad (10)$$

For  $m$  sufficiently large and sufficiently divisible, we claim that

$$h^0(E_i, (\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}) \leq \frac{m_0}{d!} c_1(\mathcal{L})^d \cdot c_1(\pi^* \mathcal{M}') m^d + o_{\mathcal{L}}(m^d) \quad (11)$$

with an error term independent of  $i$ . To prove this, we may assume  $\mathcal{L}'$  ample. Indeed, we may replace  $\mathcal{L}'$  by  $\mathcal{L}' \otimes \mathcal{H}^{\otimes \lambda}$  for an ample line bundle  $\mathcal{H}$  on  $\mathcal{X}$  and rational  $\lambda > 0$ . The Nakai–Moishezon criterion shows that  $\mathcal{L}' \otimes \mathcal{H}^{\otimes \lambda}$  is ample. Then continuity of the right hand side for  $\lambda \rightarrow 0$  proves (11) in general.

The ideal sheaf of  $E_i$  is isomorphic to  $\mathcal{O}_{\mathcal{X}}(-E_i) \cong \pi^*(\mathcal{M}')^{-m_0}$  and hence we get the following long exact cohomology sequence:

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{X}, \pi^*(\mathcal{M}')^{-m_0} \otimes (\mathcal{L}')^{\otimes m}) & \rightarrow H^0(\mathcal{X}, (\mathcal{L}')^{\otimes m}) \rightarrow H^0(E_i, (\mathcal{L}')^{\otimes m}) \\ & \rightarrow H^1(\mathcal{X}, \pi^*(\mathcal{M}')^{-m_0} \otimes (\mathcal{L}')^{\otimes m}) \rightarrow \dots \end{aligned}$$

Since we may assume  $\mathcal{L}'$  ample, we have  $H^1(\mathcal{X}, \pi^*(\mathcal{M}')^{-m_0} \otimes (\mathcal{L}')^{\otimes m}) = \{0\}$  for  $m$  sufficiently large and we get independence of  $h^0(E_i, (\mathcal{L}')^{\otimes m})$  from  $i$ . By our above considerations in (10), this dimension equals  $h^0(E_i, (\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m})$ . Now we apply the Hilbert–Samuel formula in (10) to prove (11) and the independence of the error term from  $i$ .

Using (11) in (9), we get

$$\begin{aligned} h^0(\mathcal{X}, (\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}) & \geq h^0(\mathcal{X}, (\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}) \\ & \quad - \frac{1}{d!} c_1(\mathcal{L})^d \cdot c_1(\pi^* \mathcal{M}') m^{d+1} + o_{\mathcal{L}}(m^{d+1}). \end{aligned} \quad (12)$$

We may use the generic sections  $t_i$  to compute the following intersection products

$$c_1(\pi^* \mathcal{M}')^2 = 0, \quad c_1(\pi^* \mathcal{M}') \cdot c_1(\mathcal{N}) = 0$$

in  $CH(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$  and hence we get

$$\deg_{\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon}}(\mathcal{X}) = \deg_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(\mathcal{X}) + (d+1) c_1(\mathcal{L})^d \cdot c_1(\pi^* \mathcal{M}'). \quad (13)$$

For  $v \in C$ , we have

$$\begin{aligned} d_v(\|\cdot\|_{\mathcal{L}',v}, \|\cdot\|_{0,v}) &= d_v(\|\cdot\|_{\mathcal{L},v}, \|\cdot\|_{0,v}) + d_v(\|\cdot\|_{\mathcal{M}',v}, \|\cdot\|_{\mathcal{O}_C,v}) \\ &= d_v(\|\cdot\|_{\mathcal{L},v}, \|\cdot\|_{0,v}) + \delta_v + d_v(\|\cdot\|_{\mathcal{M},v}, \|\cdot\|_{\mathcal{O}_C,v}) \end{aligned}$$

and hence we get

$$d(\mathcal{L}') \leq 2d(\mathcal{L}) + 2d(\mathcal{M}) \quad (14)$$

by (7). We deduce from (6) and (14) that

$$H_{\mathcal{L}'} \leq 3^{d+1} (H_0 + 2(d+1)(d(\mathcal{L}) + d(\mathcal{M})) D_L). \quad (15)$$

Using (13) and (15) in (8), we get a lower bound of  $h^0(\mathcal{X}, (\mathcal{L}' \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m})$  which we put in (12) to deduce (4).  $\square$

**Remark 5.7** The final argument shows that the error term  $(d(\mathcal{L}) + 1)O(\varepsilon^2)$  in Lemma 5.6 may be bounded by  $C_1\varepsilon^2$  for

$$C_1 := c_1(d)(H_0 + (d(\mathcal{L}) + d(\mathcal{M})) D_L). \quad (16)$$

The constant  $C_1 = C_1(\mathcal{L}, \mathcal{N}, \|\cdot\|_0)$  depends only on  $(\mathcal{L}, \mathcal{N}, \|\cdot\|_0)$  and the corresponding  $\mathcal{M}, \mathcal{L}_1, \mathcal{L}_2$  in the following way:

- (a) For  $m > 0$ , we have  $C_1(\mathcal{L}^{\otimes m}, \mathcal{N}^{\otimes m}, \|\cdot\|_0^{\otimes m}) = m^{d+1}C_1(\mathcal{L}, \mathcal{N}, \|\cdot\|_0)$ .
- (b)  $C_1$  is a universal positive continuous function in  $(d, H_0, d(\mathcal{L}), d(\mathcal{M}), D_L)$  with  $C_1 = O_{d, D_L}(d(\mathcal{L}) + d(\mathcal{M}) + H_0)$ .

**Remark 5.8** We need an absolute version of Lemma 5.6. Let  $K'$  be a finite extension of  $K$ . Then  $K'$  is the function field of an irreducible regular projective curve  $C'$  over  $k$  such the extension  $K'/K$  is induced by a finite morphism  $p: C' \rightarrow C$ .

We still have the  $C$ -model  $\mathcal{X}$  of  $X$  with the line bundle  $\mathcal{N}$  on  $\mathcal{X}$  which is trivial on the generic fibre  $X$  and a fixed semipositive admissible  $M_C$ -metric  $\|\cdot\|_0$  of  $L$ . Now we consider a model  $\mathcal{L}'$  of  $L' := L \otimes_K K'$ , defined on a  $C'$ -model  $\mathcal{X}'$  of  $X' := X \otimes_K K'$ . By Remark 3.7,  $d(\mathcal{L}') := d(\|\cdot\|_{\mathcal{L}'}, \|\cdot\|_0)$  is absolute, i.e. invariant under base change.

There is always a  $C'$ -model  $\mathcal{X}''$  of  $X'$  lying over  $\mathcal{X}$  and  $\mathcal{X}'$ . Hence we may assume  $\mathcal{X}' = \mathcal{X}''$  and that  $\mathcal{N}$  has a canonical pull-back  $\mathcal{N}'$  to  $\mathcal{X}'$ . Then Lemma 5.6 holds with  $\mathcal{X}', \mathcal{L}', \mathcal{N}'$  replacing  $\mathcal{X}, \mathcal{L}, \mathcal{N}$  and with an absolute error term  $(d(\mathcal{L}') + 1)O(\varepsilon^2)$ . Indeed, the invariants  $d, H_0, d(\mathcal{L}), d(\mathcal{M})$  and  $D_L$  are absolute and hence Remark 5.7 shows that the error term may be bounded by  $C_1\varepsilon^2$  with a constant  $C_1$  independent of  $K'$ .

In the following, we will use  $C_2, C_3, \dots$  for constants with the properties (a) and (b) from Remark 5.7 and  $c_2(d), c_3(d), \dots$  denote explicitly computable constants depending on  $d$ . Now we are ready to prove the *fundamental inequality for algebraic metrics*.

**Lemma 5.9** *We keep the assumptions and notations from 5.5. There is a constant  $c > 0$  such that for all vertically nef  $C$ -models  $\mathcal{L}$  of  $L$  on  $\mathcal{X}$  and all rational  $\varepsilon$  with  $|\varepsilon| \leq c$ , we have*

$$\frac{h_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(X)}{(d+1) \deg_L(X)} \leq e_1(X, \mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}) + (d(\mathcal{L}) + 1) O(\varepsilon^2),$$

where the implicit constant in  $O(\varepsilon^2)$  is independent of the choice of  $\mathcal{L}$  and  $\varepsilon$ . More precisely, the constants and the inequality are absolute in the sense of Remark 5.8.

**Proof:** We use the notation of the proof of Lemma 5.6. For  $r \in \mathbb{Q}$ , we consider

$$\mathcal{L}_r := \mathcal{L} \otimes \pi^*(\mathcal{M}')^{\otimes r} \in \text{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For  $m \in \mathbb{N}$  sufficiently large and sufficiently divisible, we would like to ensure the existence of a non-trivial global section  $s$  of  $(\mathcal{L}_r \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}$  using Lemma 5.6. By Remark 5.7, we have to choose  $r$  such that

$$0 < \stackrel{(1)}{h_{\mathcal{L}_r \otimes \mathcal{N}^{\otimes \varepsilon}}(X)} - C_1(\mathcal{L}_r, \mathcal{N}, \|\cdot\|_0)\varepsilon^2, \quad (17)$$

where we have used that the degree is equal to the height. The same computation as for (14) shows

$$d(\mathcal{L}_r) \leq (|r| + 1)d(\mathcal{L}) + 2|r|d(\mathcal{M})$$

and hence (16) yields

$$C_1(\mathcal{L}_r, \mathcal{N}, \|\cdot\|_0) \leq C_2|r| + C_1 \quad (18)$$

for  $C_1 := C_1(\mathcal{L}, \mathcal{N}, \|\cdot\|_0)$  and  $C_2 := c_2(d)(d(\mathcal{L}) + d(\mathcal{M}))D_L$ . Similarly as in (13) and by the projection formula, we have

$$h_{\mathcal{L}_r \otimes \mathcal{N}^{\otimes \varepsilon}}(X) = h_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(X) + r(d+1)\deg_{\mathcal{M}'}(C)\deg_L(X). \quad (19)$$

By (18) and (19), the positivity assumption (17) is satisfied if

$$0 < h_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(X) + r((d+1)\deg_{\mathcal{M}'}(C)\deg_L(X) \mp C_2\varepsilon^2) - C_1\varepsilon^2, \quad (20)$$

where the “ $-$ ” is used if and only if  $r \geq 0$ . Now we choose  $\varepsilon$  such that

$$C_2\varepsilon^2 \leq \frac{1}{2}(d+1)\deg_{\mathcal{M}'}(C)\deg_L(X). \quad (21)$$

By definition of  $\mathcal{M}'$ ,

$$R := (d+1)\deg_{\mathcal{M}'}(C)\deg_L(X)$$

is bounded below by

$$C_3 := (d+1)(d(\mathcal{L}) + d(\mathcal{M}))\deg_L(X)$$

and hence (21) holds for all

$$|\varepsilon| \leq c := \min \left\{ \sqrt{\frac{C_3}{2C_2}}, 1 \right\} = c_3(d) \cdot \sqrt{\frac{\deg_L(X)}{D_L}}. \quad (22)$$

Moreover, we get

$$\frac{1}{R} \leq \frac{1}{R - C_2\varepsilon^2} \leq \frac{1}{R} \left( 1 + \frac{2C_2\varepsilon^2}{R} \right) \leq \frac{1}{R} \left( 1 + \frac{\varepsilon^2}{c^2} \right) \quad (23)$$

and a similar estimate for  $(R + C_2\varepsilon^2)^{-1}$ . The same arguments as in the first part of the proof of Lemma 5.6 show that

$$|h_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(X)| \leq |h_{\mathcal{L}}(X)| + 2^{d+1}H_{\mathcal{L}}\varepsilon \leq c_4(d)(H_0 + d(\mathcal{L})D_L) =: C_4. \quad (24)$$

By (23) and (24), we conclude that (20) holds for

$$r > \frac{-h_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(X) + C_5\varepsilon^2}{(d+1)\deg_{\mathcal{M}'}(C)\deg_L(X)}, \quad (25)$$

where  $C_5 := 2C_1 + C_4/c^2$ . For such an  $r$ , we get a non-zero  $s \in H^0(\mathcal{X}, (\mathcal{L}_r \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m})$ . Let  $Y$  be the support of  $\text{div}(s)$ . For  $P \in X(\overline{K}) \setminus Y$ , there is a finite

morphism  $C' \rightarrow C$  of irreducible regular projective curves such that  $P \in X(K')$  for the function field  $K' = k(C')$ . By Theorem 3.5(e), we get

$$h_{\mathcal{L}_r \otimes \mathcal{N}^{\otimes \varepsilon}}(P) = -\frac{1}{m} \sum_{w \in C'} \mu(w) \log \|s(P)\|_{\mathcal{L}_r \otimes \mathcal{N}^{\otimes \varepsilon}, w} \geq 0$$

using that the metric of a global section of a model is bounded by 1. We conclude that

$$e_1(X, \mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}) + r \deg(\mathcal{M}') = e_1(X, \mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon} \otimes \pi^*(\mathcal{M}')^{\otimes r}) \geq 0. \quad (26)$$

If  $r$  approaches the right hand side of (25) and if we put this into (26), then we get

$$\frac{h_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(X)}{(d+1) \deg_L(X)} \leq e_1(X, \mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}) + S\varepsilon^2 \quad (27)$$

for  $S := C_5 / \{(d+1) \deg_L(X)\}$ . Since the constants  $C_1, C_2, \dots$  and  $c$  are absolute, we deduce as in Remark 5.8 that the fundamental inequality is absolute.  $\square$

Recall that  $L$  is a big semiample line bundle on  $X$ . Again,  $\|\cdot\|_0$  denotes a fixed semipositive admissible  $M_C$ -metric on  $L$  and  $\|\cdot\|_f$  is a formal  $M_C$ -metric on  $O_X$ .

The following variational form of the *fundamental inequality* was proved by Yuan in the number field case (with  $L$  ample, see [Yu], 5.2 and 5.3). Note that the absolute version of the error term is new here.

**Proposition 5.10** *There is an absolute constant  $c > 0$  with the following property: For every  $\varepsilon \in (-c, c)$  and every semipositive admissible  $M_C$ -metric  $\|\cdot\|$  of  $L$ , we have*

$$\frac{h_{(L, \|\cdot\| \otimes \|\cdot\|_f^{\otimes \varepsilon})}(X)}{(d+1) \deg_L(X)} \leq e_1(X, (L, \|\cdot\| \otimes \|\cdot\|_f^{\otimes \varepsilon})) + (d(\|\cdot\|, \|\cdot\|_0) + 1)O(\varepsilon^2), \quad (28)$$

where the implicit constant in  $O(\varepsilon^2)$  may depend on  $L$  and  $\|\cdot\|_f$  but is independent of  $\|\cdot\|$  and  $\varepsilon$ . Moreover, the implicit constant is absolute, i.e. it holds also for all semipositive admissible  $M_{C'}$ -metrics on  $L \otimes_K K'$  independently of the finite extension  $K' = k(C')$  of  $K$ .

**Proof:** If we use  $d(\overline{L}) := d(\|\cdot\|, \|\cdot\|_0)$  instead of  $d(\mathcal{L})$ , then property (b) in Remark 5.7 shows that the constants  $C_1, C_2, \dots$  and  $S$  from our previous results make sense also for  $\overline{L} = (L, \|\cdot\|)$ . We will prove that the fundamental inequality

$$\frac{h_{(L, \|\cdot\| \otimes \|\cdot\|_f^{\otimes \varepsilon})}(X)}{(d+1) \deg_L(X)} \leq e_1(X, (L, \|\cdot\| \otimes \|\cdot\|_f^{\otimes \varepsilon})) + S\varepsilon^2 \quad (29)$$

holds for  $\overline{L}$  and  $\varepsilon \in (-c, c)$ . This implies the absolute nature of the fundamental inequality. Passing to a finite base extension, we may assume that  $\|\cdot\|_f$  is algebraic (Proposition 3.4), i.e. there is a projective flat scheme  $\mathcal{X}$  over  $C$  with generic fibre  $X$  and a line bundle  $\mathcal{N}$  on  $\mathcal{X}$  which is trivial on  $X$  such that  $\|\cdot\|_f = \|\cdot\|_{\mathcal{N}}$ .

First, we prove the claim if  $\|\cdot\|$  is a formal metric and  $\varepsilon \in \mathbb{Q}$ . By Proposition 3.4, there is a finite extension  $K' = k(C')/K$  such that the base change  $\|\cdot\|'$  of  $\|\cdot\|$  to  $K'$  is an algebraic metric and the claim follows from (27).

Next, we prove that (29) holds for roots of formal metrics and  $\varepsilon \in \mathbb{Q}$ . Indeed, the definition of  $c$  in (22) shows that  $c$  depends only on  $(L, \mathcal{N})$  and this homogeneously of degree 0. Using property (a) of the constant  $C_5$ , we easily deduce that  $S$  from (27) is a homogeneous function of  $(\mathcal{L}, \mathcal{N}, \|\cdot\|_0)$  of degree 1. This allows us to deduce (29) for roots of formal metrics from the case of formal metrics.

In general,  $\|\cdot\|$  is the limit of roots of semipositive formal  $M_C$ -metrics  $\|\cdot\|_n$  with respect to the distance of metrics on  $L$ . We choose rational approximations  $\varepsilon_n$  of  $\varepsilon$ . By the special case above, inequality (29) holds for  $(L, \|\cdot\|_n)$ ,  $\varepsilon_n$  and  $S_n$  instead of  $(L, \|\cdot\|)$ ,  $\varepsilon$  and  $S$ . By Theorem 3.5(e), we deduce easily

$$\lim_{n \rightarrow \infty} e_1(X, (L, \|\cdot\|_n \otimes \|\cdot\|_f^{\otimes \varepsilon_n})) = e_1(X, (L, \|\cdot\| \otimes \|\cdot\|_f^{\otimes \varepsilon})). \quad (30)$$

Since  $\mathcal{X}$  is projective, we may write  $\mathcal{N}$  as the "difference" of two ample line bundles on  $\mathcal{X}$ . Using multilinearity of the height and Corollary 3.8, we get

$$\lim_{n \rightarrow \infty} h_{(L, \|\cdot\|_n \otimes \|\cdot\|_f^{\otimes \varepsilon_n})}(X) = h_{(L, \|\cdot\| \otimes \|\cdot\|_f^{\otimes \varepsilon})}(X). \quad (31)$$

Since  $c$  is independent of the metrics and  $S_n \rightarrow S$  (by property (b) in Remark 5.7), we deduce (29) for  $(L, \|\cdot\|)$  immediately from (30) and (31).  $\square$

**Proposition 5.11** *The fundamental inequality in the form given in Proposition 5.10 holds for arbitrary function fields.*

**Proof:** The heights and the essential minimum are invariant under algebraic base extension of  $K$  and hence we may assume that  $K$  is the function field of an irreducible projective variety  $B$  over the algebraically closed field  $k$  such that  $B$  is regular in codimension 1. Clearly, we may assume that  $\mathbf{c}$  is very ample. Let  $C := B_{\mathbf{c}}$  be the generic curve over  $k''$  constructed in 3.11. We have seen in 3.12 that base change to the function field  $K' := k''(B_{\mathbf{c}})$  induces a canonical semipositive admissible  $M_C$ -metric  $\|\cdot\|_{\mathbf{c}}$  on  $L' := L \otimes_K K'$  such that heights are invariant under base change to  $K'$ . Since the fundamental inequality holds for  $K'$  by Proposition 5.10 and since the distance between the metrics is invariant under base change, it is enough to prove that the essential minimum decreases under base change to  $K'$ .

Let  $Y'$  be a proper closed subset of  $X' = X \otimes_K K'$ . Using a suitable specialization, it is easy to construct a proper closed subset  $Y$  of  $X$  such that  $P \notin Y(\overline{K})$  implies  $P \notin Y'$ . We get

$$\inf_{P \in X(\overline{K}) \setminus Y} h_{(L, \|\cdot\|)}(P) \geq \inf_{P \in X(\overline{K}) \setminus Y'} h_{(L, \|\cdot\|)}(P) \geq \inf_{P \in X'(\overline{K}') \setminus Y'} h_{(L, \|\cdot\|_{\mathbf{c}})}(P)$$

proving immediately that the essential minimum decreases under base change to  $K'$ .  $\square$

## 6 Equidistribution theorems

In this section,  $K$  is always a function field. First, we will deduce Theorem 1.1 from the fundamental inequality. Then we apply our methods to dynamical systems. In this case, we will generalize equidistribution to small generic nets of subvarieties. As examples, we will consider abelian varieties and multiplicative tori.

**Proof of Theorem 1.1:** We have seen in 2.3 that formal functions generate a dense  $\mathbb{Q}$ -subspace of  $C(X_v^{\text{an}})$ , hence it is enough to check

$$\lim_m \frac{1}{|O(P_m)|} \sum_{P_m^\sigma \in O(P_m)} f(P_m^\sigma) = \frac{1}{\deg_L(X)} \int_{X_v^{\text{an}}} f c_1(L, \|\cdot\|_v)^{\wedge d} \quad (32)$$

for a formal function  $f$  on  $X_v^{\text{an}}$ . Then  $f = -\log \|1\|_{f,v}$  for a formal  $M_B$ -metric  $\|\cdot\|_f$  of  $O_X$  which is trivial for all places different from  $v$ . Since the net  $(P_m)_{m \in I}$  is generic in  $X$ , the fundamental inequality (Proposition 5.11) yields

$$\frac{1}{(d+1) \deg_L(X)} h_{(L, \|\cdot\| \otimes \|\cdot\|_f^{\otimes \varepsilon})}(X) \leq \liminf_m h_{(L, \|\cdot\| \otimes \|\cdot\|_f^{\otimes \varepsilon})}(P_m) + O(\varepsilon^2), \quad (33)$$

where the implicit constant is independent of  $\varepsilon \in (-c, c)$ . By Theorem 3.5, we easily deduce

$$h_{(L, \|\cdot\|_{\otimes} \|\cdot\|_f^{\otimes \varepsilon})}(X) = h_{\mathcal{L}}(X) + \varepsilon(d+1)\mu(v) \int_{X_v^{\text{an}}} f c_1(L, \|\cdot\|_v)^{\wedge d} + O(\varepsilon^2)$$

and

$$h_{(L, \|\cdot\|_{\otimes} \|\cdot\|_f^{\otimes \varepsilon})}(P_m) = h_{\mathcal{L}}(P_m) + \varepsilon \frac{\mu(v)}{|O(P_m)|} \sum_{P_m^\sigma \in O(P_m)} f(P_m^\sigma) + O(\varepsilon^2).$$

If we insert both identities in (33), then  $\lim_m h_{\mathcal{L}}(P_m) = \frac{h_{\mathcal{L}}(X)}{(d+1)\deg_L(X)}$  leads to

$$\frac{\varepsilon}{\deg_L(X)} \int_{X_v^{\text{an}}} f c_1(L, \|\cdot\|_v)^{\wedge d} \leq \liminf_m \frac{\varepsilon}{|O(P_m)|} \sum_{P_m^\sigma \in O(P_m)} f(P_m^\sigma) + O(\varepsilon^2).$$

Choosing  $\varepsilon > 0$  and letting  $\varepsilon \rightarrow 0$ , we get “ $\geq$ ” in (32) with “ $\lim$ ” replaced by “ $\liminf$ ”. Using the same argument for  $-f$  instead of  $f$ , we get also “ $\leq$ ” for the “ $\limsup$ ” and hence the claim.  $\square$

**6.1** We consider a dynamical system  $(W, \Phi, L)$ , where  $L$  is an ample line bundle on the irreducible projective variety  $W$  over the function field  $K = k(B)$  and  $\Phi : W \rightarrow W$  is a morphism with

$$\Phi^*(L) \cong L^{\otimes q}$$

for some  $q \in \mathbb{N}$  with  $q > 1$ . This is an isometry with respect to a unique admissible  $M_B$ -metric  $\|\cdot\|_{\text{can}}$  of  $L$ . In fact, the canonical metric is given as the uniform limit of recursively defined metrics

$$\|\cdot\|_j := \left( \Phi^* \|\cdot\|_{j-1} \right)^{1/q}$$

on  $L$ , where  $\|\cdot\|_0$  is any admissible  $M_B$ -metric on  $L$  (see [BG], Theorem 9.5.4 and its proof). We may choose  $\|\cdot\|_0$  as a root of an algebraic metric associated to an ample model and hence  $\|\cdot\|_{\text{can}}$  is semipositive, even uniform limit of roots of ample metrics. We call

$$\hat{h}_L(Z) := h_{(L, \|\cdot\|_{\text{can}})}(Z)$$

the *canonical height* of the cycle  $Z$  of  $W$ . Since  $\|\cdot\|_{\text{can}}$  is uniform limit of roots of ample metrics, the canonical height is non-negative for effective cycles. By functoriality in Theorem 3.5, we deduce easily  $\hat{h}_L(W) = 0$ .

**6.2** In the number field case, Yuan ([Yu], Theorem 5.6) proved a general *equidistribution theorem of small subvarieties* motivated by the case of abelian varieties due to Autissier [Au] and Baker–Ih [BI]. Next, we will give a similar equidistribution theorem for the dynamical system  $(W, \Phi, L)$  over the function field  $K = k(B)$ . A similar result was proved independently by Faber ([Fa], Theorem 4.1) in the case of function fields of curves.

Let  $X$  be an irreducible  $d$ -dimensional closed subvariety of  $W$ . We consider a *generic net*  $(Y_m)_{m \in I}$  of irreducible  $t$ -dimensional closed subvarieties of  $X \otimes_K \overline{K}$ , i.e. for every proper closed subset  $Y$  of  $X$ , there is  $m_0 \in I$  such that  $Y_m$  is not contained in  $Y$  for all  $m \geq m_0$ . We assume that the net is *small* in the sense of

$$\lim_m \frac{1}{(t+1)\deg_L(Y_m)} \hat{h}_L(Y_m) = 0. \quad (34)$$

Let us denote the  $\text{Gal}(\overline{K}/K)$ -orbit of  $Y_m$  by  $O(Y_m)$ . We fix a place  $v \in M_B$  and an embedding  $\overline{K} \hookrightarrow \mathbb{K}_v$  over  $K$  to identify  $Y_m \otimes_K \overline{K}$  with a subvariety of  $X_v^{\text{an}}$ . Then we may view  $\deg_L(Y_m)^{-1} c_1(L|_{Y_m}, \|\cdot\|_{\text{can}, v})^{\wedge t}$  as a regular probability measure on  $X_v^{\text{an}}$  with support in  $(Y_m)_v^{\text{an}}$ .

**Theorem 6.3** *Under the hypothesis of 6.3, we have  $\hat{h}_L(X) = 0$  and the following weak limit of regular probability measures on  $X_v^{\text{an}}$ :*

$$\frac{1}{|O(Y_m)| \deg_L(Y_m)} \sum_{Y_m^\sigma \in O(Y_m)} c_1(L|_{Y_m^\sigma}, \|\cdot\|_{\text{can},v})^{\wedge t} \xrightarrow{w} \frac{1}{\deg_L(X)} c_1(L|_X, \|\cdot\|_{\text{can},v})^{\wedge d}.$$

**Proof:** We need some preliminary considerations using the methods of §5. We first assume that  $K$  is the function field of an irreducible projective curve  $C$  over an algebraically closed field  $k$  and that  $\mathbf{c}$  has degree 1. We also assume that  $\|\cdot\|$  is an algebraic  $M_C$ -metric on  $L$  induced by an ample model  $\mathcal{L}$  on a flat projective scheme  $\pi: \mathcal{X} \rightarrow C$  with generic fibre  $X$ . As in 5.5, we consider a line bundle  $\mathcal{N}$  on  $\mathcal{X}$  with  $\mathcal{N}|_X = O_X$  and  $\varepsilon \in \mathbb{Q}$ . We use the notation of 5.6–5.9. For  $\varepsilon \in (-c, c)$ , there is a non-zero global section  $s$  of  $(\mathcal{L}_r \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}$  if

$$r > \frac{-h_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(X) + C_5 \varepsilon^2}{(d+1) \deg_{\mathcal{M}'}(C) \deg_L(X)} \quad (35)$$

and if  $m$  is sufficiently large and sufficiently divisible (see (25)). Let  $Y$  be a generic  $t$ -dimensional closed subvariety of  $X \otimes_K \bar{K}$  and  $Z := \text{div}(s|_Y) \in Z_{t-1}(X)$ . Let  $K'$  be a finite extension of  $K$  such that  $Y$  is defined over  $K'$  and let  $X', L'$  be the base change of  $X, L$  to  $K'$ . By the induction formula for heights of subvarieties (see [Gu2], Remark 9.5) or by an easy direct calculation of intersection numbers, we get

$$\begin{aligned} & h_{\mathcal{L}, \dots, \mathcal{L}, (\mathcal{L}_r \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}}(Y) \\ &= h_{\mathcal{L}}(Z) - \sum_{w \in M_{K'}} \mu(w) \int_{Y_w^{\text{an}}} \log \|s\|'_w c_1(L'|_Y, \|\cdot\|_{\mathcal{L},w})^{\wedge t}, \end{aligned} \quad (36)$$

where  $\|\cdot\|'_w$  is induced by the model  $(\mathcal{L}_r \otimes \mathcal{N}^{\otimes \varepsilon})^{\otimes m}$ . Since  $\|s\|'_w \leq 1$ , we get

$$h_{\mathcal{L}}(Y) + r \deg_{\mathcal{M}'}(C) \deg_L(Y) + \varepsilon h_{\mathcal{L}, \dots, \mathcal{L}, \mathcal{N}}(Y) \geq \frac{1}{m} h_{\mathcal{L}}(Z).$$

Using  $h_{\mathcal{L}}(Z) \geq 0$  for the effective cycle  $Z$ , we deduce

$$\frac{1}{\deg_L(Y)} h_{\mathcal{L}}(Y) \geq -r \deg_{\mathcal{M}'}(C) - \varepsilon \frac{h_{\mathcal{L}, \dots, \mathcal{L}, \mathcal{N}}(Y)}{\deg_L(Y)}.$$

Now the right hand side is also independent of  $s$ . If  $r$  approaches the right hand side of (35), then we get

$$\frac{1}{\deg_L(Y)} h_{\mathcal{L}}(Y) \geq \frac{h_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(X) - C_5 \varepsilon^2}{(d+1) \deg_L(X)} - \varepsilon \frac{h_{\mathcal{L}, \dots, \mathcal{L}, \mathcal{N}}(Y)}{\deg_L(Y)}.$$

If we use Corollary 3.8 for  $h_{\mathcal{L} \otimes \mathcal{N}^{\otimes \varepsilon}}(X)$  in the same way as in the proof of Lemma 5.6, then the right hand side has the lower bound

$$\frac{h_{\mathcal{L}}(X)}{(d+1) \deg_L(X)} + \varepsilon \left( \frac{h_{\mathcal{L}, \dots, \mathcal{L}, \mathcal{N}}(X)}{\deg_L(X)} - \frac{h_{\mathcal{L}, \dots, \mathcal{L}, \mathcal{N}}(Y)}{\deg_L(Y)} \right) - \frac{C_6 \varepsilon^2}{\deg_L(X)}.$$

Similarly as for the fundamental inequality, we can extend the resulting inequality to arbitrary semipositive admissible  $M_C$ -metrics on  $L$  and to any formal metric  $\|\cdot\|_f$  on  $O_X$  to get

$$\begin{aligned} \frac{1}{\deg_L(Y)} h_{\bar{L}}(Y) &\geq \frac{h_{\bar{L}}(X)}{(d+1) \deg_L(X)} \\ &+ \varepsilon \left( \frac{h_{\bar{L}, \dots, \bar{L}, \bar{O}_X^f}(X)}{\deg_L(X)} - \frac{h_{\bar{L}, \dots, \bar{L}, \bar{O}_X^f}(Y)}{\deg_L(Y)} \right) - \frac{C_6 \varepsilon^2}{\deg_L(X)}, \end{aligned} \quad (37)$$

where  $\overline{O}_X^f := (O_X, \|\cdot\|_f)$ . Again, the constant  $C_6$  is absolute. Next, we note that (37) holds for arbitrary function fields  $K = k(B)$ . Indeed, we may assume  $\mathbf{c}$  very ample and  $k$  algebraically closed. Then the base change to the function field  $F$  of the generic curve does not change the heights (see 3.12). By the usual specialization argument, the subvariety  $Y \otimes_{\overline{K}} \overline{F}$  remains generic in  $X \otimes_K F$  and hence (37) follows from the corresponding inequality over  $F$ .

We apply (37) now with  $Y = Y_m$ ,  $\overline{L} = (L, \|\cdot\|_{\text{can}})$  and we pass to the limit with respect to  $m$ . Setting  $\varepsilon = 0$  and using non-negativity of the height, we deduce from (34) that  $\hat{h}_L(X) = 0$ . For arbitrary  $\varepsilon > 0$ , we get

$$\varepsilon \frac{h_{\overline{L}, \dots, \overline{L}, \overline{O}_X^f}(X)}{\deg_L(X)} \leq \varepsilon \liminf_m \frac{h_{\overline{L}, \dots, \overline{L}, \overline{O}_X^f}(Y_m)}{\deg_L(Y_m)} + O(\varepsilon^2).$$

Letting  $\varepsilon \rightarrow 0$ , we conclude

$$\frac{h_{\overline{L}, \dots, \overline{L}, \overline{O}_X^f}(X)}{\deg_L(X)} \leq \liminf_m \frac{h_{\overline{L}, \dots, \overline{L}, \overline{O}_X^f}(Y_m)}{\deg_L(Y_m)}. \quad (38)$$

Now let  $f$  be a formal function on  $X_v^{\text{an}}$  inducing a formal  $M_B$ -metric  $\|\cdot\|_f$  on  $O_X$  as at the beginning of this section. Then Theorem 3.5 shows

$$h_{\overline{L}, \dots, \overline{L}, \overline{O}_X^f}(X) = \mu(v) \int_{X_v^{\text{an}}} f c_1(\overline{L})^{\wedge d}. \quad (39)$$

Let us choose a finite normal subextension  $K'/K$  of  $\overline{K}/K$  such that  $Y$  is defined over  $K'$ . Again Theorem 3.5 yields

$$h_{\overline{L}, \dots, \overline{L}, \overline{O}_X^f}(Y_m) = \sum_{w|v} \mu(w) \int_{(Y_m)_w^{\text{an}}} f c_1(\overline{L})^{\wedge t}, \quad (40)$$

where  $w$  ranges over all places of  $K'$  over  $v$ . Since  $K'/K$  is normal,  $\text{Aut}(K'/K)$  acts transitively on these places and hence  $\mu(w)$  is independent of  $w$ . Moreover,  $(Y_m)_w^{\text{an}}$  ranges over the analytic spaces associated to the conjugates  $Y_m^\sigma \in O(Y)$  and every  $Y_m^\sigma$  is attained the same number of times. We conclude that

$$h_{\overline{L}, \dots, \overline{L}, \overline{O}_X^f}(Y_m) = \lambda_m \sum_{Y_m^\sigma \in O(Y)} \int_{(Y_m^\sigma)_v^{\text{an}}} f c_1(\overline{L})^{\wedge t} \quad (41)$$

for some  $\lambda_m > 0$ . To determine the number  $\lambda_m$ , we use the constant function  $f = 1$  in (40). Then the integral is equal to  $\deg_L((Y_m)_w^{\text{an}}) = \deg_L(Y)$  by Proposition 2.6. The normalizations satisfy  $\sum_{w|v} \mu(w) = \mu(v)$  (see [BG], Example 1.4.13) and hence we get  $\lambda_m = \mu(v) |O(Y_m)|^{-1}$ . Using (39) and (41), Theorem 6.3 follows from (38) in the same way as we proved Theorem 1.1.  $\square$

**Remark 6.4** If  $W$  is an abelian variety  $A$  over  $K$  and  $L$  is an ample even line bundle, then the theorem of the cube shows that multiplication by  $m \in \mathbb{N} \setminus \{0, 1\}$  leads to a dynamical system  $(A, [m], L)$  with canonical height equal to the Néron–Tate height. For a  $d$ -dimensional irreducible closed subvariety  $X$  of  $A$ , the equidistribution measure  $c_1(L|_X, \|\cdot\|_{\text{can}, v})^{\wedge d}$  is explicitly described in terms of convex geometry by [Gu5], Theorem 6.7.

**Remark 6.5** If we use the multiplicative torus  $\mathbb{G}_m^n$  over  $K$  instead of an abelian variety, then  $\mathbf{x} \mapsto \mathbf{x}^m$  extends to a dynamical system  $(\mathbb{P}_K^n, \Phi, O_{\mathbb{P}^n}(1))$  with canonical height equal to the classical Weil height of  $\mathbb{P}_K^n$ . Since  $\mathbb{P}_K^n$  has good reduction with respect to any  $v \in M_K$ , the canonical equidistribution measure in Theorem 6.3 is the Dirac measure in the unique point of  $(\mathbb{P}_K^n)_v^{\text{an}}$  whose reduction modulo  $v$  is the generic point. In the number field case, this non-archimedean analogue of Bilu's theorem ([Bi], Theorem 1.1) was proved by Chambert-Loir (see [Ch], Exemple 3.2).

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